Automatic structures, chain Monadic Second-Order logic, Epistemic planning synthesis.

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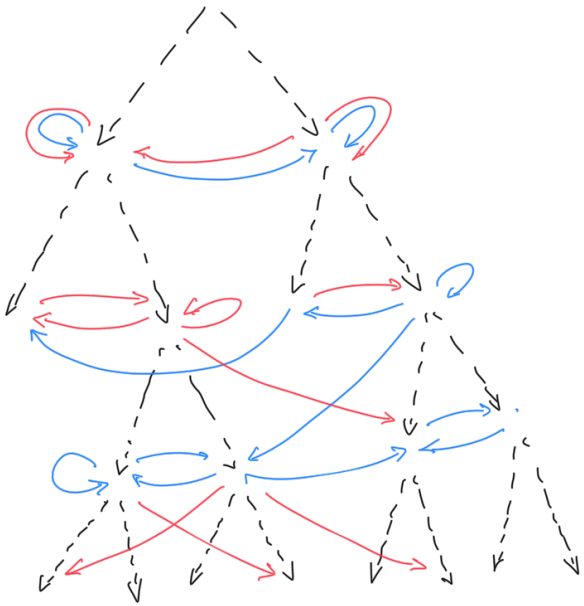
Outline

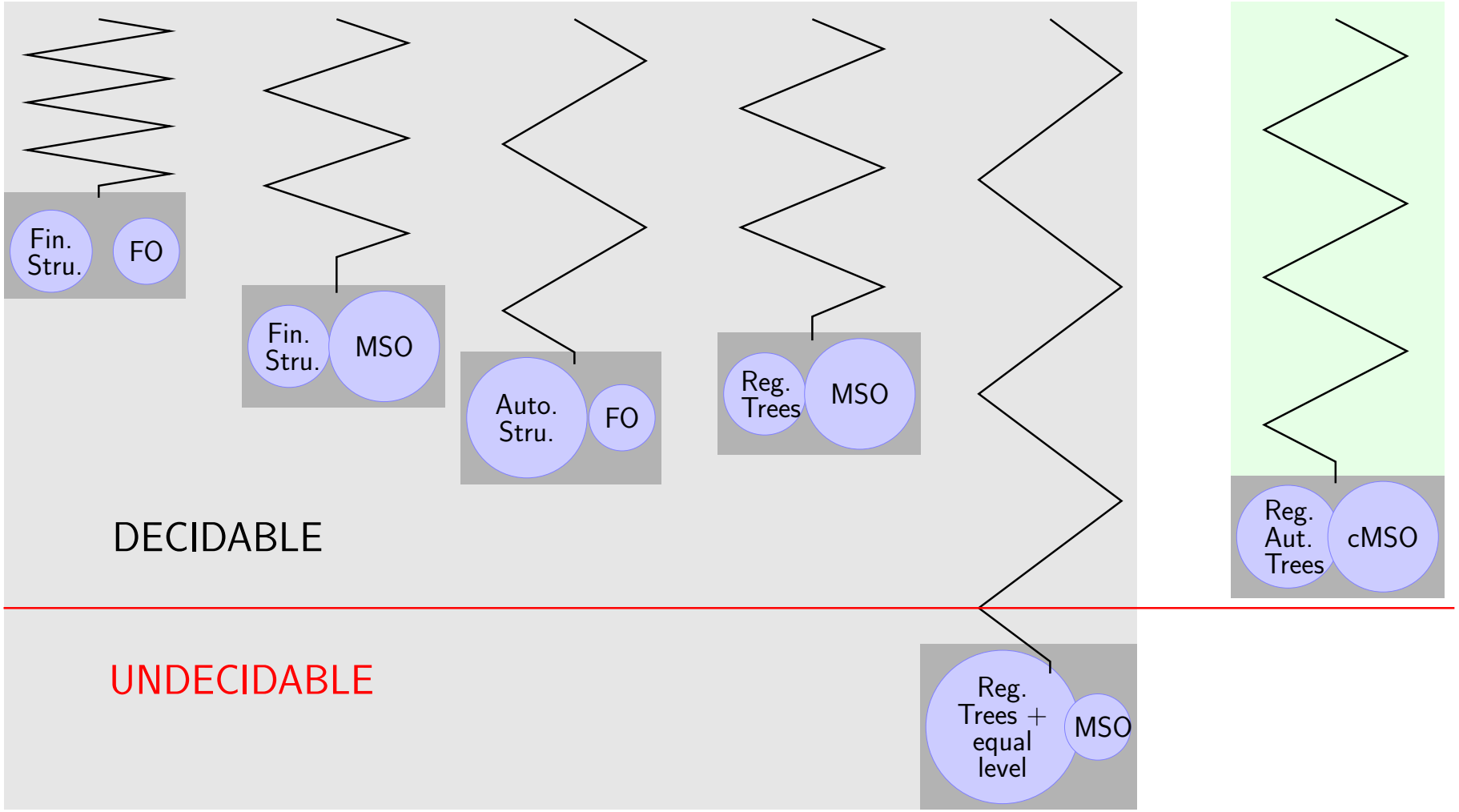
Motivations



- Uniform strategies in imperfect information games (Maubert 2014)
- Models of epistemic temporal logic, interpreted systems (Halpern-Vardi 1989)
- Epistemic planning (Pinchinat et al. 2018)

Trade-off between $\begin{cases} \text{the class of infinite structures} \\ \text{the considered logic} \end{cases}$

Contribution



Outline

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(Relational) structures

Interpretations for logics like FO and MSO

$\mathcal{S} = \langle D, R_1 \dots R_p \rangle$

- Domain $D \neq \emptyset$
- Relations $R_i \subseteq D^{r_i}$

Example

$\langle \mathbb{N}, \leq \rangle$

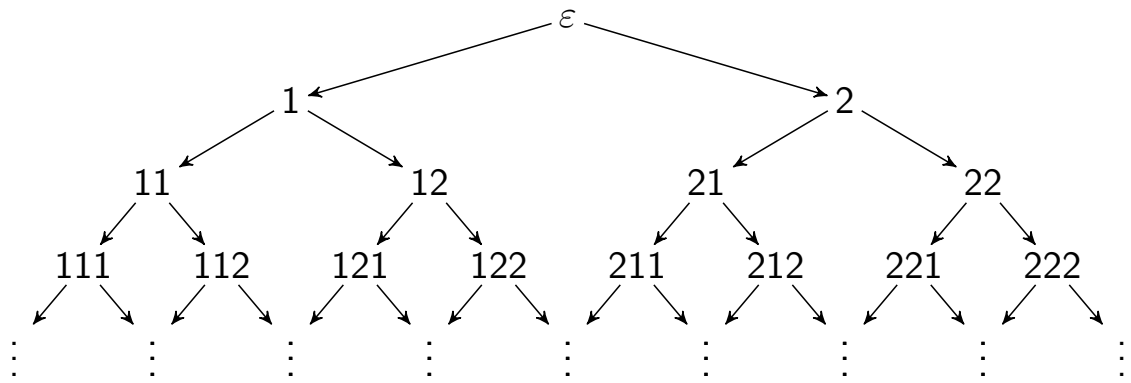
Definition (Tree structures)

$\mathcal{T} = \langle D, r, S_1, \dots, S_n, R_1, \dots, R_p \rangle$ (n-ary tree)

- $D \subseteq \{1, \dots, n\}^*$ prefix-closed (addresses of the nodes)
- $r(\varepsilon)$ only (root node)
- $S_j(u, v.j)$ whenever $u.j \in D$
- $R_1, \dots, R_p$ additional relations on D such as the "at equal level" binary relation.

Full binary trees

- $\mathcal{T}_2 = \langle \{1,2\}^*, S_1, S_2 \rangle$



$S_i(u, u.i)$ for every $u \in \{1,2\}^*$

- $\mathcal{T}_2^{\mathrm{el}} = \langle \{1,2\}^*, S_1, S_2, \mathrm{el} \rangle$ with relation "at equal level":

$$\mathrm{el}(u,v) \text{ whenever } |u| = |v|$$

Outline

An example

Take structure $\langle \mathbb{N}, \leq \rangle$.

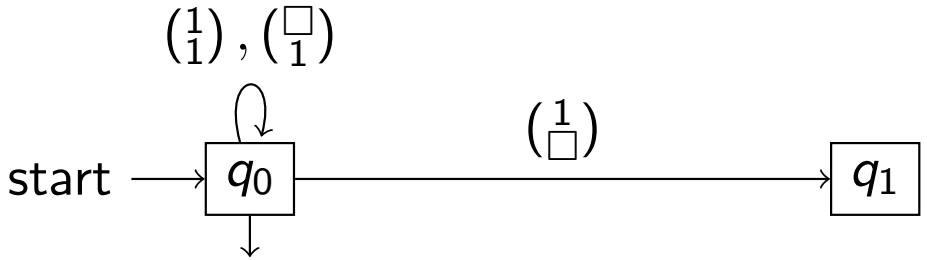
$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

- Encode $n \in \mathbb{N}$ by $\overbrace{11\ldots1}^{n} = enc(n)$
- Encoding of relation $\leq$: the pair $(1^2, 1^3)$ is represented by

$$1^2 \otimes 1^3 := \binom{1}{1}\binom{1}{1}\binom{\square}{1} \qquad \text{(convolution)}$$

Use two-tape automaton to accept input $1^n \otimes 1^m$ whenever $n \leq m$.

$\mathcal{A}_{\leq}$:



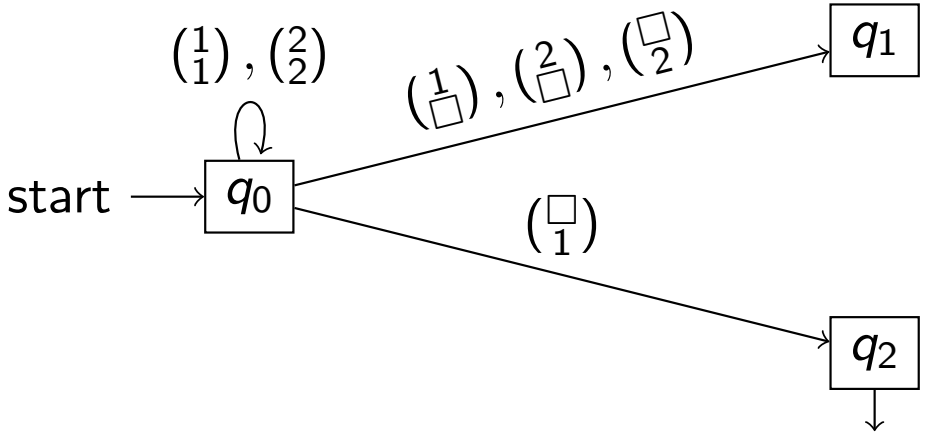
- $1^2 \otimes 1^3 \in \mathcal{L}(\mathcal{A}_{\leq})$
- $1^3 \otimes 1^2 = \binom{1}{1}\binom{1}{1}\binom{1}{\square} \notin \mathcal{L}(\mathcal{A}_{\leq})$

$\mathcal{T}_2$ and $\mathcal{T}_2^{\mathrm{el}}$ are automatic

Recall $\mathcal{T}_2 = \langle \{1,2\}^*, S_1, S_2 \rangle$ and $\mathcal{T}_2^{\mathrm{el}} = \langle \{1,2\}^*, S_1, S_2, \mathrm{el} \rangle$

- Encode nodes with their address $u \in \{1,2\}^*$
- $S_1(u,v)$ iff $v = u.1$

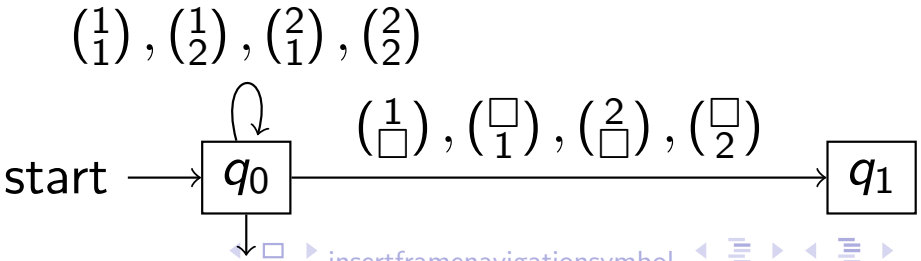
$\mathcal{A}_{S_1}$:



Similary $\mathcal{A}_{S_2}$...

- $\mathrm{el}(u,v)$ iff $|u| = |v|$

$\mathcal{A}_{\mathrm{el}}$:



Automatic presentations

Definition

An automatic presentation of structure $\mathcal{S} = \langle D, R_1 \dots R_p \rangle$ is $(\mathcal{A}_D, \mathcal{A}_1, \dots, \mathcal{A}_p)$ a tuple of (finite-state) automata with

- a bijective encoding function $enc : D \to \mathcal{L}(\mathcal{A}_D)$
- $\mathcal{L}(\mathcal{A}_i) = enc(R_i)$:

$$u_1 \otimes \cdots \otimes u_{r_i} \in \mathcal{L}(\mathcal{A}_i)$$
$$\text{iff}$$
$$u_j \in \mathcal{L}(\mathcal{A}_D) \text{ and } (enc^{-1}(u_1), \dots, enc^{-1}(u_{r_i})) \in R_i$$

$(\mathcal{A}_D, \mathcal{A}_1, \dots, \mathcal{A}_p) \rightsquigarrow$ structure $\mathcal{S}_{(\mathcal{A}_D, \mathcal{A}_1, \dots, \mathcal{A}_p)}$

Outline

Logics FO and MSO

- $\mathcal{V}_1 = \{x, x_1, x_2, \ldots\}$ set of for first-order variables
- $\mathcal{V}_2 = \{X, X_1, \ldots, Y, \ldots\}$ set of second-order variables

$$\text{MSO} \ni \varphi ::= \overbrace{R_i(x_1 \ldots x_{r_i}) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \varphi}^{\text{FO}} \mid x \in X \mid \exists X \varphi$$

Model checking on automatic structures against FO

$\text{FO} \ni \varphi ::= R_i(x_1 \dots x_{r_i}) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \varphi$

$$\text{MC-FO} \begin{cases} \textbf{Input :} \ (\mathcal{A}_D, \mathcal{A}_1, \dots, \mathcal{A}_p) \text{ an A.S., FO-formula } \varphi. \\ \textbf{Output :} \ \mathcal{S}_{(\mathcal{A}_D, \mathcal{A}_1, \dots, \mathcal{A}_p)} \models \varphi? \end{cases}$$

Theorem ([Khoussainov and Nerode 1995, Grädel and Bumensath 2000, Rubin 2008)

MC-FO *is decidable.*

Main ingredient: for $\varphi(x_1 \dots x_n) \in \text{FO}$, structure $\mathcal{S}$,

$$\varphi^{\mathcal{S}} := \{(d_1 \dots d_n) \in D^n \mid \mathcal{S}, [x_i \mapsto d_i] \models \varphi[x_1 \dots x_n]\}$$

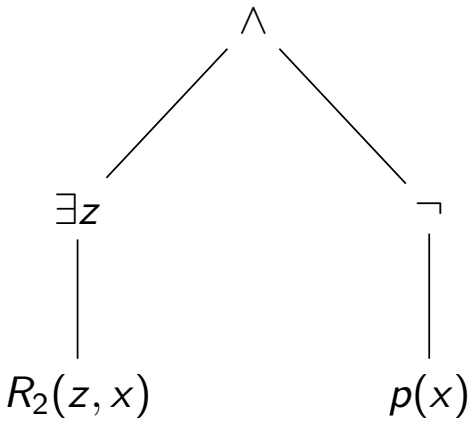
Proposition

Effective construction of automaton $\mathcal{A}_\varphi$ s.t. $\mathcal{L}(\mathcal{A}_\varphi) = enc(\varphi^{\mathcal{S}})$.

Bottom-up construction of $\mathcal{A}_\varphi$: intuitive example

Take $\varphi(x) := \exists z R_2(z,x) \wedge \neg p(x)$

$\varphi^{\mathcal{S}} := \{d \in D \mid \mathcal{S}, [x \mapsto d] \models \varphi[x]\}$



1. Project $\mathcal{A}_{R_2(x,z)}$ given by the automatic presentation, onto the first component and get $\mathcal{A}_{\exists z R_2(x,z)}$;
2. Take $\mathcal{A}_{p(x)} \in A$, complement it and get $\mathcal{A}^c_{p(x)}$ and compute $\mathcal{A}_D \cap \mathcal{A}^c_{p(x)}$ to get the automaton $\mathcal{A}_{\neg p(x)}$;
3. Compute $\mathcal{A}_{\exists z R_2(x,z)} \cap \mathcal{A}_{\neg p(x)}$ to get automaton $\mathcal{A}_{\exists z R_2(z,x) \wedge \neg p(x)}$.

$\mathcal{L}(\mathcal{A}_{\varphi(x)}) = \{enc(d) \mid d \in \varphi^{\mathcal{S}}\}$.

Model checking on automatic structures against MSO

$$\text{MSO} \ni \varphi ::= \overbrace{R_i(x_1 \dots x_{r_i}) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \varphi}^{\text{FO}} \mid x \in X \mid \exists X \varphi$$

$$\text{MC-MSO} \begin{cases} \textbf{Input :} \ (\mathcal{A}_D, \mathcal{A}_1, \dots, \mathcal{A}_p), \text{ MSO-formula } \varphi \\ \textbf{Output :} \ \mathcal{S}_{(\mathcal{A}_D, \mathcal{A}_1, \dots, \mathcal{A}_p)} \models \varphi ? \end{cases}$$

Proposition (Barany 2007)

MC-MSO *is decidable if* $\mathcal{L}(\mathcal{A}_D) \subseteq \{1\}^*$ *(unary alphabet, e.g.,* $(\mathbb{N}, \leq)$*)*

Theorem (Rabin 1969)

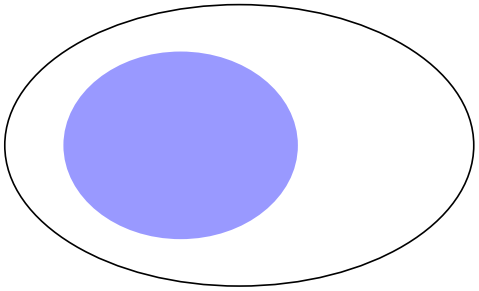
MSO-theory of $\mathcal{T}_2 = \langle \{1,2\}^*, S_1, S_2 \rangle$ *is decidable* *(full binary tree)*

Theorem (Thomas 1990)

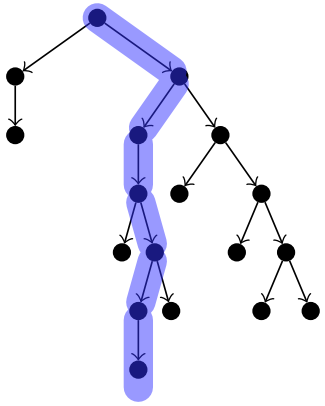
MSO-theory of $\mathcal{T}_2^{el} = \langle \{1,2\}^*, S_1, S_2, el \rangle$ *is undecidable* *(equal level)*

Variants of MSO over trees

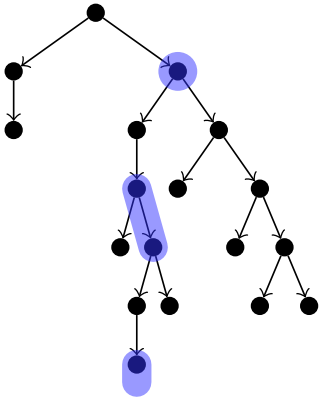
Different restrictions of second-order quantifications



(a) MSO
quantification over any subset



(b) path-MSO
quantification over any path in a tree



(c) cMSO
quantification over any chain in a tree

Outline

Regular automatic trees (RegAutTrees)

A particular encoding!

Definition (RegAutTree)

Tree $\mathcal{T} = \langle D, r, S_1, \ldots, S_n, R_1, \ldots, R_p \rangle$ is regular automatic if

- its language of node addresses is regular
- $enc_{address} : D \to \{1, \ldots, n\}^*$ allows an automatic presentation of $\mathcal{T}$

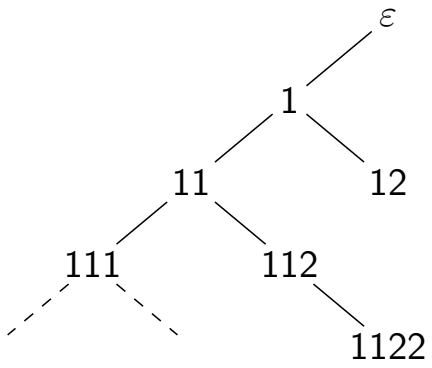
$$\langle \mathcal{A}_D, \mathcal{A}_r, (\mathcal{A}_{S_i})_{1 \leq i \leq n}, (\mathcal{A}_{R_i})_{1 \leq i \leq p} \rangle$$

The canonical representation of $\mathcal{T}$.

Theorem

RegAutTrees is a proper subclass of automatic trees.

Regular automatic trees $\subsetneq$ Automatic trees

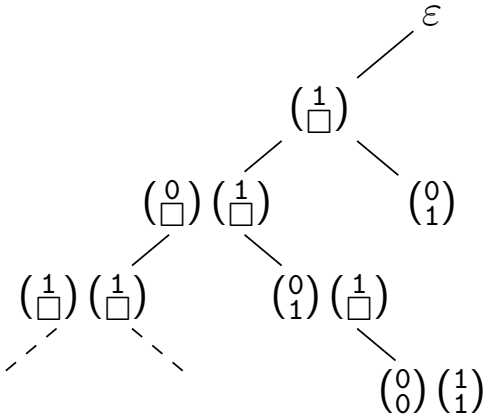


Set of addresses
$\{1^m 2^k \mid 0 \leq k \leq m\}$
$\rightsquigarrow$ not a regular automatic tree

Write $\mathtt{bin}(n)$ for the binary encoding of n with least significant digit first.

$$enc(1^m 2^k) := \mathtt{bin}(m) \otimes \mathtt{bin}(k)$$

$enc(112) = enc(1^2 2^1)$
$= \mathtt{bin}(2) \otimes \mathtt{bin}(1)$
$= \binom{0}{1}\binom{1}{\square}$



$enc(D)$, $enc(S_1)$, $enc(S_2)$ and $enc(\mathrm{el})$ are regular languages.

Properties of RegAutTrees

In $\mathcal{T} = \langle D, r, S_1, \ldots, S_n, R_1, \ldots, R_p \rangle$, define

- Generalized successor relation $S := \bigcup_{i=1}^{n} S_i$
- Reflexive and transitive closure S^* of S
- Binary relation $\preccurlyeq$ for 'deeper in the tree'
- Binary relation el for "at equal level'
- Equality relation $=$

Lemma

$\mathcal{T} = \langle D, r, S_1, \ldots, S_n, R_1, \ldots, R_p \rangle \in$ RegAutTrees implies $\langle D, r, S_1, \ldots, S_n, R_1 \ldots R_p, S^*, \preccurlyeq, el, = \rangle \in$ RegAutTrees.

FO is decidable on RegAutTrees.

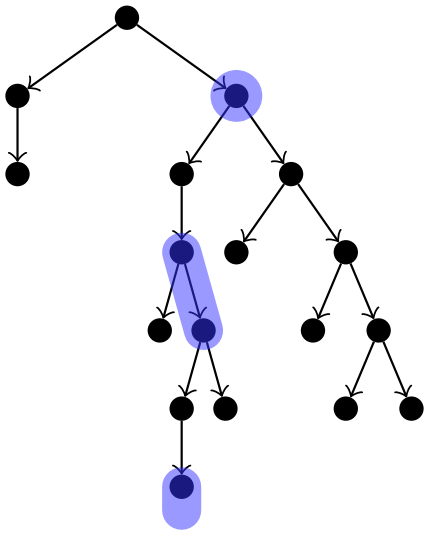
Chains in trees

Tree $\mathcal{T} = \langle D, r, S_1, \ldots, S_n, R_1, \ldots, R_p \rangle$.

Definition

$C \subseteq D$ is a chain if it is totally ordered with respect to S^*:

$$\text{for all } u, v \in C, \text{ either } S^*(u, v) \text{ or } S^*(u, v).$$



Outline

Logic cMSO

$$\text{cMSO} \ni \varphi ::= R_i(x_1 \dots x_{r_i}) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x\varphi \mid x \in X \mid \exists X\varphi$$

interpreted over $\mathcal{T} = \langle D, r, S_1, \dots, S_n, R_1, \dots, R_p \rangle$ + assignment σ

$\mathcal{T}, \sigma \models \exists X\varphi$ iff there exists a chain $C \in Chains(\mathcal{T})$ s.t. $\mathcal{T}, \sigma[X \mapsto C] \models \varphi$.

Example ("X is a maximal path starting at node x_0")

$x_0 \in X \wedge$
$\forall x \{x \in X \to [(\exists y S(x,y) \to \exists y(S(x,y) \wedge y \in X)) \wedge \neg S(x, x_0)]\}$

Corollary

cMSO subsumes path-MSO.

Model checking on RAT against cMSO

$$\text{MC-RATcMSO} \begin{cases} \textbf{Input :} \ \mathcal{T} \in \text{RAT (canon. pres.)}, \ \varphi \in \text{cMSO} \\ \textbf{Output :} \ \mathcal{T} \models \varphi? \end{cases}$$

Theorem

MC-RATcMSO is decidable.

The proof uses automata constructions, inspired from (Thomas 1997).

Corollaries of MC-RATcMSO decidability

- Since over a unary alphabet every set is a chain:

Corollary (Barany 2007)

The MSO-theory of automatic structures on a unary alphabet is decidable.

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- Since in RAT, relations $S^*, \preccurlyeq, \mathrm{el}, =$ are regular:

Corollary

The cMSO$[r, S_1, \ldots, S_n, R_1, \ldots, R_p, S^*, \preccurlyeq, el, =]$-theory of RATs is decidable.

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Corollary

The cMSO$[r, S_1, \ldots, S_n, R_1, \ldots, R_p, S^*, \preccurlyeq, el, =]$-theory of RATs is decidable.

- Since one can express in cMSO that a chain is a path:

Corollary

The path-MSO-theory of RATs is decidable.

Outline

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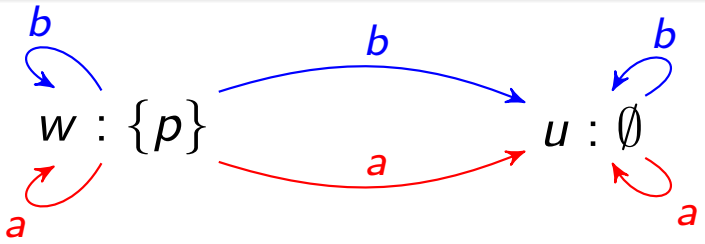
Dynamic Epistemic Logic: DEL presentation $(\mathcal{M}, w), \mathcal{E}$

(Baltag et al. 1998, van Ditmarsch et al. 2007)

$\mathcal{M}$ an epistemic model



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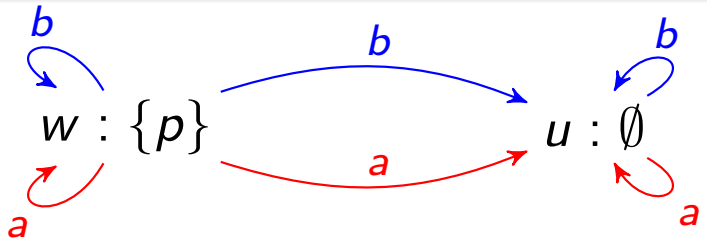
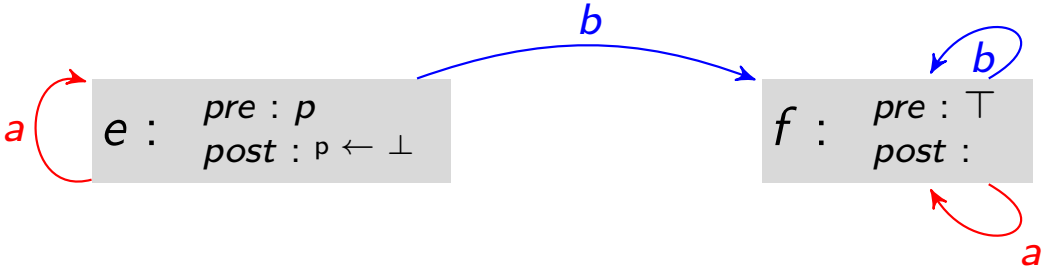


$\mathcal{E}$ an event model



$\mathcal{M} \otimes \mathcal{E}$

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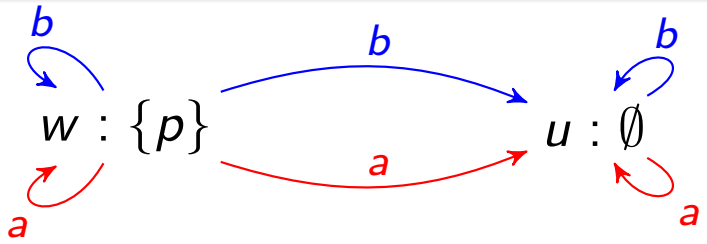
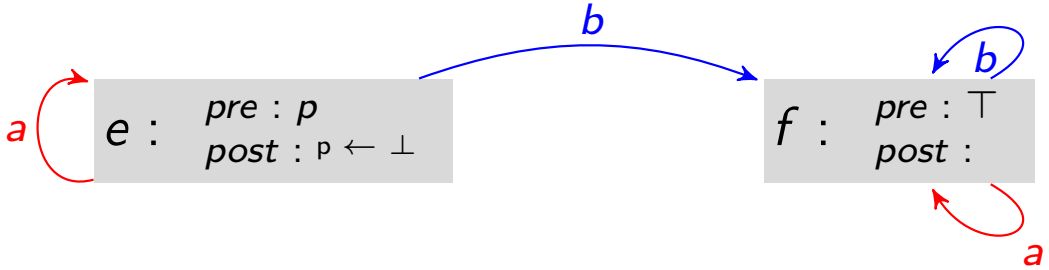


$\mathcal{E}$ an event model



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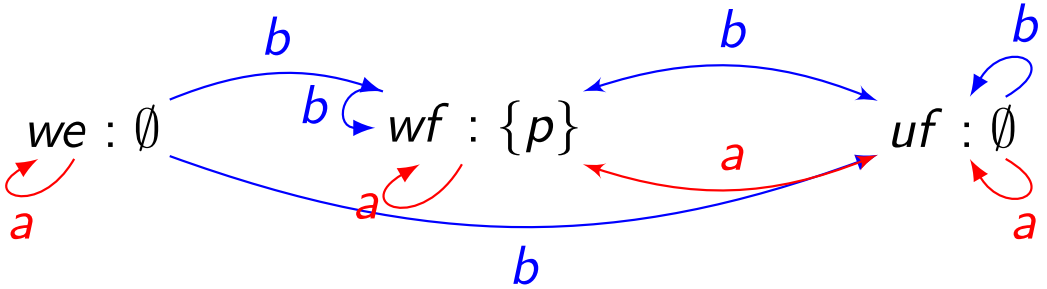

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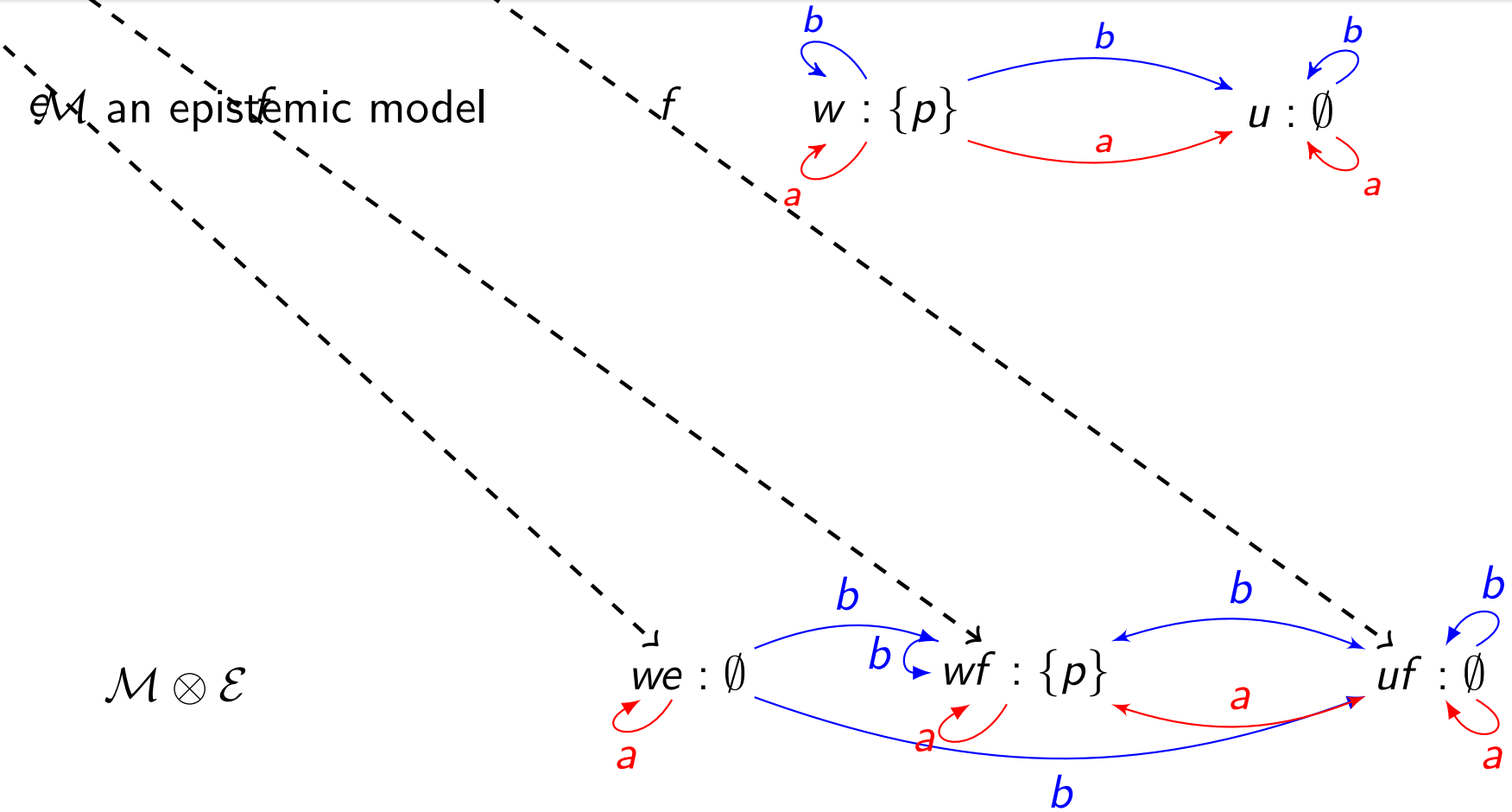

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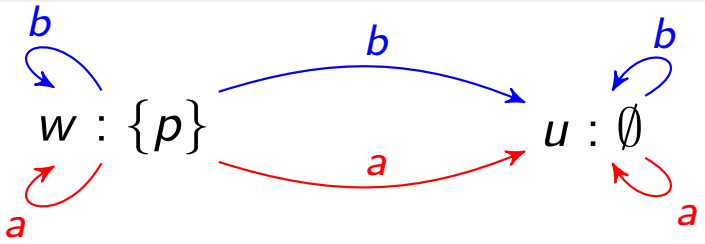

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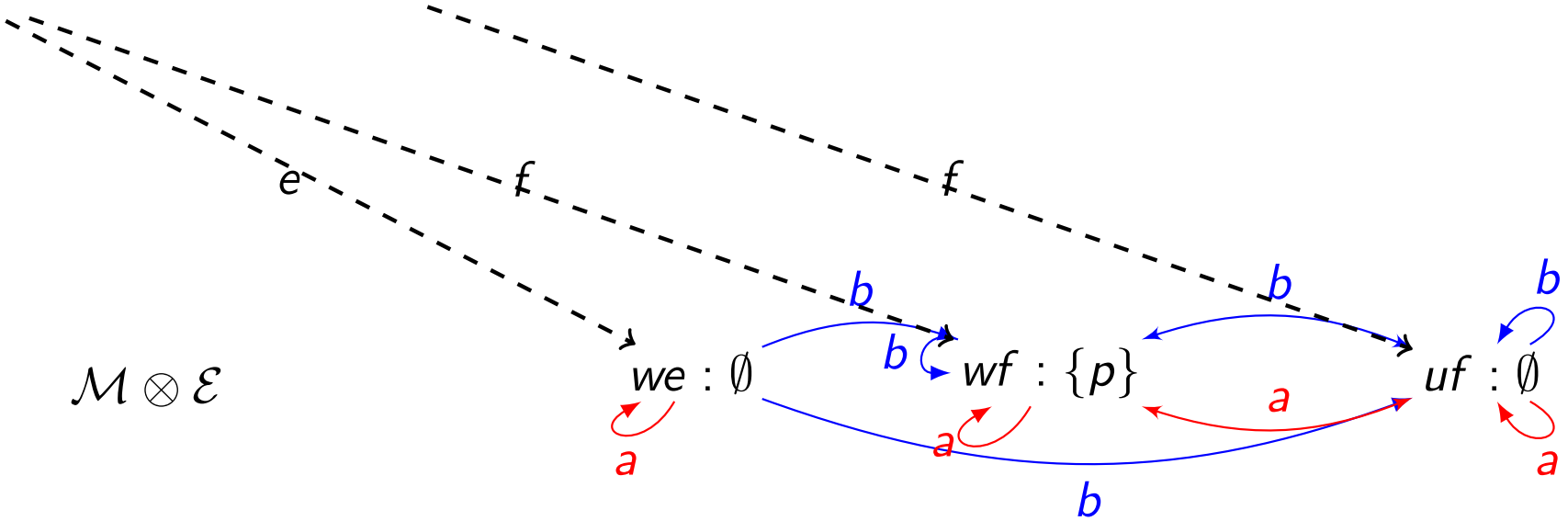

$\mathcal{M}, w \not\models K_a \neg p$

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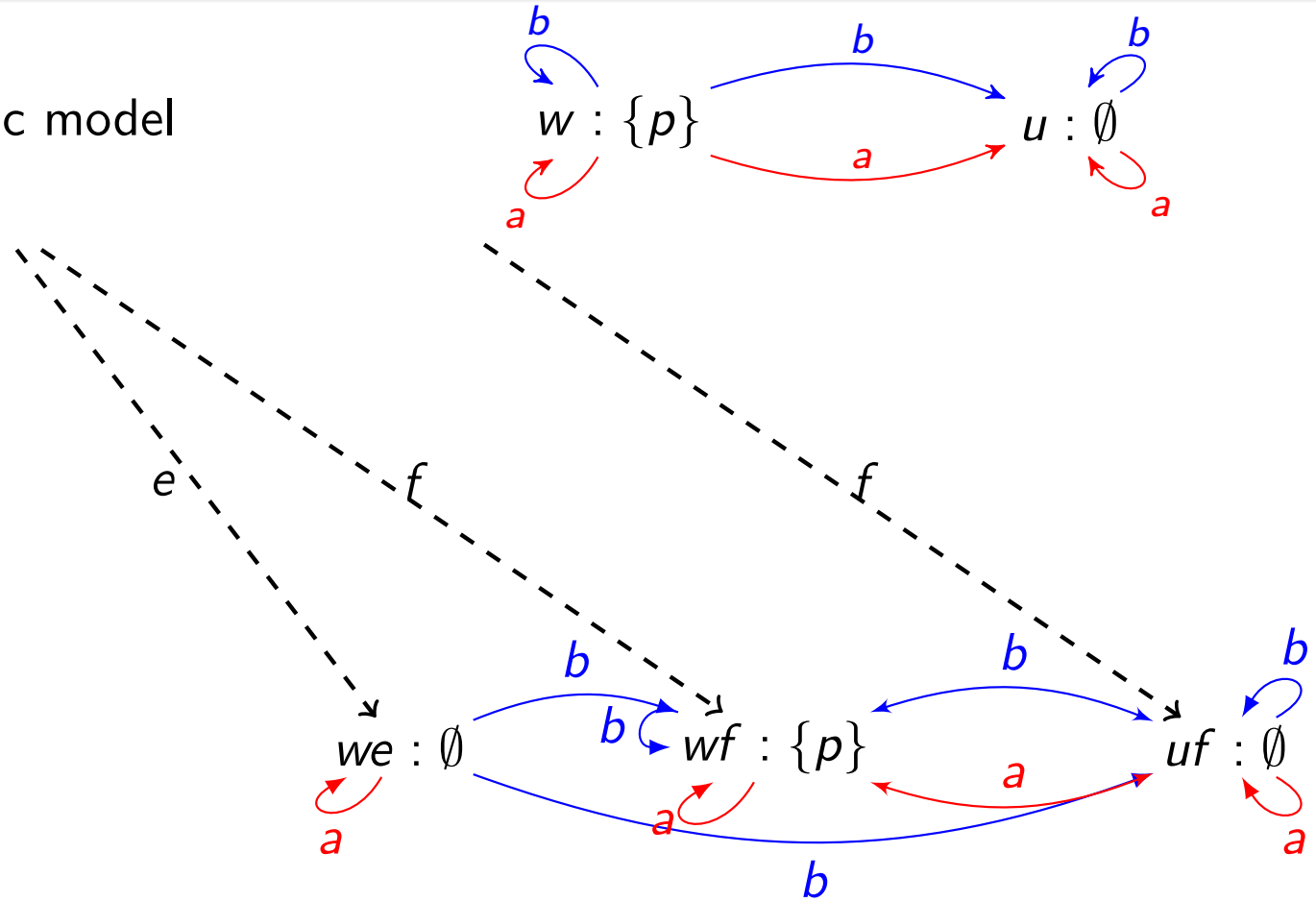

$\mathcal{M} \otimes \mathcal{E}$

$\mathcal{M}, w \not\models K_a \neg p$, but $\mathcal{M}, w \models < \mathcal{E}, e > K_a \neg p$

Dynamic Epistemic Logic: DEL presentation $(\mathcal{M}, w), \mathcal{E}$

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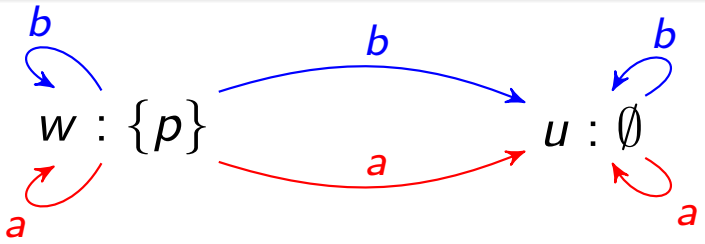
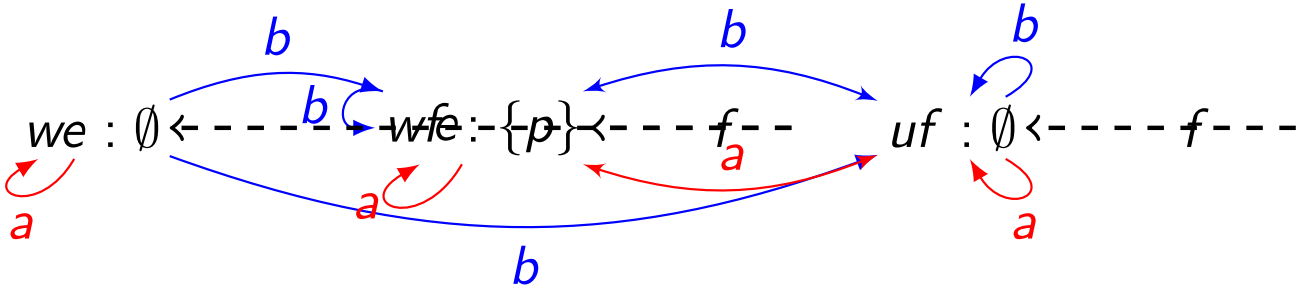
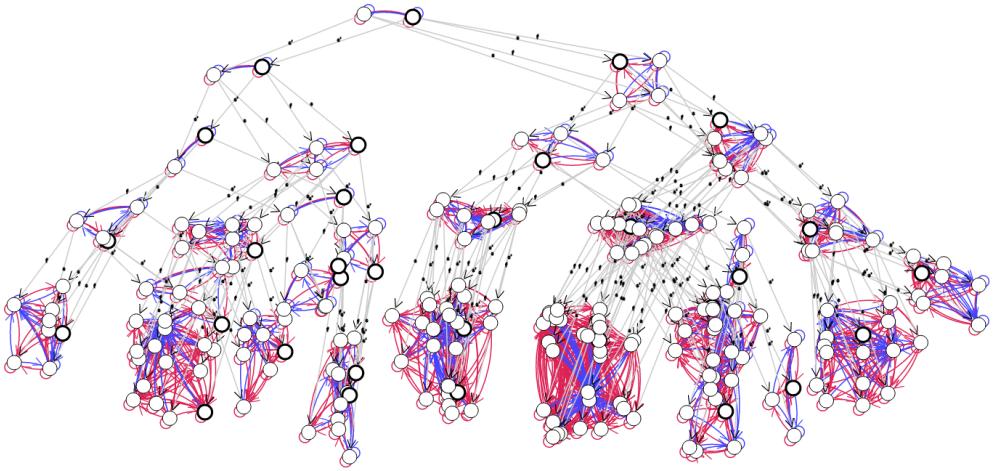


$\mathcal{M} \otimes \mathcal{E}$



$\mathcal{M}, w \not\models K_a \neg p$, but $\mathcal{M}, w \models < \mathcal{E}, e > K_a \neg p$ $\qquad$ $\mathcal{M} \otimes \mathcal{E}, we \models K_a \neg p$

Epistemic planning



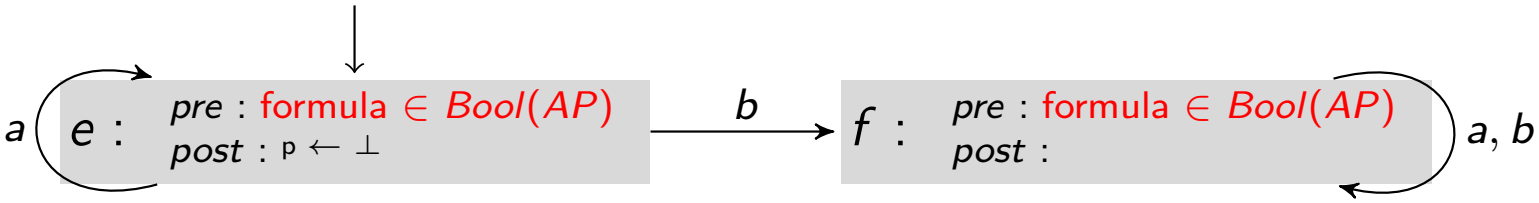
EPISTEMIC PLANNING

$$\begin{cases} \textbf{Input:} \text{ DEL presentation } (\mathcal{M}, w),\ \mathcal{E}, \text{ epistemic formula } \psi \\ \textbf{Output:} \text{ Is there } e_1, \ldots, e_n \text{ in } \mathcal{E} \text{ s.t. } \mathcal{M}\mathcal{E}^n, we_1 \ldots e_n \models \psi? \end{cases}$$

UNDECIDABLE by (Bolander et al. 2011, Yu et al. 2013)

Propositional DEL presentations

Events have propositional preconditions



Theorem (Maubert et al. 2014)

Propositional DEL structures ∈ RegAutTrees.

Epistemic planning for PDEL specifications

Corollary (Maubert et al. 2014, AiML2018)

EPISTEMIC PLANNING *is decidable for propositional DEL presentations.*

- On the PDEL structure, model check the property $\exists x \varphi_G(x)$.
- Automaton $\mathcal{A}_{\varphi(x)}$ provides all solution plans, since the encoding of nodes in the PDEL structure is the sequence of events that has led to that node, i.e. the plan.

Corollary

The set of solution plans is a regular language regular.

- One can exploit $\mathcal{A}_{\varphi_G(x)}$ to decide other properties, e.g. if there are infinitely many plans.

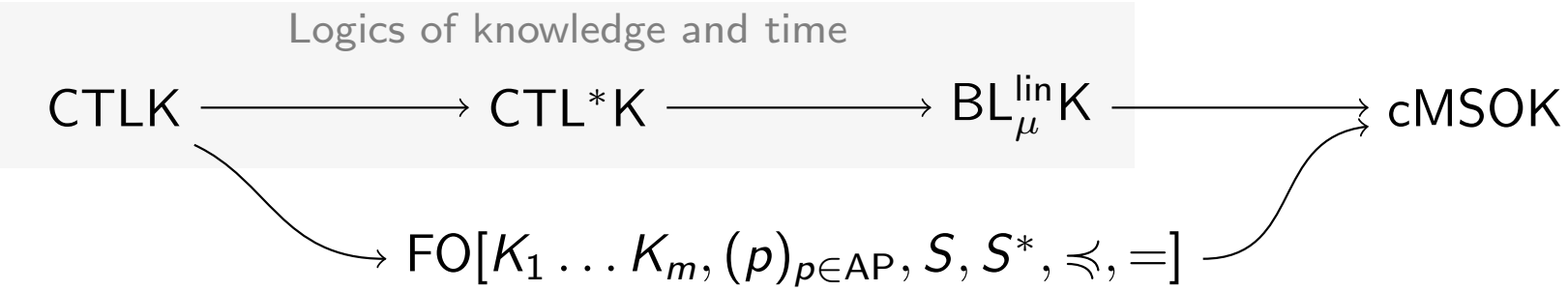
Outline

Logics of knowledge and time (Halpern-Vardi 1989)

The models are particular trees $\mathcal{T} = \langle D, r, S_1, \ldots, S_n, K_1 \ldots K_m, (p)_{p \in AP} \rangle$

- time progess is given by $S := \bigcup_{i=1}^{n} S_i$ (temporal logics, CTL, etc.)
- atomic propositions in $AP = \{p, q, \ldots\}$ are unary relations
- knowledge modalities $K_1 \ldots K_m$ are binary relations.

Write cMSOK for cMSO$[r, S_1, \ldots, S_n, (p)_{p \in AP}, K_1 \ldots K_m]$



Corollary

Model checking on RATs against any of these logics is decidable.