# Automata Techniques for Epistemic Protocol Synthesis

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In this work we aim at applying automata techniques to problems studied in Dynamic Epistemic Logic, such as epistemic planning. To do so, we first remark that repeatedly executing *ad infinitum* a propositional event model from an initial epistemic model yields a relational structure that can be finitely represented with automata. This correspondence, together with recent results on *uniform strategies*, allows us to give an alternative decidability proof of the epistemic planning problem for propositional events, with as by-products accurate upper-bounds on its time complexity, and the possibility to synthesize a finite word automaton that describes the set of all solution plans. In fact, using automata techniques enables us to solve a much more general problem, that we introduce and call *epistemic protocol synthesis*.

## 1 Introduction

Automated planning, as defined and studied in [6], consists in computing a finite sequence of actions that takes some given system from its initial state to one of its designated "goal" states. The Dynamic Epistemic Logic (DEL) community has recently investigated a particular case of automated planning, called *epistemic planning* [5, 8, 1]. In DEL, epistemic models and event models can describe accurately how agents perceive the occurrence of events, and how their knowledge or beliefs evolve. Given initial epistemic states of the agents, a finite set of available events, and an epistemic objective, the epistemic planning problem consists in computing (if any) a finite sequence of available events whose occurrence results in a situation satisfying the objective property. While this problem is undecidable in general [5, 1], restricting to *propositional events* (those whose pre and postconditions are propositional) yields decidability [14].

In this paper, preliminary to our main results we bring a new piece to the merging of various frameworks for knowledge and time. Some connections between DEL and Epistemic Temporal Logics (ETL) are already known [7, 3, 2, 13]. We establish that structures generated by iterated execution of an event model from an epistemic model are regular structures, *i.e.* they can be finitely represented with automata, in case the event model is propositional. This allows us to reduce the epistemic planning problem for propositional events to the *uniform strategy problem*, as studied in [10, 11, 9]. The automata techniques developed for uniform strategies then provide an alternative proof of [14], with the additional advantage of bringing accurate upper-bounds on the time complexity of the problem, as well as an effective synthesis procedure to generate the recognizer of all solution plans. In fact, our approach allows us to solve a generalized problem in DEL, that we call *epistemic protocol synthesis problem*, and that is essentially the problem of synthesizing a protocol from an epistemic temporal specification; its semantics relies on the interplay between DEL and ETL. We then make use of the connections with regular structures and uniform strategies to solve this latter general problem.

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## 2 DEL and epistemic planning problem

For this paper we fix $Ag$, a finite set of *agents* and $AP$ be a set of atomic propositions. The epistemic language $\mathscr{L}^{EL}$ is simply the language of propositional logic extended with "knowledge" modalities, one for each agent. Intuitively, $K_i\varphi$ reads as "agent $i$ knows $\varphi$". The syntax of $\mathscr{L}^{EL}$ is given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_i\varphi, \qquad (\text{where } p \in AP \text{ and } i \in Ag)$$

The semantics of $\mathscr{L}^{EL}$ is given in terms of epistemic models. Intuitively, a (pointed) epistemic model $(\mathscr{M}, w)$ represents how the agents perceive the actual world $w$.

**Definition 1** *An* epistemic model *is a tuple* $\mathscr{M} = (W, \{R_i\}_{i\in Ag}, V)$ *where* $W$ *is a finite set of possible* worlds*,* $R_i \subseteq W \times W$ *is an* accessibility relation *on* $W$ *for agent* $i \in Ag$*, and* $V : AP \to 2^W$ *is a* valuation function*.*

We write $w \in \mathscr{M}$ for $w \in W$, and we call $(\mathscr{M}, w)$ a *pointed epistemic model*. Formally, given a pointed epistemic model $(\mathscr{M}, w)$, we define the semantics of $\mathscr{L}^{EL}$ by induction on its formulas: $\mathscr{M}, w \models p$ if $w \in V(p)$, $\mathscr{M}, w \models \neg\varphi$ if it is not the case that $\mathscr{M}, w \models \varphi$, $\mathscr{M}, w \models \varphi \vee \psi$ if $\mathscr{M}, w \models \varphi$ or $\mathscr{M}, w \models \psi$, and $\mathscr{M}, w \models K_i\varphi$ if for all $w'$ such that $wR_iw'$, $\mathscr{M}, w' \models \varphi$.

**Definition 2** *An* event model *is a tuple* $\mathscr{E} = (\mathsf{E}, \{\mathsf{R}_i\}_{i\in Ag}, pre, post)$ *where* $\mathsf{E}$ *is finite set of* events*,* $\mathsf{R}_i \subseteq \mathsf{E} \times \mathsf{E}$ *is an* accessibility relation *on* $\mathsf{E}$ *for agent* $i$*,* $pre : \mathsf{E} \to \mathscr{L}^{EL}$ *is a* precondition function *and* $post : \mathsf{E} \to AP \to \mathscr{L}^{EL}$ *is a* postcondition function*.*

We write $e \in \mathscr{E}$ for $e \in \mathsf{E}$, and call $(\mathscr{E}, e)$ a *pointed event model*. For an event $e \in \mathscr{E}$, the precondition $\mathrm{pre}(e)$ and the postconditions $\mathrm{post}(e)(p)$ ($p \in AP$) are epistemic formulas. They describe the set of worlds where event $e$ may take place and the set of worlds where proposition $p$ will hold after event $e$ has occurred. In case all pre and postconditions of an event model $\mathscr{E}$ lie in the propositional fragment of $\mathscr{L}^{EL}$, $\mathscr{E}$ is *propositional*.

**Definition 3** *Let* $\mathscr{M} = (W, \{R_i\}_{i\in Ag}, V)$ *be an epistemic model and* $\mathscr{E} = (\mathsf{E}, \{\mathsf{R}_i\}_{i\in Ag}, pre, post)$ *be an event model, the* update product *of* $\mathscr{M}$ *and* $\mathscr{E}$ *is the epistemic model* $\mathscr{M} \otimes \mathscr{E} = (W^{\otimes}, \{R_i^{\otimes}\}_{i\in Ag}, V^{\otimes})$*, where* $W^{\otimes} = \{(w,e) \in W \times \mathsf{E} \mid \mathscr{M}, w \models pre(e)\}$*,* $R_i^{\otimes}(w,e) = \{(w',e') \in W^{\otimes} \mid w' \in R_i(w) \text{ and } e' \in \mathsf{R}_i(e)\}$*, and* $V^{\otimes}(p) = \{(w,e) \in W^{\otimes} \mid \mathscr{M}, w \models post(e)(p)\}$*.*

The update product of a pointed epistemic model $(\mathscr{M}, w)$ with a pointed event model $(\mathscr{E}, e)$ is $(\mathscr{M}, w) \otimes (\mathscr{E}, e) = (\mathscr{M} \otimes \mathscr{E}, (w,e))$ if $\mathscr{M}, w \models \mathrm{pre}(e)$, and it is undefined otherwise.

## 3 Trees, forests and $\mathsf{CTL}^*\mathsf{K}_n$

A *tree alphabet* is a finite set of *directions* $\Upsilon = \{d_1, d_2 \ldots\}$ . A $\Upsilon$-*tree*, or *tree* for short when $\Upsilon$ is clear from the context, is a set of words $\tau \subseteq \Upsilon^+$ that is closed for nonempty prefixes, and for which there is a direction $r = \tau \cap \Upsilon$, called the *root*, such that for all $x \in \tau$, $x = r \cdot x'$ for some $x' \in \Upsilon^*$. A $\Upsilon$-*forest*, or *forest* when $\Upsilon$ is understood, is defined likewise, except that it can have several roots. Alternatively a forest can be seen as a union of trees.

We classically allow nodes of trees and forests to carry additional information via labels: Given a *labelling alphabet* $\Sigma$ and a tree alphabet $\Upsilon$, a $\Sigma$-*labelled* $\Upsilon$-*tree*, or $(\Sigma, \Upsilon)$-*tree* for short, is a pair $t = (\tau, \ell)$, where $\tau$ is a $\Upsilon$-tree and $\ell : \tau \to \Sigma$ is a *labelling*. The notion of $(\Sigma, \Upsilon)$-*forest* $\mathscr{U} = (u, \ell)$ is defined likewise.

Note that we use forests to represent the universe in the semantics of $\mathsf{CTL}^*\mathsf{K}_n$, hence the notations $\mathscr{U}$ and $u$. Given a $\Upsilon$-forest $u$ and a node $x = d_1 \dots d_n$ in the forest $u$, we define the tree $u_x$ to which this node belongs as the "greatest" tree in the forest $u$ that contains the node $x$: $u_x = \{y \in u \mid d_1 \preccurlyeq y\}$. Similarly, given a $(\Sigma,\Upsilon)$-forest $\mathscr{U} = (u,\ell)$ and a node $x \in u$, $\mathscr{U}_x = (u_x,\ell_x)$, where $u_x$ is as above and $\ell_x$ is the restriction of $\ell$ to the tree $u_x$.

The set of well-formed $\mathsf{CTL}^*\mathsf{K}_n$ formulas is given by the following grammar:

State formulas: $\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{A}\psi \mid K_i\varphi$ (where $p \in AP$ and $i \in Ag$)

Path formulas: $\psi ::= \varphi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\psi \mid \psi\mathbf{U}\psi$,

Let $\Upsilon$ be a finite set of directions, and let $\Sigma = 2^{AP}$ be the set of possible valuations. A $\mathsf{CTL}^*\mathsf{K}_n$ (state) formula is interpreted in a node of a $(\Sigma,\Upsilon)$-tree, but the semantics is parameterized by, first, for each agent $i \in Ag$, a binary relation $\rightsquigarrow_i$ between finite words over $\Sigma$, and second, a forest of $(\Sigma,\Upsilon)$-trees which we see as the universe. Preliminary to defining the semantics of $\mathsf{CTL}^*\mathsf{K}_n$, we let the *node word* of a node $x = d_1 d_2 \dots d_n \in \tau$ be $w(x) = \ell(d_1)\ell(d_1 d_2)\dots\ell(d_1 \dots d_n) \in \Sigma^*$, made of the sequence of labels of all nodes from the root to this very node. Now given a family $\{\rightsquigarrow_i\}_{i \in Ag}$ of binary relations over $\Sigma^*$, a $(\Sigma,\Upsilon)$-forest $\mathscr{U}$, two nodes $x, y \in \mathscr{U}$ and $i \in Ag$, we let $x \rightsquigarrow_i y$ denote that $w(x) \rightsquigarrow_i w(y)$.

A state formula of $\mathsf{CTL}^*\mathsf{K}_n$ is interpreted over a $(\Sigma,\Upsilon)$-tree $t = (\tau,\ell)$ in a node $x \in \tau$, with an implicit universe $\mathscr{U}$ and relations $\{\rightsquigarrow_i\}_{i \in Ag}$, usually clear from the context: the notation $t, x \models \varphi$ means that $\varphi$ holds at the node $x$ of the labelled tree $t$. Because all inductive cases but the knowledge operators follow the classic semantics of $\mathsf{CTL}^*$ on trees, we only give the semantics for formulas of the form $K_i\varphi$:

$$t, x \models K_i\varphi \quad \text{if for all } y \in \mathscr{U} \text{ such that } x \rightsquigarrow y,\ \mathscr{U}_y, y \models \varphi$$

## 4 DEL-generated models and regular structures

A *relational structure* is a tuple $\mathscr{S} = (D, \{\rightsquigarrow_i\}_{i \in Ag}, V)$ where $D$ is the (possibly infinite) *domain* of $\mathscr{S}$, for each $i \in Ag$, $\rightsquigarrow_i \subseteq D \times D$ is a binary relation and $V : AP \to 2^D$ is a valuation function. $V$ can alternatively be seen as a set of predicate interpretations for atomic propositions in $AP$. We recall that *regular relations* are relations recognized by synchronous finite state transducers (see [4]).

**Definition 4** *A relational structure $\mathscr{S} = (D, \{\rightsquigarrow_i\}_{i \in Ag}, V)$ is a* regular structure *over a finite alphabet $\Sigma$ if its domain $D \subseteq \Sigma^*$ is a regular language over $\Sigma$, for each $i$, $\rightsquigarrow_i \subseteq \Sigma^* \times \Sigma^*$ is a regular relation and for each $p \in AP$, $V(p) \subseteq D$ is a regular language. Given deterministic word automata $\mathscr{A}_{\mathscr{S}}$ and $\mathscr{A}_p$ ($p \in AP$), as well as transducers $T_i$ for $i \in Ag$, we say that $(\mathscr{A}_{\mathscr{S}}, \{T_i\}_{i \in Ag}, \{\mathscr{A}_p\}_{p \in AP})$ is a* representation *of $\mathscr{S}$ if $\mathscr{L}(\mathscr{A}_{\mathscr{S}}) = D$, for each $i \in Ag$, $[T_i] = \rightsquigarrow_i$ and for each $p \in AP$, $\mathscr{L}(\mathscr{A}_p) = V(p)$.*

**Definition 5** *For an epistemic model $\mathscr{M} = (W, \{R_i\}_{i \in Ag}, V)$ and an event model $\mathscr{E} = (\mathsf{E}, \{\mathsf{R}_i\}_{i \in Ag}, pre, post)$, we define the family of epistemic models $\{\mathscr{M}\mathscr{E}^n\}_{n \geq 0}$ by letting $\mathscr{M}\mathscr{E}^0 = \mathscr{M}$ and $\mathscr{M}\mathscr{E}^{n+1} = \mathscr{M}\mathscr{E}^n \otimes \mathscr{E}$. Letting, for each $n$, $\mathscr{M}\mathscr{E}^n = (W^n, \{R_i^n\}_{i \in Ag}, V^n)$, we define the relational structure generated by $\mathscr{M}$ and $\mathscr{E}$ as $\mathscr{M}\mathscr{E}^* = (D, \{\rightsquigarrow_i\}_{i \in Ag}, V)$, where:*

- $D = \bigcup_{n \geq 0} W^n$,
- *$h \rightsquigarrow_i h'$ if there is some $n$ such that $h, h' \in \mathscr{M}^n$ and $h R_i^n h'$, and*
- $V(p) = \bigcup_{n \geq 0} V^n(p)$.

**Proposition 1** *If $\mathscr{M}$ is an epistemic model and $\mathscr{E}$ is a propositional event model, then $\mathscr{M}\mathscr{E}^*$ is a regular structure.*

**Proof** See Appendix A. □

## 5 Epistemic protocol synthesis

We first consider the problem of epistemic planning [5, 8] studied in the Dynamic Epistemic Logic community. Note that our formulation slightly differs from the classic one as we consider a unique event model, but both problems can easily be proved inter-reducible in linear time.

**Definition 6 (Epistemic planning problem)** *Given a pointed epistemic model* $(\mathscr{M}_\iota, w_\iota)$*, an event model* $\mathscr{E}$*, a set of events* $\mathsf{E} \subseteq \mathscr{E}$ *and a goal formula* $\varphi \in \mathscr{L}^{EL}$*, decide if there exists a finite series of events* $e_1 \dots e_n$ *in* $\mathsf{E}$ *such that* $(\mathscr{M}_\iota, w_\iota) \otimes (\mathscr{E}, e_1) \otimes \dots \otimes (\mathscr{E}, e_n) \models \varphi$*. The* propositional epistemic planning problem *is the restriction of the epistemic planning problem to propositional event models.*

The epistemic planning problem is undecidable [5, 1]. However, [5] proved that the problem is decidable in the case of one agent and equivalence accessibility relations in epistemic and event models. More recently, [1] and [14] proved independently that the one agent problem is also decidable for K45 accessibility relations. [14] also proved that restricting to propositional event models yields decidability of the epistemic planning problem, even for several agents and arbitrary accessibility relations.

**Theorem 2 ([14])** *The propositional epistemic planning problem is decidable.*

Proposition 1 allows us to establish an alternative proof of this result, with two side-benefits. First, using automata techniques, our decision procedure can synthesize as a by-product a finite word automaton that generates exactly the (possibly infinite) set of all solution plans. Second, we obtain accurate upper-bounds on the time complexity.

**Theorem 3** *The propositional epistemic planning problem is in* $k$-EXPTIME *for formulas of nesting depth k. Moreover, it is possible to build in the same time a finite word automaton* $\mathscr{P}$ *such that* $\mathscr{L}(\mathscr{P})$ *is the set of all solution plans.*

**Proof** Let $(\mathscr{M}, \mathscr{E}, \mathsf{E}, \varphi)$ be an instance of the problem. By Proposition 1 we obtain an automatic representation of the forest $\mathscr{M}\mathscr{E}^*$: the set of possible histories, as well as their valuations, are represented by a finite automaton $\mathscr{A}$, and the epistemic relations are given by finite state transducers. Because the epistemic relations are rational, we can use the powerset construction presented in [10] in the context of uniform strategies [10, 11, 9]. Indeed, this construction easily generalizes to the case of $n$ relations, and even though in [10] it is defined on game arenas it can, in our context, be adapted to regular structures. Letting $k$ be the maximal nesting depth of knowledge operators in $\varphi$, this construction yields an automaton $\widehat{\mathscr{A}}$ of size $k$-exponential, that still represents $\mathscr{M}\mathscr{E}^*$, and in which $\varphi$ can be evaluated positionally. Keeping only transitions labelled by events in $\mathsf{E}$, and choosing for accepting states those that verify $\varphi$, we obtain the automaton $\mathscr{P}$ that recognizes the set of solution plans. Furthermore, solving the epistemic planning problem amounts to solving the nonemptiness problem for $\mathscr{L}(\mathscr{P})$; this can be done in time linear in the size of $\mathscr{P}$, which is $k$-exponential in the size of the input $(\mathscr{M}, \mathscr{E}, \mathsf{E}, \varphi)$. □

In fact, the correspondence between the DEL framework and automatic structures established in Proposition 1 allows us to solve a much more general problem than epistemic planning.

We generalize the notion of epistemic planning in three directions. First, we no longer consider finite sequences of actions but infinite ones. As a consequence, we need not stick to reachability objectives as in planning (where the aim is to reach a state of the world that verifies some formula), and we therefore allow for any epistemic temporal formula as objective, which is the second generalization. Finally, we no longer look for a single series of events, but we try to synthesize a *protocol*, *i.e.* a set of plans.

**Definition 7** *Given an epistemic model $\mathscr{M}$ and an event model $\mathscr{E}$, an* epistemic protocol *is a forest $P \subseteq \mathscr{M}\mathscr{E}^*$; it is* rooted *if it is a tree.*

**Definition 8 (Epistemic protocol synthesis problem)** *Given an initial pointed epistemic model $(\mathscr{M}, w)$, a propositional event model $\mathscr{E}$ and a $\mathsf{CTL}^*\mathsf{K}_n$ formula $\varphi$, letting $\mathscr{U} = \mathscr{M}\mathscr{E}^*$ be the universe, decide if there is an epistemic protocol $P \subseteq \mathscr{U}$ rooted in $w$ such that $P \models \varphi$, and synthesize such a protocol if any.*

Again making use of Proposition 1, the epistemic protocol synthesis problem can be reduced to synthesizing a uniform strategy in a game arena with regular relations between plays. This can be solved with the powerset construction from [10] and classic automata techniques for solving games with $\mathsf{CTL}^*$ winning condition. We finally obtain the following result.

**Theorem 4** *The epistemic protocol synthesis problem is decidable. If the nesting depth of the goal formulas is bounded by k, then the problem is in* $\max(2,k)$-EXPTIME.

## References


[1] Guillaume Aucher & Thomas Bolander (2013): *Undecidability in Epistemic Planning*. In Rossi [12]. Available at `http://www.aaai.org/ocs/index.php/IJCAI/IJCAI13/paper/view/6903`.

[2] Guillaume Aucher & Andreas Herzig (2011): *Exploring the power of converse events*. In: Dynamic formal epistemology, Springer, pp. 51–74.

[3] Johan van Benthem, Jelle Gerbrandy, Tomohiro Hoshi & Eric Pacuit (2009): *Merging frameworks for interaction*. Journal of Philosophical Logic 38(5), pp. 491–526.

[4] Jean Berstel (1979): *Transductions and context-free languages*. 4, Teubner Stuttgart.

[5] Thomas Bolander & Mikkel Birkegaard Andersen (2011): *Epistemic planning for single and multi-agent systems*. Journal of Applied Non-Classical Logics 21(1), pp. 9–34. Available at `http://dx.doi.org/10.3166/jancl.21.9-34`.

[6] Malik Ghallab, Dana S. Nau & Paolo Traverso (2004): *Automated planning - theory and practice*. Elsevier.

[7] Tomohiro Hoshi & Audrey Yap (2009): *Dynamic epistemic logic with branching temporal structures*. Synthese 169(2), pp. 259–281. Available at `http://dx.doi.org/10.1007/s11229-009-9552-6`.

[8] Benedikt Löwe, Eric Pacuit & Andreas Witzel (2011): *DEL Planning and Some Tractable Cases*. In Hans P. van Ditmarsch, Jérôme Lang & Shier Ju, editors: LORI, Lecture Notes in Computer Science 6953, Springer, pp. 179–192. Available at `http://dx.doi.org/10.1007/978-3-642-24130-7_13`.

[9] Bastien Maubert & Sophie Pinchinat (2013): *Jumping Automata for Uniform Strategies*. In: FSTTCS'13, pp. 287–298, doi:http://dx.doi.org/10.4230/LIPIcs.FSTTCS.2013.287.

[10] Bastien Maubert & Sophie Pinchinat (2014): *A General Notion of Uniform Strategies*. International Game Theory Review 16(01). Available at `http://www.irisa.fr/prive/bmaubert/Publications/IGTR2013.pdf`. To appear.

[11] Bastien Maubert, Sophie Pinchinat & Laura Bozzelli (2013): *The Complexity of Synthesizing Uniform Strategies*. In Fabio Mogavero, Aniello Murano & Moshe Y. Vardi, editors: SR, EPTCS 112, pp. 115–122. Available at `http://dx.doi.org/10.4204/EPTCS.112.17`.

[12] Francesca Rossi, editor (2013): *IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China, August 3-9, 2013*. IJCAI/AAAI.

[13] Yanjing Wang & Guillaume Aucher (2013): *An Alternative Axiomatization of DEL and Its Applications*. In Rossi [12]. Available at `http://www.aaai.org/ocs/index.php/IJCAI/IJCAI13/paper/view/6802`.

[14] Quan Yu, Ximing Wen & Yongmei Liu (2013): *Multi-Agent Epistemic Explanatory Diagnosis via Reasoning about Actions*. In: IJCAI, pp. 1183–1190. Available at `http://ijcai.org/papers13/Papers/IJCAI13-178.pdf`.

# A Appendix

**Proposition 5** *If $\mathscr{M}$ is an epistemic model and $\mathscr{E}$ is a propositional event model, then $\mathscr{M}\mathscr{E}^*$ is a regular structure.*

**Proof** Let $\mathscr{M} = (W,R,V)$ be an epistemic model, let $\mathscr{E} = (\mathsf{E},\mathsf{R},\mathrm{pre},\mathrm{post})$ be a propositional event model, and let $\mathscr{M}\mathscr{E}^* = (D,\{\rightsquigarrow_i\}_{i\in Ag},V_D)$.

Define the word automaton $\mathscr{A}_D = (\Sigma,Q,\delta,q_\iota,F)$, where $\Sigma = W \cup \mathsf{E}$, $F = \{q_\nu \mid \nu \subseteq AP\}$ and $Q = F \uplus \{q_\iota\}$. For a world $w \in W$, we define its valuation as $\nu(w) := \{p \in AP \mid w \in V(p)\}$. We now define $\delta$, which is the following partial transition function:

$$
\begin{array}{ll}
\forall w \in W,\ \forall e \in \mathsf{E}, & \\
\delta(q_\iota,w) = q_{\nu(w)} & \delta(q_\iota,e) \text{ is undefined,} \\
\delta(q_\nu,w) \text{ is undefined} & \delta(q_\nu,e) = \begin{cases} q_{\nu'}, \text{ with } \nu' = \{p \mid \nu \models \mathrm{post}(e)(p)\} & \text{if } \nu \models \mathrm{pre}(e) \\ \text{undefined} & \text{otherwise.} \end{cases}
\end{array}
$$

It is not hard to see that $\mathscr{L}(\mathscr{A}_D) = D$, hence $D$ is a regular language.

Concerning valuations, take some $p \in AP$. Let $\mathscr{A}_p = (\Sigma,Q,\delta,q_\iota,F_p)$, where $F_p = \{q_\nu \mid p \in \nu\}$. Clearly, $\mathscr{L}(\mathscr{A}_p) = V_D(p)$, hence $V_D(p)$ is a regular language. For a state $q \in Q$, define $\nu_D(q) = \{p \mid q \in F_p\}$.

For the relations, let $i \in Ag$ and consider the one-state synchronous transducer $T_i = (\Sigma,Q',\Delta_i,q_\iota,F')$, where $Q' = \{q\}$, $q_\iota = q$, $F' = \{q\}$, and $\Delta_i = \{(q,w,w',q) \mid w\,R_i\,w'\} \cup \{(q,e,e',q) \mid e\,\mathsf{R}_i\,e'\}$. It is easy to see that $\rightsquigarrow_i = [T_i] \cap D \times D$. Since $[T_i]$ is a regular relation and $D$ is a regular language, $\rightsquigarrow_i$ is a regular relation recognized by $T_i' = T_D \circ T_i \circ T_D$, where $T_D$ is a synchronous transducer that recognizes the identity relation over $D$ (easily obtained from $\mathscr{A}_D$). Therefore, $\mathscr{M}\mathscr{E}^*$ is a regular structure that accepts $(\mathscr{A}_D,\{T_i'\}_{i\in Ag},\nu_D)$ as a regular representation. $\square$