Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

- syntax and semantics of LTL
- automata-based LTL model checking ⟵
- complexity of LTL model checking

Computation-Tree Logic

Equivalences and Abstraction

LTL model checking problem

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given: finite transition system $\mathcal{T}$ over AP
(without terminal states)
LTL-formula φ over AP

question: does $\mathcal{T} \models \varphi$ hold **?**

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$$\pi \not\models \varphi, \text{ i.e., } \pi \models \neg\varphi$$

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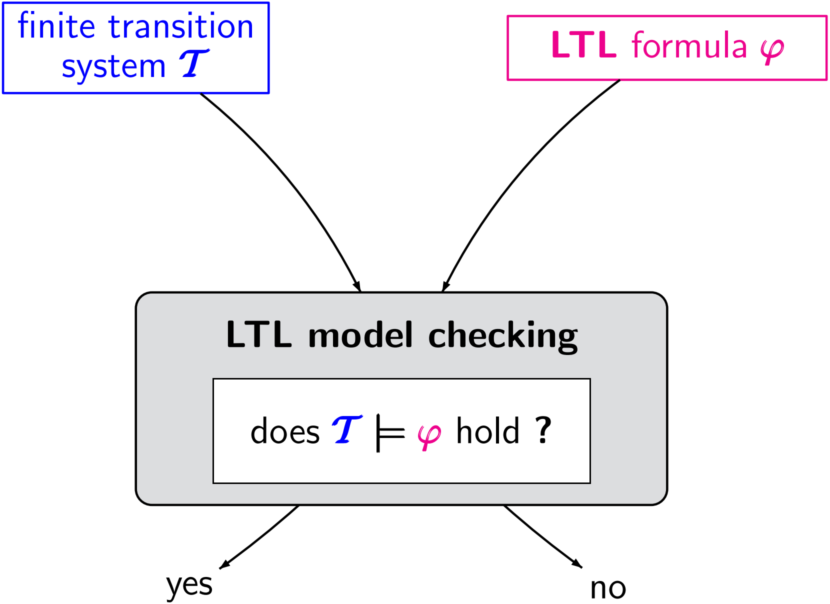
↑

construct the product-TS $\mathcal{T} \otimes \mathcal{A}$
search a path in the product that meets
the acceptance condition of $\mathcal{A}$

Automata-based LTL model checking



finite transition system $\mathcal{T}$

LTL formula φ

LTL model checking

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yes

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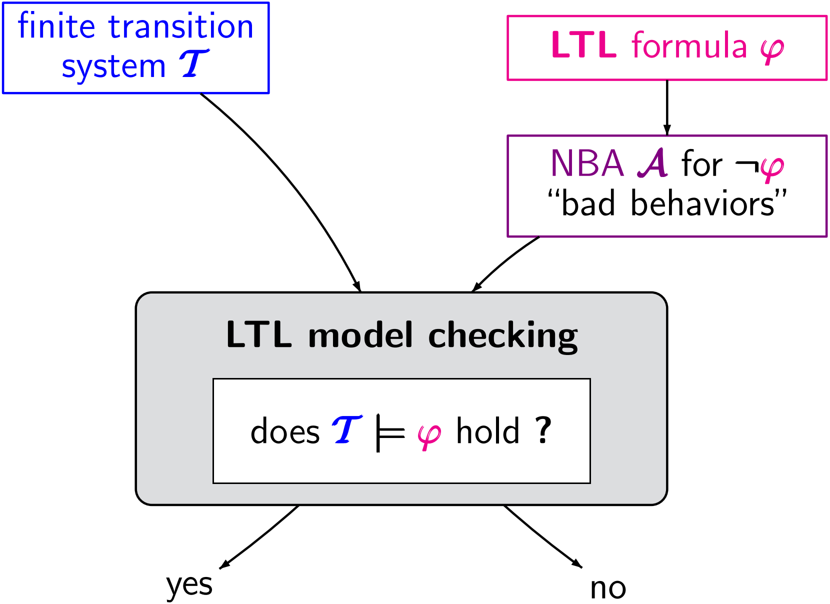

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"bad behaviors"

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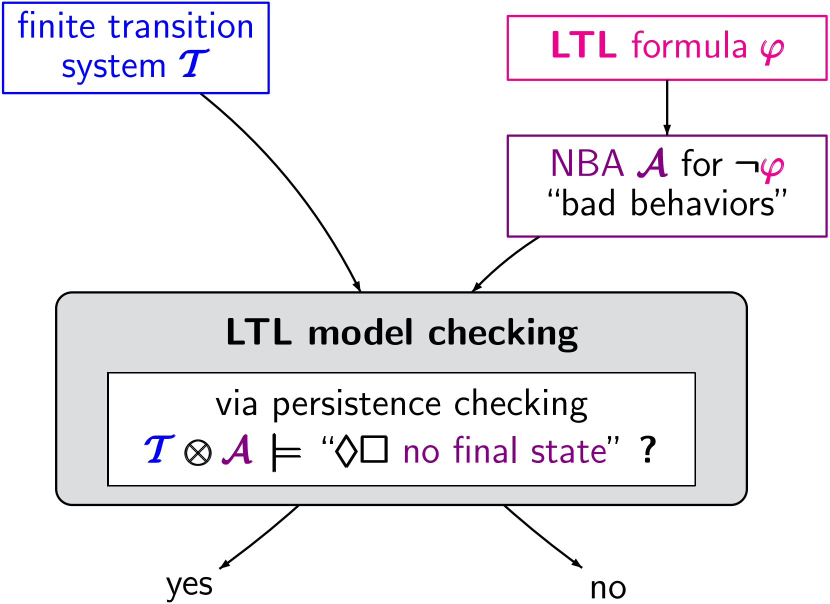

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via persistence checking
$\mathcal{T} \otimes \mathcal{A} \models$ "$\Diamond\Box$ no final state" **?**

yes

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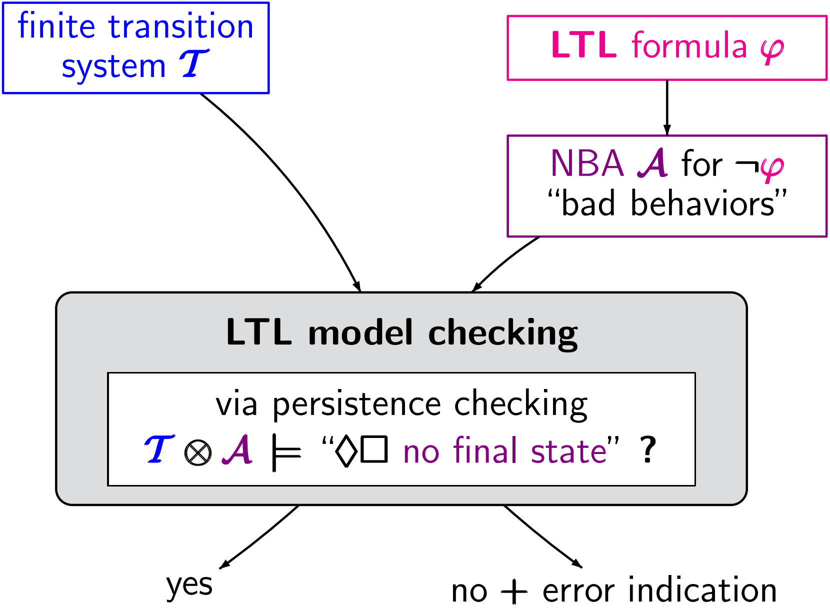

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yes

no + error indication

Safety and LTL model checking

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Safety vs LTL model checking

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$\mathcal{T} \models$ safety property E

iff $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \cap \mathcal{L}(\mathcal{A}) = \varnothing$

where $\mathcal{A}$ is an NFA for the bad prefixes

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---

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iff $\mathcal{T} \otimes \mathcal{A} \models \Diamond \Box \neg F$ ⟵ persistence checking

Recall: nondeterministic Büchi automata

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \to 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
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run for a word $A_0 \, A_1 \, A_2 \, \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0 \, q_1 \, q_2 \, \ldots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

run π is accepting if $\overset{\infty}{\exists} \, i \in \mathbb{N}. \; q_i \in F$

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From LTL to NBA

LTLMC3.2-THM-LTL-2-NBA

From LTL to NBA

For each **LTL** formula φ over AP there is an **NBA** $\mathcal{A}$ over the alphabet 2^{AP} such that

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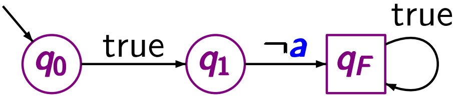
proof: ... later ...

NBA for LTL formulas



$\mathcal{L}_\omega(\mathcal{A}) = \,$ **?**

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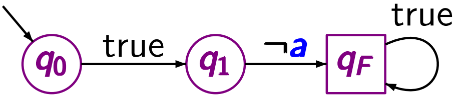

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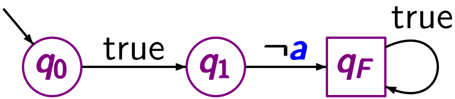

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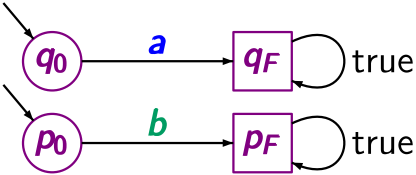

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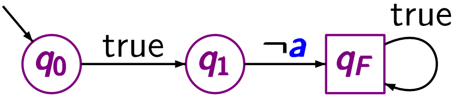

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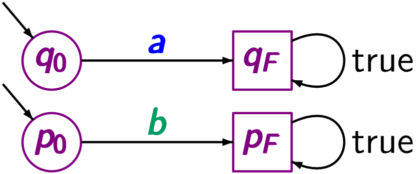

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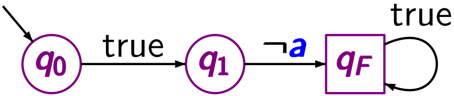

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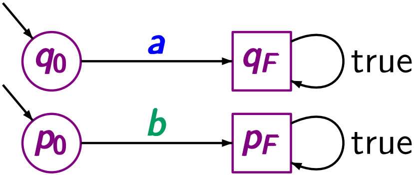

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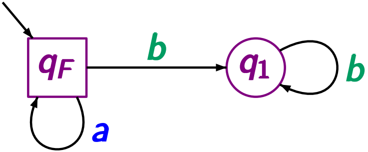

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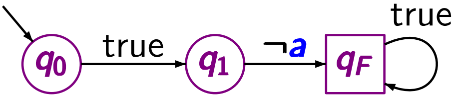

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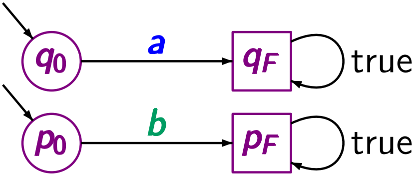

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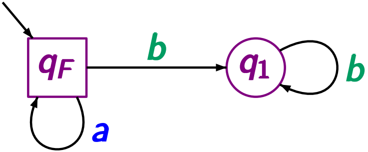

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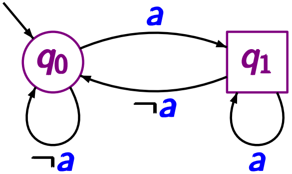
NBA for LTL formulas



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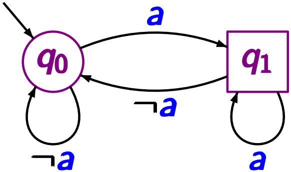

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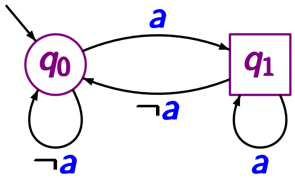

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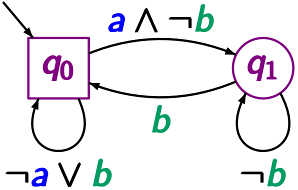

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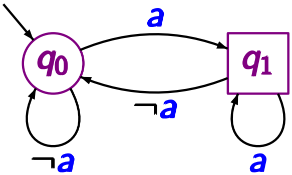

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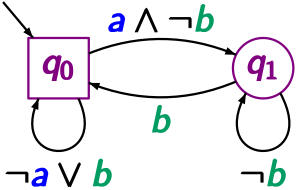

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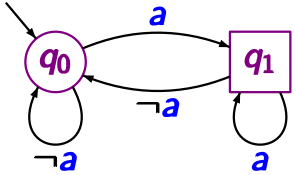
e.g., $$\left.\begin{array}{l} \varnothing\,\varnothing\,\varnothing\,\varnothing\ldots = \varnothing^\omega \\ (\{a\}\,\{b\})^\omega \end{array}\right\}\ \text{are accepted by } \mathcal{A}$$

NBA for LTL formulas



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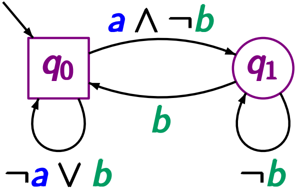

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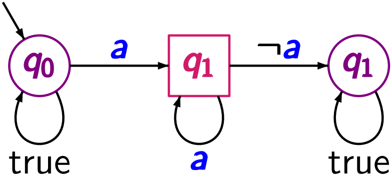
NBA for LTL formula



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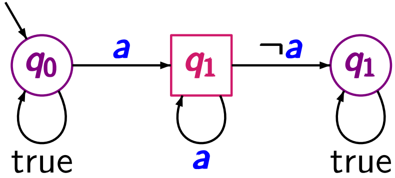

$$\mathcal{L}_\omega(\mathcal{A}) = Words(\Diamond\Box a)$$

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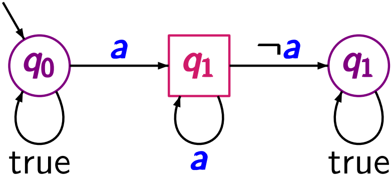

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possible runs for $\{a\}^\omega$

$q_0\ q_0\ q_0\ q_0\ q_0\ q_0\ \ldots$ not accepting

$q_0\ q_1\ q_1\ q_1\ q_1\ q_1\ \ldots$ accepting

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$\vdots$

NFA and NBA for safety properties

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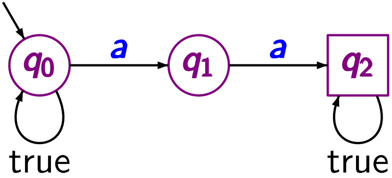

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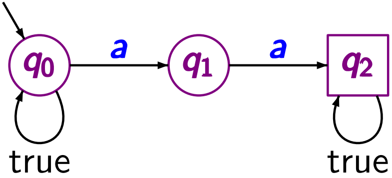

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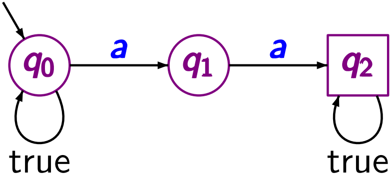

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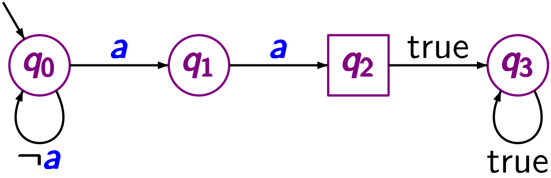

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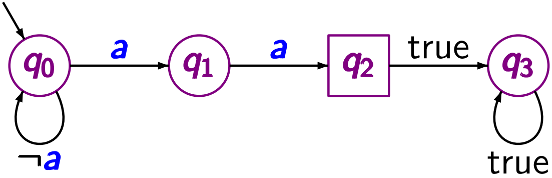

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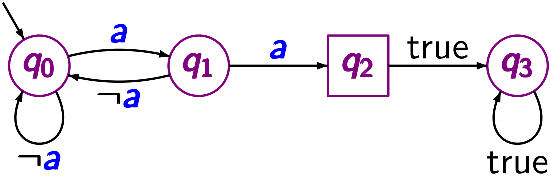

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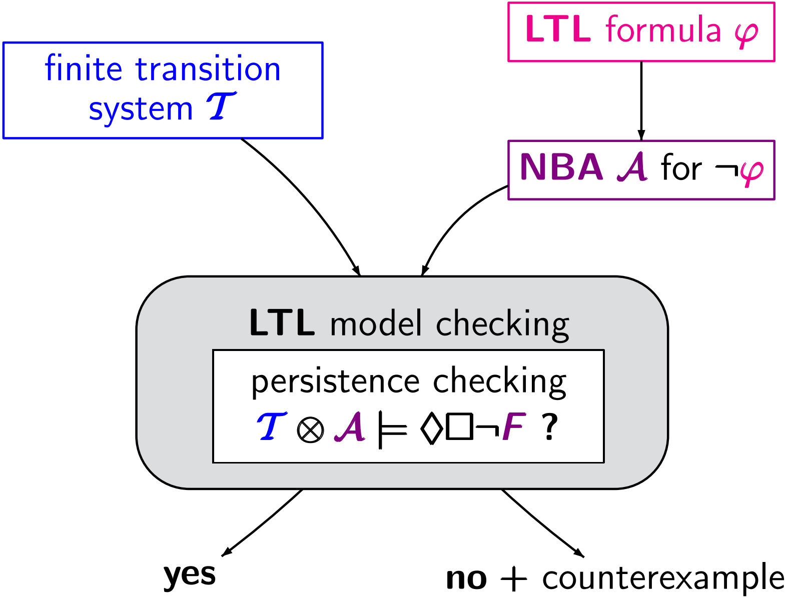
LTL model checking



finite transition system $\mathcal{T}$

LTL formula φ

NBA $\mathcal{A}$ for $\neg\varphi$

LTL model checking

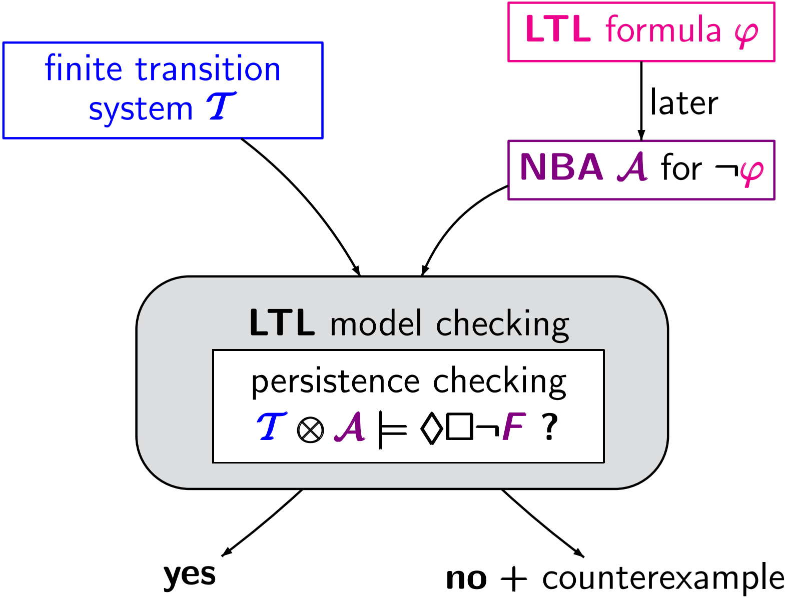
persistence checking

$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$?

yes

no + counterexample

LTL model checking



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Recall: product transition system

$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ TS without terminal states

$\mathcal{A} = (Q, 2^{AP}, \delta, Q_0, F)$ NBA or NFA

non-blocking, $Q_0 \cap F = \varnothing$

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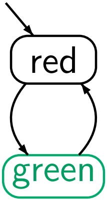
transition relation:

$$\frac{s \xrightarrow{\alpha} s' \;\wedge\; q' \in \delta(q, L(s'))}{\langle s, q \rangle \xrightarrow{\alpha}{}' \langle s', q' \rangle}$$

Example: LTL model checking

TS $\mathcal{T}$



LTL formula $\varphi = \Box\Diamond\text{green}$

Example: LTL model checking

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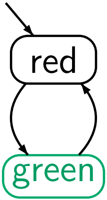

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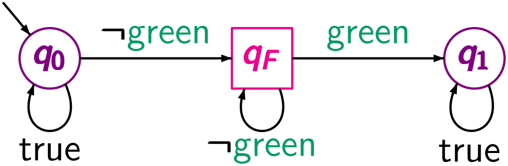

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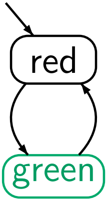

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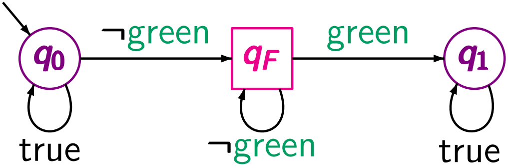

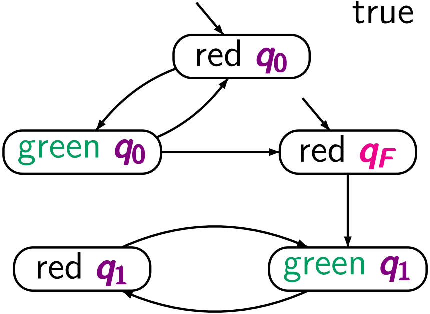

reachable fragment of the product TS $\mathcal{T} \otimes \mathcal{A}$

Example: LTL model checking

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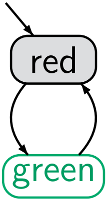

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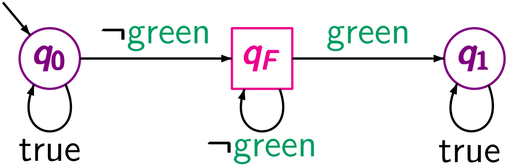
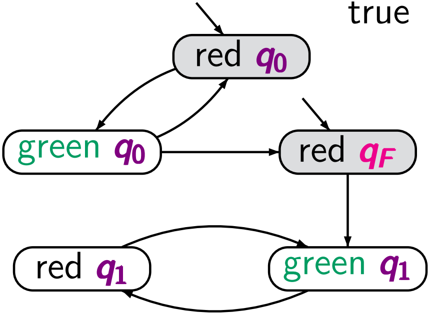




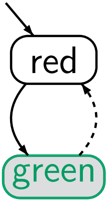
initial states:

$\langle \text{red}, q \rangle$ where

$$\begin{aligned} q &\in \delta(q_0, L(\text{red})) \\ &= \delta(q_0, \varnothing) \\ &= \{q_0, q_F\} \end{aligned}$$

Example: LTL model checking

TS $\mathcal{T}$



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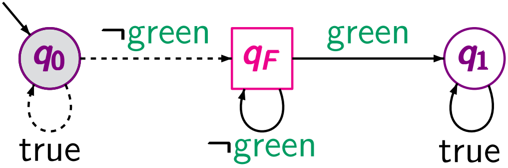

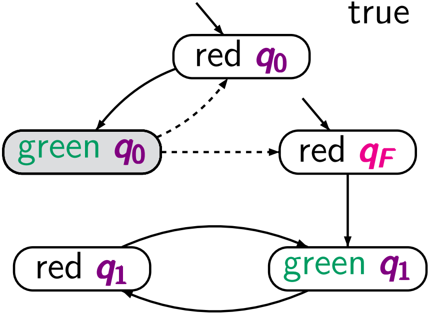

transition

$$\langle \text{green}, q_0 \rangle \rightarrow \langle \text{red}, q \rangle$$

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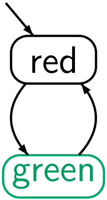

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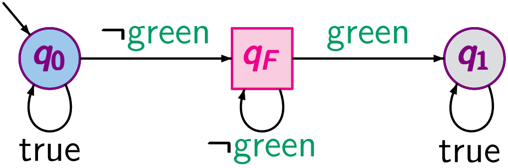
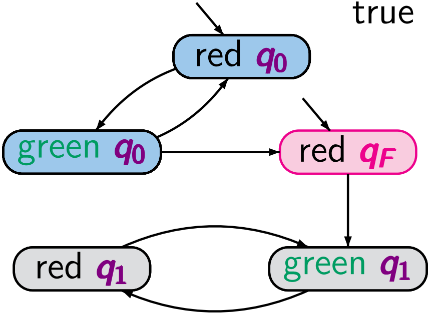




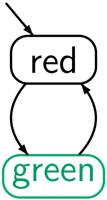
atomic propositions

$AP' = \{q_0, q_F, q_1\}$

obvious labeling function

Example: LTL model checking

TS $\mathcal{T}$



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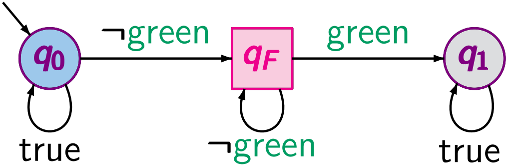

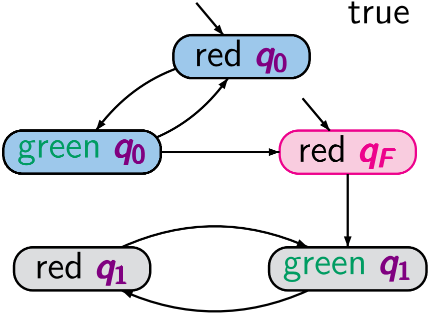

$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

Example: LTL model checking

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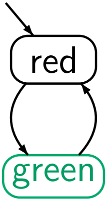

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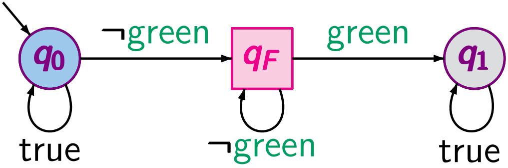

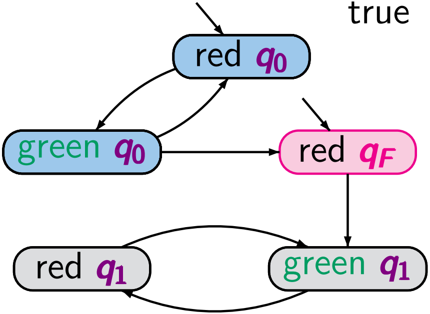

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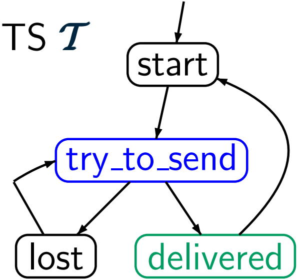
hence: $\mathcal{T} \models \varphi$

Example: LTL model checking

TS $\mathcal{T}$



LTL formula $\varphi = \Box(\textit{try} \rightarrow \Diamond \textit{del})$

"each (repeatedly) sent message will eventually be delivered"

Example: LTL model checking

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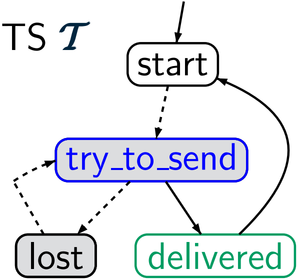

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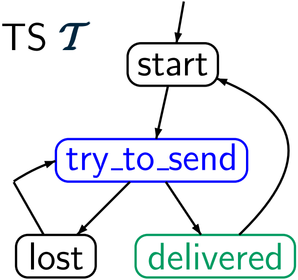

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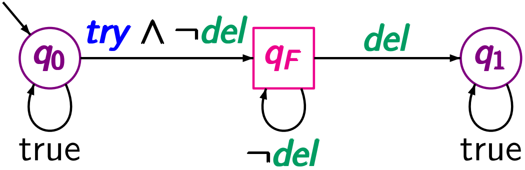

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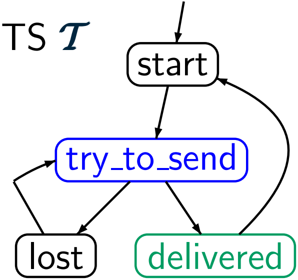
TS $\mathcal{T}$



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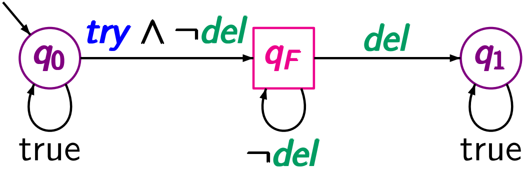
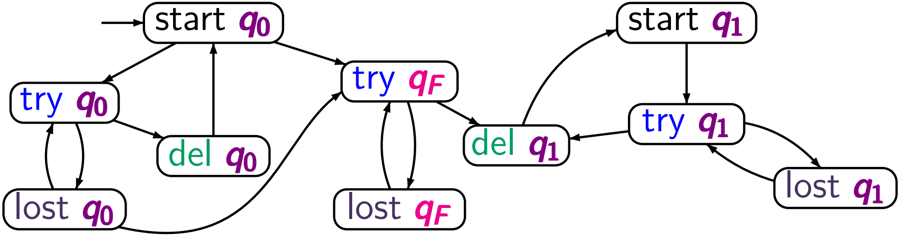
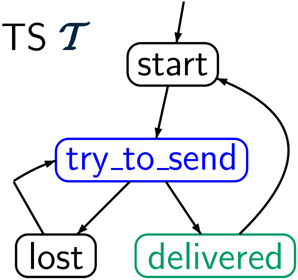




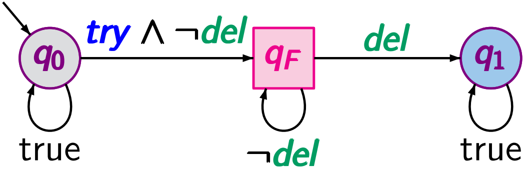
reachable fragment of the product-TS

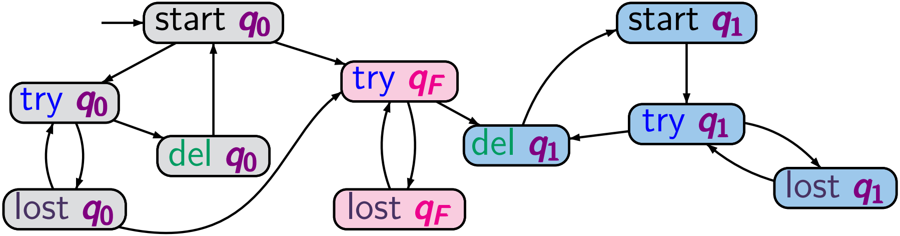
Example: LTL model checking

TS $\mathcal{T}$


start
try_to_send
lost
delivered

NBA $\mathcal{A}$ for $\neg\varphi \equiv \Diamond(\mathit{try} \wedge \Box\neg \mathit{del})$


q_0
q_F
q_1
$\mathit{try} \wedge \neg\mathit{del}$
del
true
$\neg\mathit{del}$
true

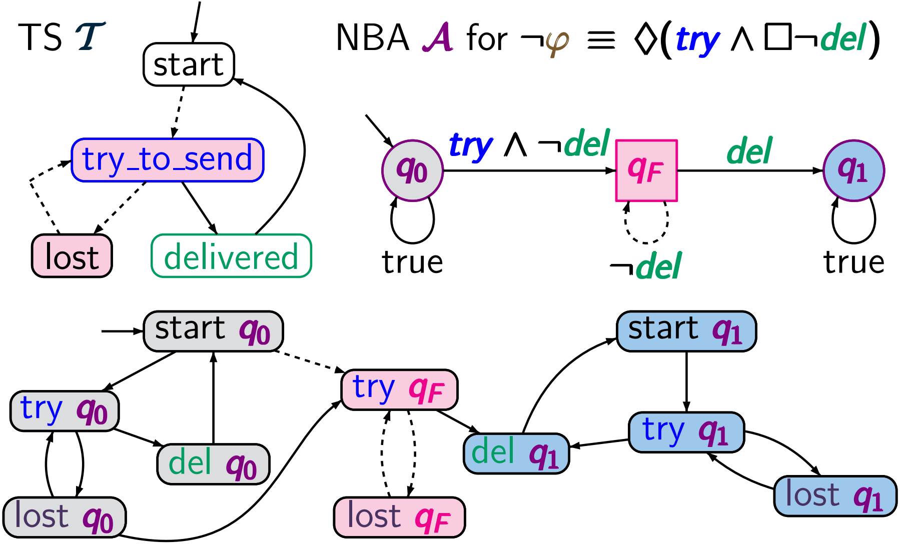

start q_0
try q_0
del q_0
lost q_0
try q_F
lost q_F
del q_1
start q_1
try q_1
lost q_1

set of atomic propositions $AP' = \{q_0, q_1, q_F\}$

Example: LTL model checking

TS $\mathcal{T}$

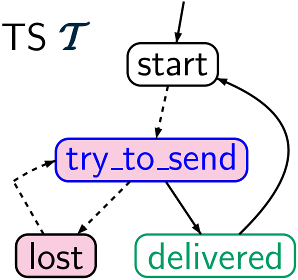
NBA $\mathcal{A}$ for $\neg\varphi \equiv \Diamond(\mathit{try} \wedge \Box\neg \mathit{del})$



$$\mathcal{T} \otimes \mathcal{A} \not\models \Diamond\Box\neg F$$

Example: LTL model checking

TS $\mathcal{T}$



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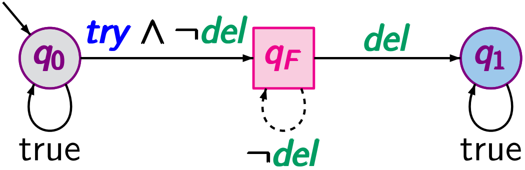
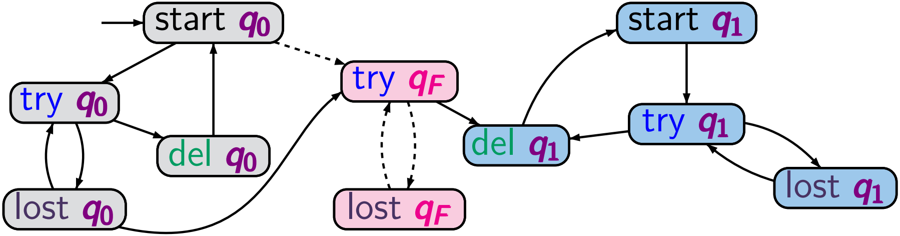




$\mathcal{T} \otimes \mathcal{A} \not\models \Diamond\Box\neg F$ hence: $\mathcal{T} \not\models \varphi$

LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

construct an NBA $\mathcal{A}$ for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$

check whether $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

construct an NBA $\mathcal{A}$ for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$

check whether $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$ ⟵

persistence checking nested **DFS**

LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

```
construct an NBA $\mathcal{A}$ for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$
check whether $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$ ⟵
 IF $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$
   THEN return "yes"
   ELSE compute a counterexample
            $\langle s_0, p_0 \rangle \ldots \langle s_n, p_n \rangle \ldots \langle s_n, p_n \rangle$
            for $\mathcal{T} \otimes \mathcal{A}$ and $\Diamond\Box\neg F$
        return "no" and $s_0 \ldots s_n \ldots s_n$
```

persistence checking
nested **DFS**

Complexity of LTL model checking

given: finite TS $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \models \varphi$ hold **?**

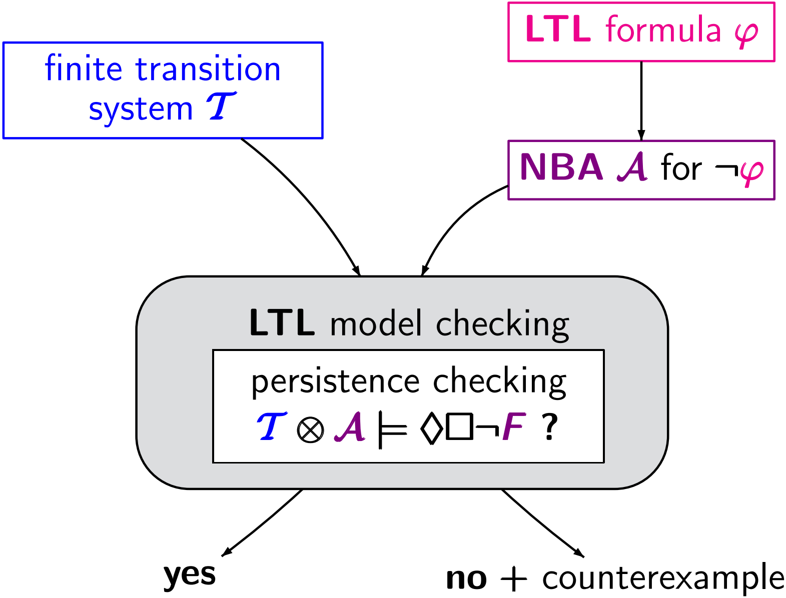
~~construct an NBA $\mathcal{A}$~~ for $\neg\varphi$ and the product $\mathcal{T} \otimes \mathcal{A}$

```
check whether $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$ ⟵
  IF $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$
    THEN  return "yes"
    ELSE  compute a counterexample
              $\langle s_0, p_0 \rangle \dots \langle s_n, p_n \rangle \dots \langle s_n, p_n \rangle$
              for $\mathcal{T} \otimes \mathcal{A}$ and $\Diamond\Box\neg F$
          return "no" and $s_0 \dots s_n \dots s_n$
```

persistence checking nested **DFS**

time complexity: $\mathcal{O}(size(\mathcal{T}) \cdot size(\mathcal{A}))$

LTL model checking



finite transition system $\mathcal{T}$

LTL formula φ

NBA $\mathcal{A}$ for $\neg\varphi$

LTL model checking

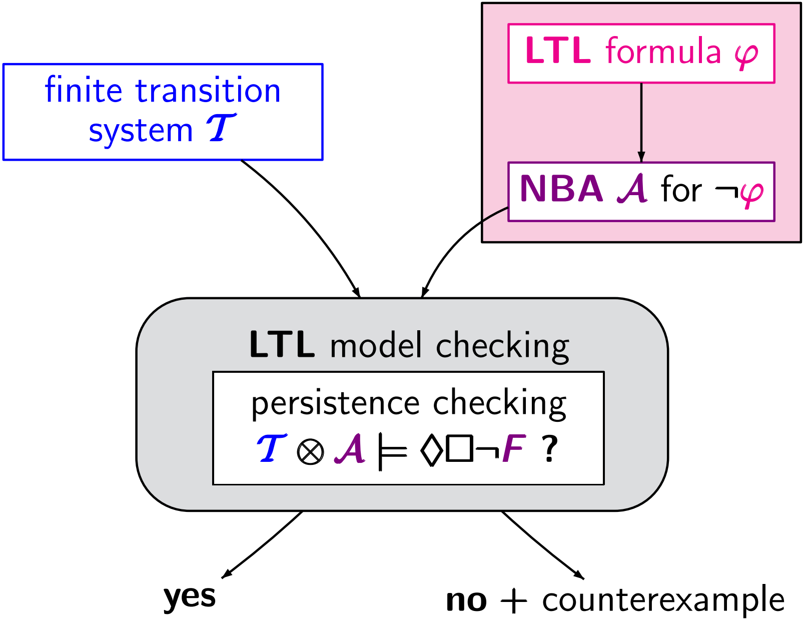
persistence checking

$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$?

yes

no + counterexample

LTL model checking



finite transition system $\mathcal{T}$

LTL formula φ

NBA $\mathcal{A}$ for $\neg\varphi$

LTL model checking

persistence checking

$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$?

yes

no + counterexample

From LTL to NBA

LTLMC3.2-46

From LTL to NBA

For each **LTL** formula φ there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) \; = \; \mathit{Words}(\varphi)$$

From LTL to NBA

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LTL formula φ

NBA $\mathcal{A}$ s.t.
$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\varphi)$

nondeterministic
Büchi automaton

From LTL to NBA

For each **LTL** formula φ there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) \;=\; \mathit{Words}(\varphi)$$

LTL formula φ

GNBA $\mathcal{G}$ s.t.
$\mathcal{L}_\omega(\mathcal{G}) = \mathit{Words}(\varphi)$

generalized NBA
several acceptance sets

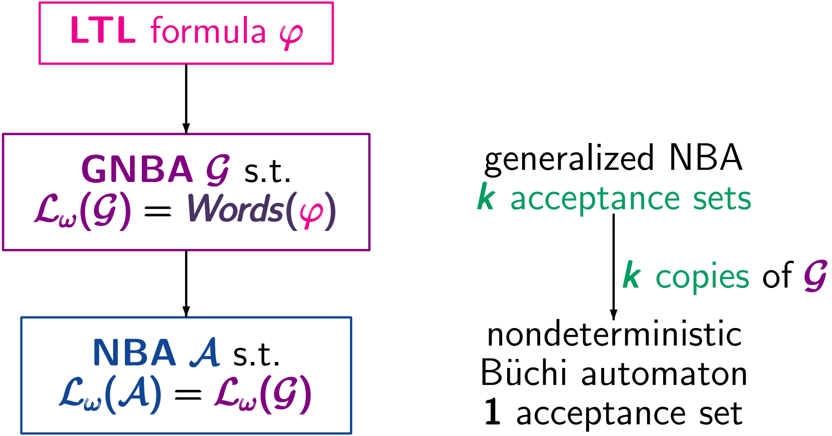
NBA $\mathcal{A}$ s.t.
$\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$

nondeterministic
Büchi automaton
1 acceptance set

From LTL to NBA

For each **LTL** formula φ there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) \;=\; \mathit{Words}(\varphi)$$



LTL formula φ

GNBA $\mathcal{G}$ s.t.
$\mathcal{L}_\omega(\mathcal{G}) = \mathit{Words}(\varphi)$

NBA $\mathcal{A}$ s.t.
$\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$

generalized NBA
k acceptance sets

k copies of $\mathcal{G}$

nondeterministic
Büchi automaton
1 acceptance set

Encoding of LTL semantics in a GNBA

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

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semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	
next $\bigcirc$	
until U	

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	
until **U**	

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until $\mathbf{U}$	

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
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next $\bigcirc$	in the transition relation
until **U**	via expansion law

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
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$$\psi_1 \,\mathsf{U}\, \psi_2 \;\equiv\; \psi_2 \vee \big(\psi_1 \wedge \bigcirc(\psi_1 \,\mathsf{U}\, \psi_2)\big)$$

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

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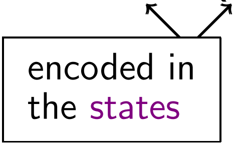


encoded in the states

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until $\mathbf{U}$	via expansion law

$$\psi_1 \,\mathbf{U}\, \psi_2 \quad \equiv \quad \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathbf{U}\, \psi_2))$$

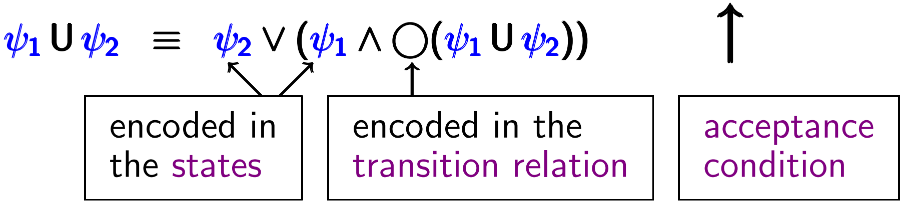
encoded in the states

encoded in the transition relation

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until $\mathbf{U}$	expansion law, **least fixed point**



$$\psi_1 \,\mathbf{U}\, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathbf{U}\, \psi_2))$$

encoded in the states

encoded in the transition relation

acceptance condition

LTL $\rightsquigarrow$ GNBA

LTLMC3.2-46A

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for ***Words*(φ)**

states of $\mathcal{G}$ $\hat{=}$ (certain) sets of subformulas of φ

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

states of $\mathcal{G} \quad \widehat{=} \quad$ (certain) sets of subformulas of φ
s.t. each word $\sigma = A_0\, A_1\, A_2 \ldots \in \mathit{Words}(\varphi)$ can be
extended to an accepting run $B_0\, B_1\, B_2 \ldots$ in $\mathcal{G}$

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$$A_0 \quad A_1 \quad A_2 \quad A_3 \quad \ldots \quad \in Words(\varphi)$$

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LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

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$$\begin{array}{cccccl} A_0 & A_1 & A_2 & A_3 & \ldots & \in \mathit{Words}(\varphi) \\ \downarrow & \downarrow & \downarrow & \downarrow & & \\ B_0 & B_1 & B_2 & B_3 & \ldots & \text{accepting run} \end{array}$$

where $B_i = \big\{\psi \in cl(\varphi) : A_i A_{i+1} A_{i+2} \ldots \models \psi\big\}$

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$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & A_2 \arrow[d] & A_3 \arrow[d] & \ldots & \in \mathit{Words}(\varphi) \\
B_0 & B_1 & B_2 & B_3 & \ldots & \text{accepting run}
\end{tikzcd}$$

where $B_i = \{\psi \in \mathit{cl}(\varphi) : A_i A_{i+1} A_{i+2} \ldots \models \psi\}$

set of subformulas of φ and their negations

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LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

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Example: $\varphi \;=\; a\,\mathsf{U}(\neg a \wedge b)$

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Example: $\varphi \; = \; a\, \mathsf{U}(\neg a \wedge b)$

$$\{a\} \quad \{a\} \quad \{a,b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

LTL $\rightsquigarrow$ GNBA

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states of $\mathcal{G}$ $\;\widehat{=}\;$ (certain) sets of subformulas of φ s.t. each word $\sigma = A_0\,A_1\,A_2 \ldots \in Words(\varphi)$ can be extended to an accepting run $B_0\,B_1\,B_2 \ldots$ in $\mathcal{G}$

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$$\begin{tikzcd}
\{a\} \arrow[d] & \{a\} \arrow[d] & \{a,b\} \arrow[d] & \{b\} \arrow[d] & \varnothing \arrow[d] & \varnothing \arrow[d] & \ldots \models \varphi \\
B_0 & B_1 & B_2 & B_3 & B_4 & B_5 &
\end{tikzcd}$$

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Example: $\varphi \;=\; a\,\mathsf{U}(\neg a \wedge b) \qquad \psi = \neg a \wedge b$

$$\begin{tikzcd}
\{a\} \arrow[d] & \{a\} \arrow[d] & \{a,b\} \arrow[d] & \{b\} \arrow[d] & \varnothing \arrow[d] & \varnothing \arrow[d] & \ldots \models \varphi \\
B_0 & B_1 & B_2 & B_3 & B_4 & B_5 &
\end{tikzcd}$$

where the B_i's are subsets of
$\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

LTL ⇝ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

states of $\mathcal{G}$ $\;\widehat{=}\;$ (certain) sets of subformulas of φ s.t. each word $\sigma = A_0\, A_1\, A_2 \ldots \in \mathit{Words}(\varphi)$ can be extended to an accepting run $B_0\, B_1\, B_2 \ldots$ in $\mathcal{G}$

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$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \quad \models \varphi$$

$$\downarrow$$

$$\boxed{\begin{array}{r} a \\ \neg b \\ \neg \psi \\ \varphi \end{array}}$$

just for better readability:
tuple rather than set notation

LTL ⇝ GNBA

LTL formula φ ⇝ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

states of $\mathcal{G}$ $\widehat{=}$ (certain) sets of subformulas of φ s.t. each word $\sigma = A_0\, A_1\, A_2 \ldots \in \mathit{Words}(\varphi)$ can be extended to an accepting run $B_0\, B_1\, B_2 \ldots$ in $\mathcal{G}$

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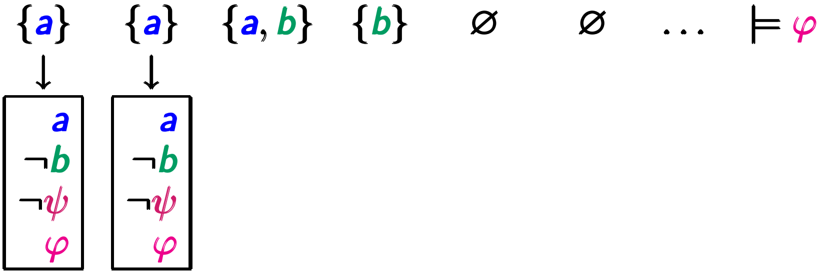

$$\{a\} \quad \{a\} \quad \{a,b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \quad \models \varphi$$

$$\begin{array}{cc} \downarrow & \downarrow \\ \boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} & \boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} \end{array}$$

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

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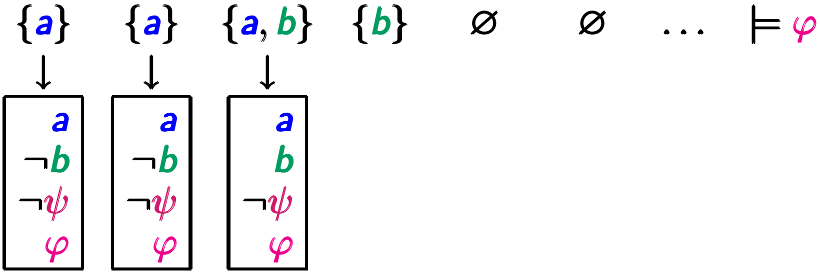

$\{a\} \quad \{a\} \quad \{a,b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} & \boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} & \boxed{\begin{array}{r} a \\ b \\ \neg\psi \\ \varphi \end{array}}
\end{array}$$

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

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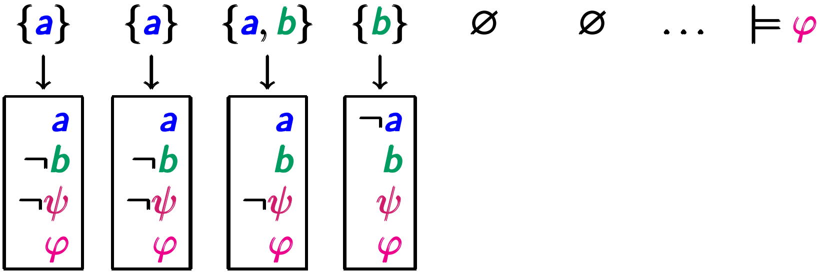

$\{a\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\ldots$	$\models \varphi$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$				
a $\neg b$ $\neg \psi$ φ	a $\neg b$ $\neg \psi$ φ	a b $\neg \psi$ φ	$\neg a$ b ψ φ				

LTL ⇝ GNBA

LTL formula φ ⇝ GNBA $\mathcal{G}$ for $Words(\varphi)$

states of $\mathcal{G}$ $\hat{=}$ (certain) sets of subformulas of φ s.t. each word $\sigma = A_0\,A_1\,A_2 \ldots \in Words(\varphi)$ can be extended to an accepting run $B_0\,B_1\,B_2\,\ldots$ in $\mathcal{G}$

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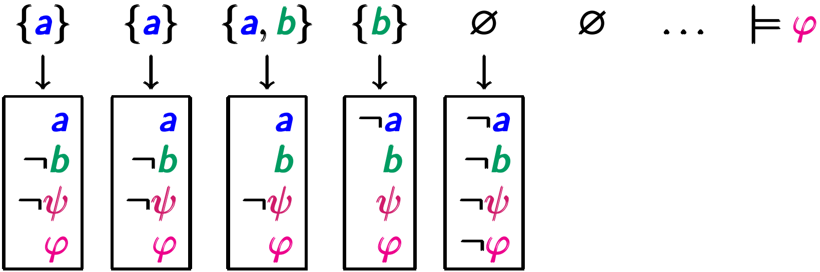

$$\begin{array}{cccccccc}
\{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots & \models \varphi \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\
\boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} &
\boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} &
\boxed{\begin{array}{r} a \\ b \\ \neg\psi \\ \varphi \end{array}} &
\boxed{\begin{array}{r} \neg a \\ b \\ \psi \\ \varphi \end{array}} &
\boxed{\begin{array}{r} \neg a \\ \neg b \\ \neg\psi \\ \neg\varphi \end{array}} & & &
\end{array}$$

LTL $\rightsquigarrow$ GNBA

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

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\{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots & \models \varphi \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
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\boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} &
\boxed{\begin{array}{r} a \\ b \\ \neg\psi \\ \varphi \end{array}} &
\boxed{\begin{array}{r} \neg a \\ b \\ \psi \\ \varphi \end{array}} &
\boxed{\begin{array}{r} \neg a \\ \neg b \\ \neg\psi \\ \neg\varphi \end{array}} &
\boxed{\begin{array}{r} \neg a \\ \neg b \\ \neg\psi \\ \neg\varphi \end{array}} &
\ldots &
\end{array}$$

Closure of LTL formulas

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Let φ be an LTL formula. Then:

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where ψ and $\neg\neg\psi$ are identified

Closure of LTL formulas

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Example: if $\varphi = a\,\mathsf{U}(\neg a \wedge b)$ then

$$cl(\varphi) = \{a, b, \neg a \wedge b, \varphi\} \cup \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$$

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Example: if $\varphi' = \Box a$

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Example: if $\varphi' = \Box a = \neg\Diamond\neg a = \neg(\mathit{true}\,\mathsf{U}\,\neg a)$

Closure of LTL formulas

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Example: if $\varphi' = \Box a = \neg\Diamond\neg a = \neg(true\,\mathsf{U}\,\neg a)$ then

$$cl(\varphi') = \{a, \neg a, true, \neg true, \Box a, \neg\Box a\}$$

Elementary formula-sets

LTLMC3.2-50

Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

(1) B is consistent w.r.t. propositional logic

(2) B is maximal consistent

(3) B is locally consistent with respect to until $\mathbf{U}$:

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if $\psi \in B$ then $\neg\psi \notin B$

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Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

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if $\psi \in B$ then $\neg\psi \notin B$

if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$

(2) B is maximal consistent

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Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

(1) B is consistent w.r.t. propositional logic

if $\psi \in B$ then $\neg\psi \notin B$
if $\psi_1 \wedge \psi_2 \in B$ then $\neg\psi_1 \notin B$ and $\neg\psi_2 \notin B$
if $\psi_1 \in B$ and $\psi_2 \in B$ then $\neg(\psi_1 \wedge \psi_2) \notin B$

(2) B is maximal consistent

(3) B is locally consistent with respect to until **U**:

Elementary formula-sets

Let $B \subseteq cl(\varphi)$. B is called elementary if:

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if $\psi \in cl(\varphi) \setminus B$ then $\neg\psi \in B$

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(3) B is locally consistent with respect to until $\mathbf{U}$:

if $\psi_1 \,\mathbf{U}\, \psi_2 \in B$ and $\neg\psi_2 \in B$ then $\neg\psi_1 \notin B$

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Let $B \subseteq cl(\varphi)$. B is called elementary if:

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(3) B is locally consistent with respect to until $\mathbf{U}$:

if $\psi_1 \,\mathbf{U}\, \psi_2 \in B$ and $\neg\psi_2 \in B$ then $\neg\psi_1 \notin B$

if $\psi_2 \in B$ and $\psi_1 \,\mathbf{U}\, \psi_2 \in cl(\varphi)$ then $\neg(\psi_1 \,\mathbf{U}\, \psi_2) \notin B$

Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
i.e., if ψ, $\psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\boxed{\begin{array}{rcl} \psi \notin B & \text{iff} & \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \text{iff} & \psi_1 \in B \text{ and } \psi_2 \in B \\ \mathit{true} \in cl(\varphi) & \text{implies} & \mathit{true} \in B \end{array}}$$

(ii) B is locally consistent with respect to until $\mathbf{U}$,
i.e., if $\psi_1 \,\mathbf{U}\, \psi_2 \in cl(\varphi)$ then:

$$\boxed{\begin{array}{l} \text{if } \psi_1 \,\mathbf{U}\, \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B \\ \text{if } \psi_2 \in B \text{ then } \psi_1 \,\mathbf{U}\, \psi_2 \in B \end{array}}$$

Elementary or not?

Let $\varphi = a \, \mathsf{U} (\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$

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propositional inconsistent

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$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Elementary or not?

Let $\varphi = a\,\mathsf{U}(\neg a \wedge b)$.

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$B_2 = \{\neg a, b, \varphi\}$ not elementary, not maximal

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$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$ elementary

Example: elementary formula-sets

closure $cl(\varphi)$:

- set of all subformulas of φ and their negations
- ψ and $\neg\neg\psi$ are identified

elementary formula-sets: subsets B of $cl(\varphi)$

- maximal consistent w.r.t. propositional logic
- locally consistent w.r.t. U

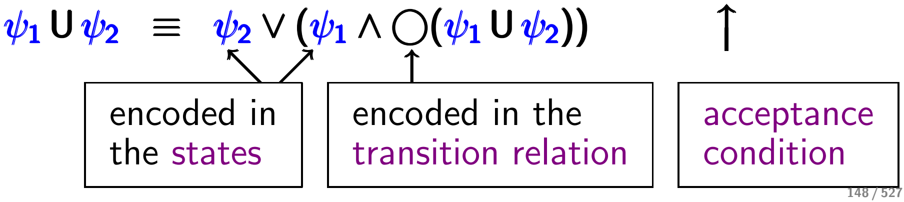
For $\varphi = a\,\mathsf{U}(\neg a \wedge b)$, the elementary sets are:

$$\begin{array}{ll}
\{\ a,\ b, \neg(\neg a \wedge b), \varphi\} & \{\ a,\ b, \neg(\neg a \wedge b), \neg\varphi\} \\
\{\ a, \neg b, \neg(\neg a \wedge b), \varphi\} & \{\ a, \neg b, \neg(\neg a \wedge b), \neg\varphi\} \\
\{\neg a,\ b,\ \neg a \wedge b\ , \varphi\} & \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}
\end{array}$$

Encoding of LTL semantics in a GNBA

idea: encode the semantics of the operators appearing in φ by appropriate components of the GNBA $\mathcal{G}$:

semantics of ...	encoding
propositional logic ***true***, $\neg$, $\wedge$	in the states
next $\bigcirc$	in the transition relation
until U	expansion law, least fixed point



$$\psi_1 \,\mathsf{U}\, \psi_2 \;\equiv\; \psi_2 \vee (\psi_1 \wedge \bigcirc(\psi_1 \,\mathsf{U}\, \psi_2))$$

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acceptance condition

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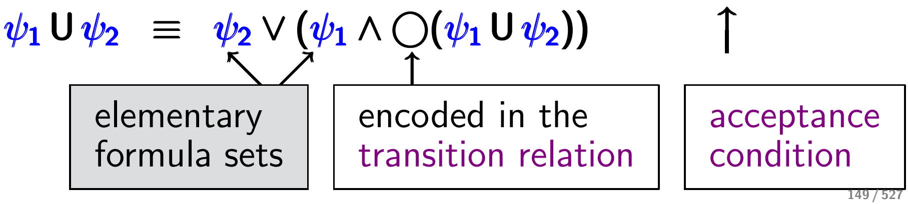

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GNBA for LTL-formula φ

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$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \,\mathsf{U}\, \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \,\mathsf{U}\, \psi_2 \in B')$$

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acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

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where $F_{\psi_1 \mathsf{U} \psi_2} = \{ B \in Q : \psi_1 \, \mathsf{U} \, \psi_2 \notin B \vee \psi_2 \in B \}$

Example: GNBA for $\varphi = \bigcirc a$

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$a, \bigcirc a$

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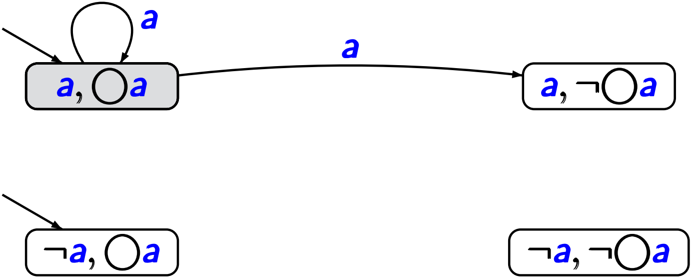

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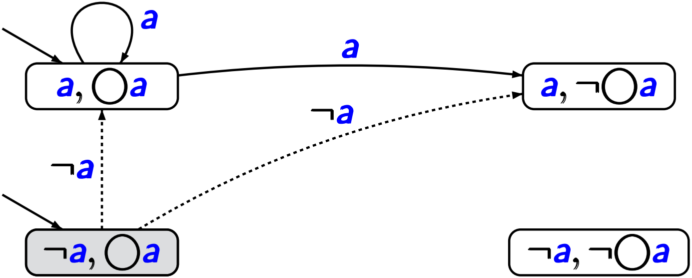

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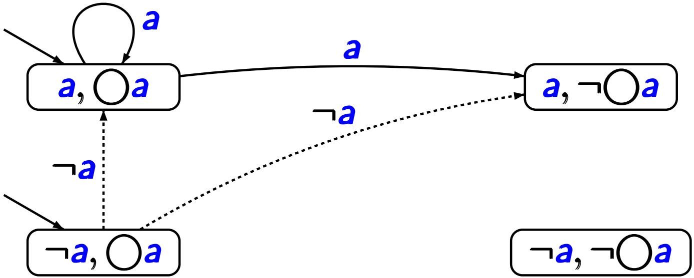

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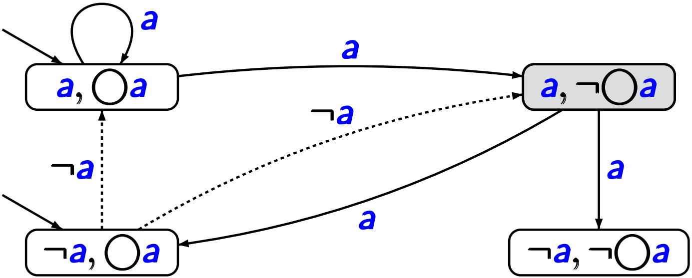

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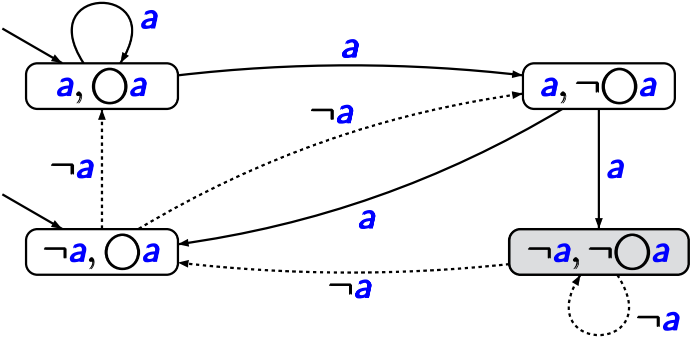

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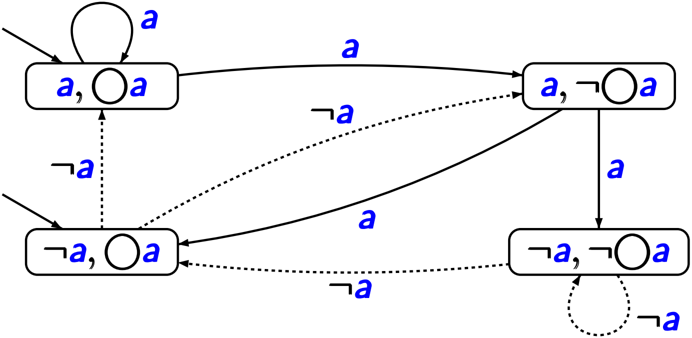
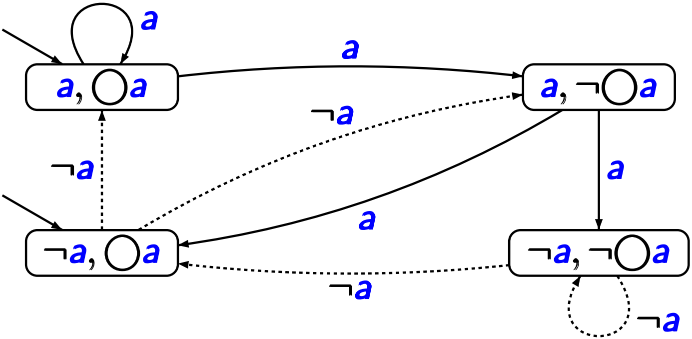


set of acceptance sets:

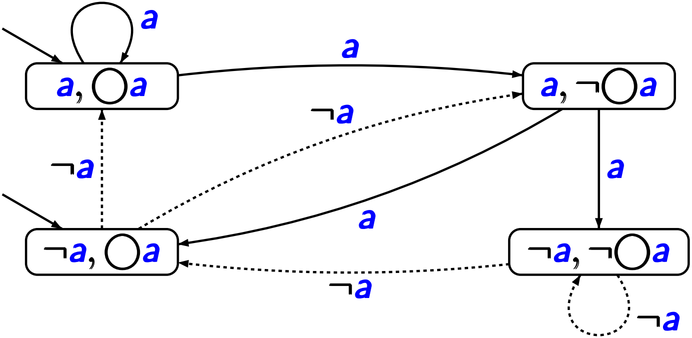
Example: GNBA for $\varphi = \bigcirc a$



set of acceptance sets: $\mathcal{F} = \varnothing$

hence: all words having an infinite run are accepted

Example: GNBA for $\varphi = \bigcirc a$



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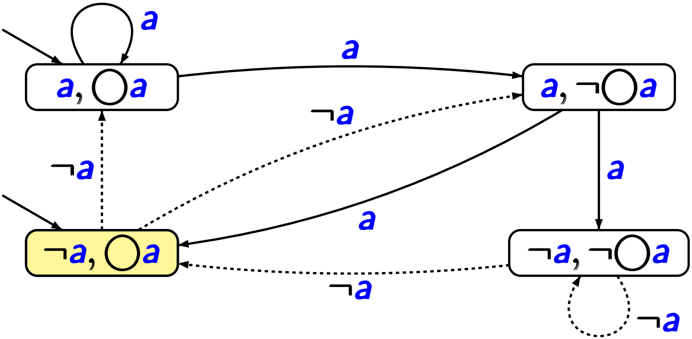

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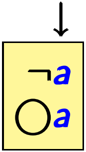

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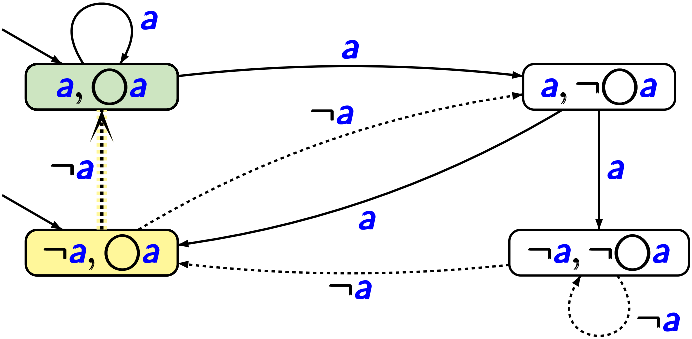

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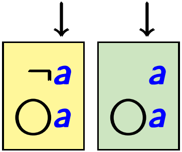

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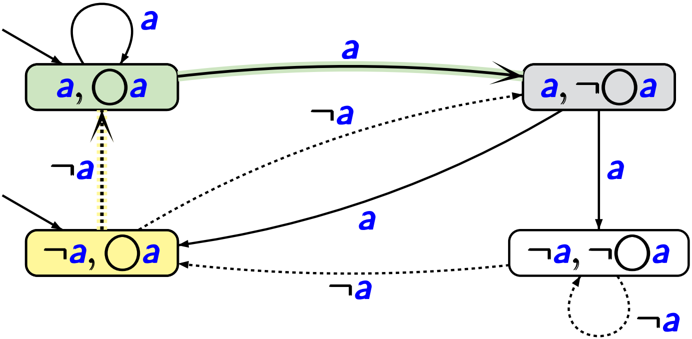

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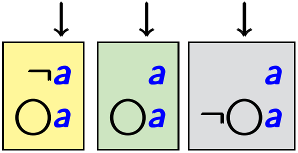

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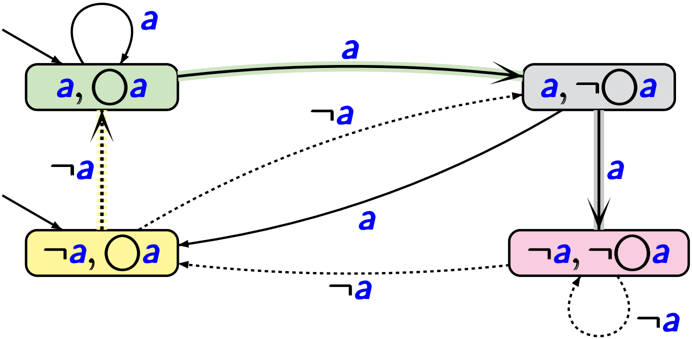

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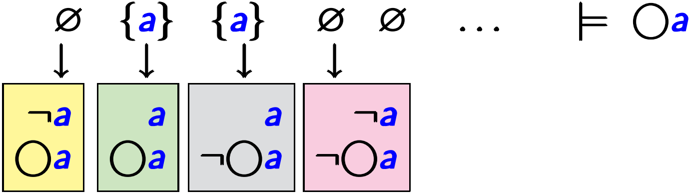

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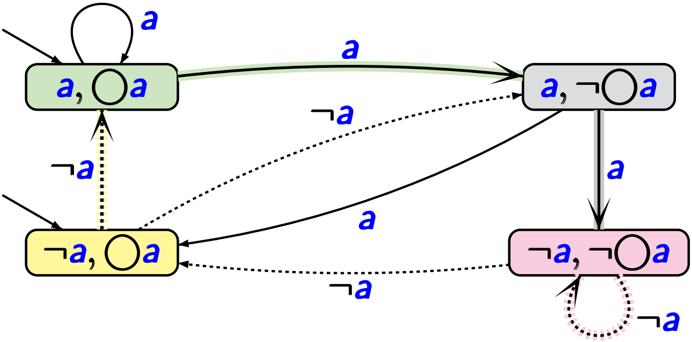

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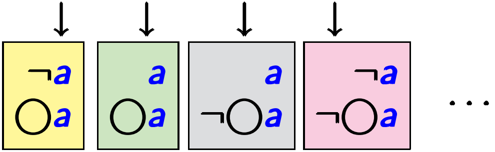
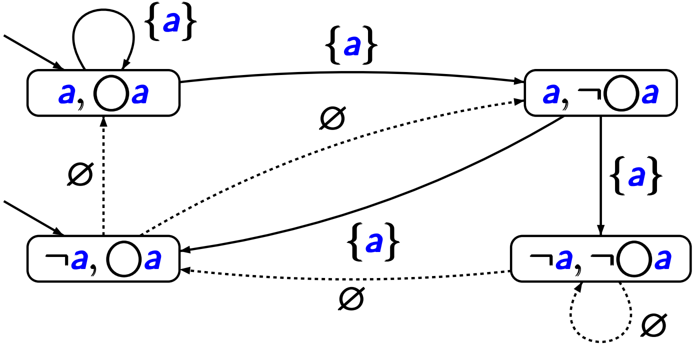


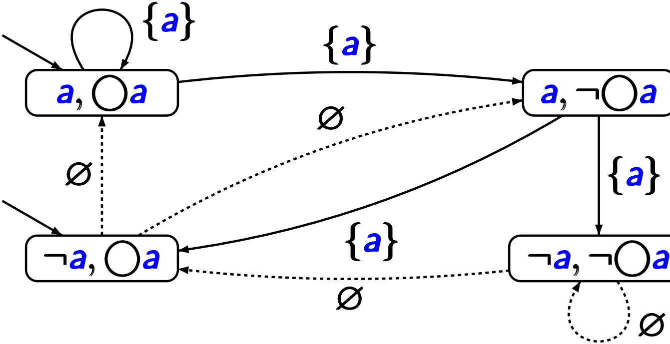
accepting
run

Soundness of the GNBA for $\varphi = \bigcirc a$



for all words $\sigma = A_0\, A_1\, A_2\, A_3 \ldots \in \mathcal{L}_\omega(\mathcal{G})$: $\quad A_1 = \{a\}$

Soundness of the GNBA for $\varphi = \bigcirc a$



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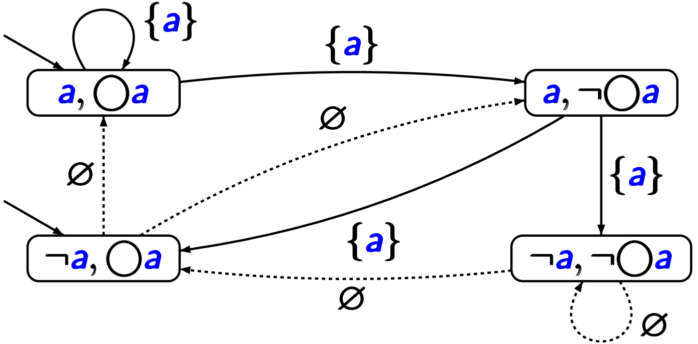

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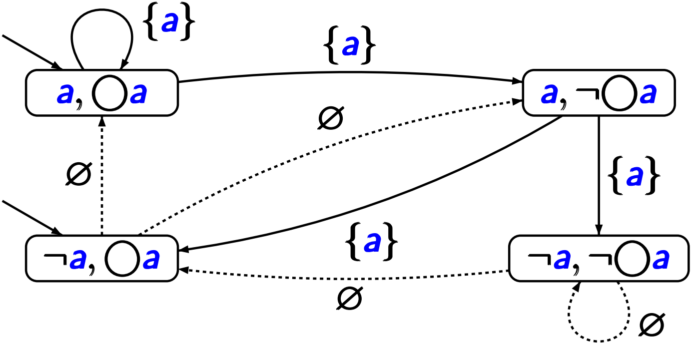

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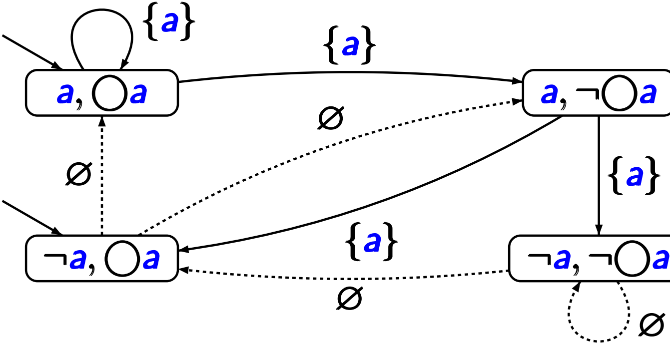
$\Longrightarrow \quad \bigcirc a \in B_0$

Soundness of the GNBA for $\varphi = \bigcirc a$



for all words $\sigma = A_0\, A_1\, A_2\, A_3 \ldots \in \mathcal{L}_\omega(\mathcal{G})$: $\quad A_1 = \{a\}$

proof: Let $B_0\, B_1\, B_2\, \ldots$ be an accepting run for σ.

$\Longrightarrow \quad \bigcirc a \in B_0$ and therefore $a \in B_1$

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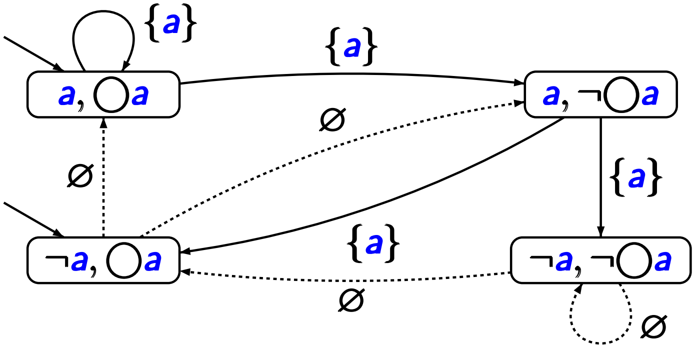

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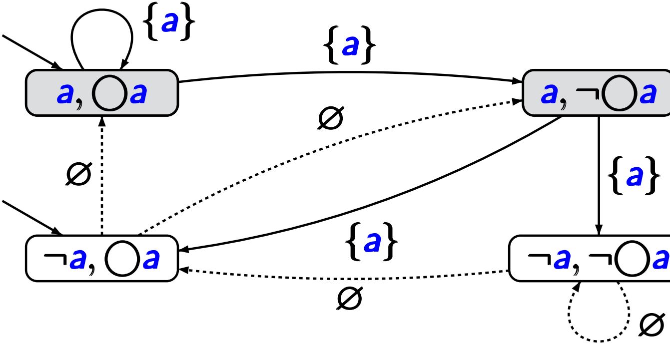

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$\Longrightarrow$ $\{a\} = B_1 \cap AP = A_1$

Example: GNBA for $\varphi = a \,\mathsf{U}\, b$

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$a, b, a \,\mathsf{U}\, b$

$\neg a, \neg b, \neg(a \,\mathsf{U}\, b)$

$a, \neg b, a \,\mathsf{U}\, b$

$a, \neg b, \neg(a \,\mathsf{U}\, b)$

$\neg a, b, a \,\mathsf{U}\, b$

locally inconsistent: $\{a, b, \neg(a \,\mathsf{U}\, b)\}$

$\{\neg a, b, \neg(a \,\mathsf{U}\, b)\}$

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initial states: B with $\varphi = a \,\mathsf{U}\, b \in B$

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F = set of all B with $\varphi \notin B$ or $b \in B$

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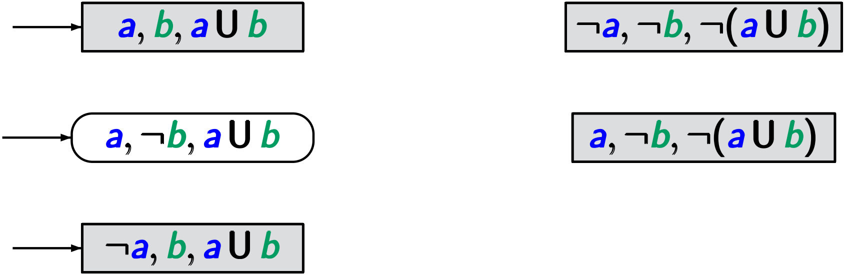

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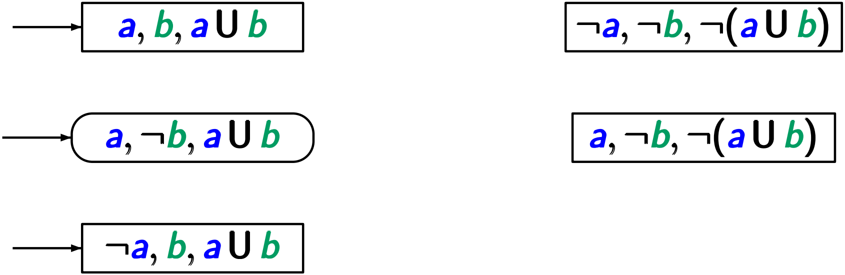

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transition relation: $B' \in \delta(B, B \cap AP)$ iff

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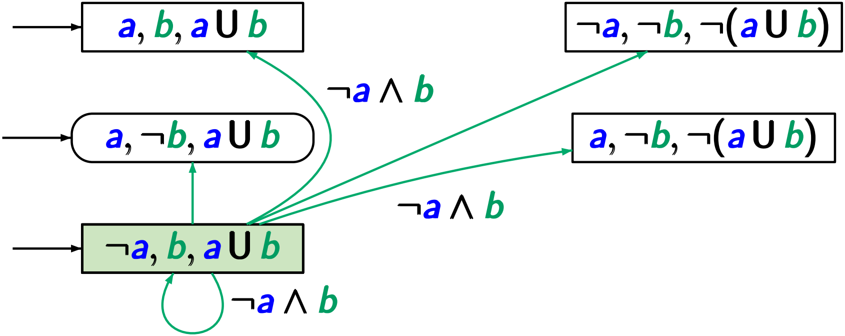

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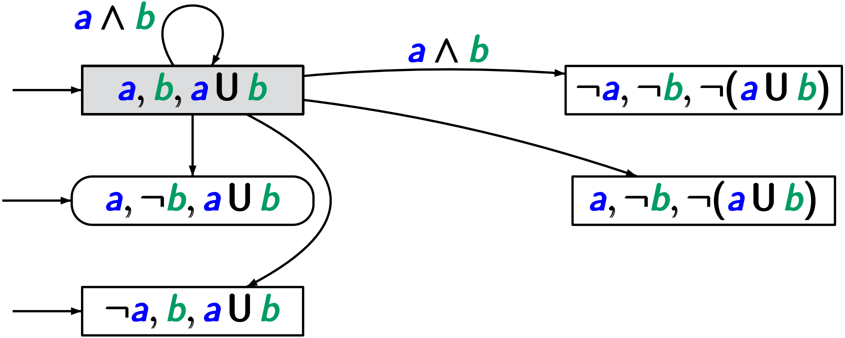

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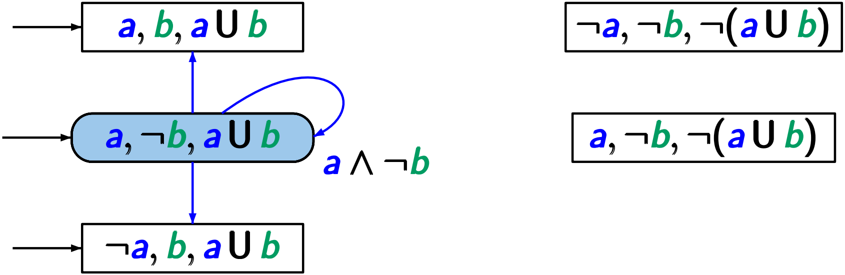

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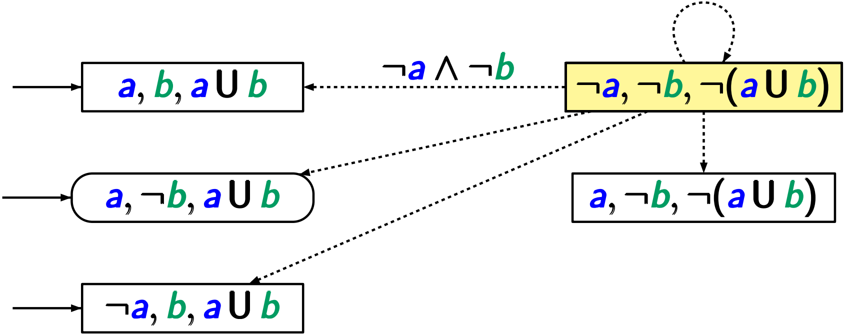

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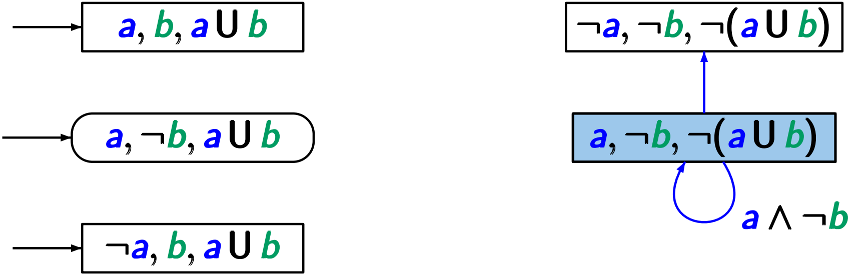

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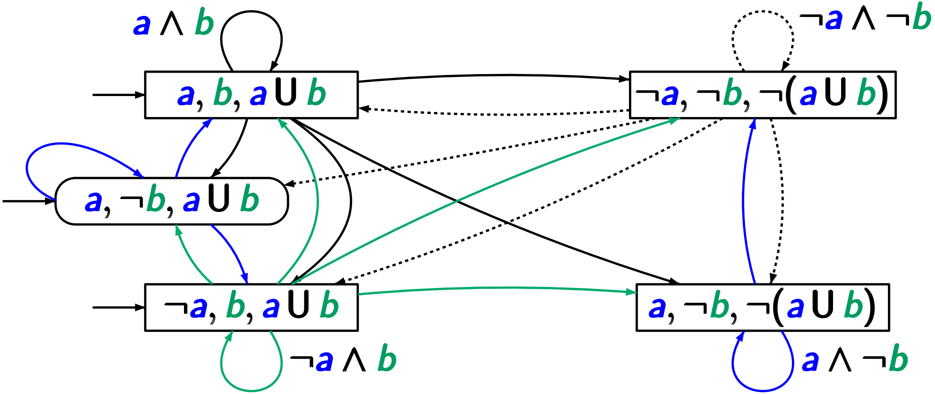
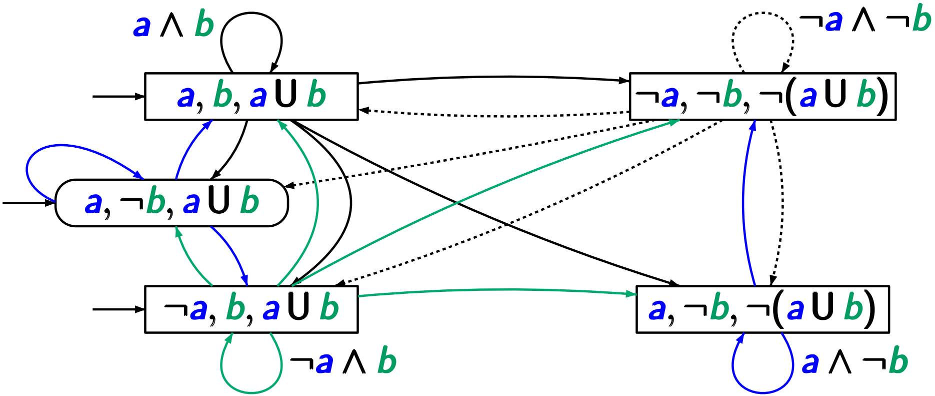
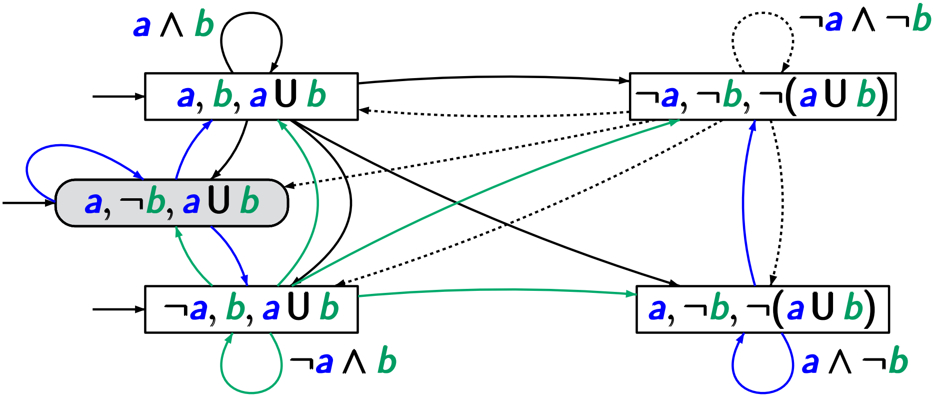


Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



$\{a\} \quad \{a\} \quad \{a, b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \,\mathsf{U}\, b$

Example: (G)NBA for $\varphi = a \, \mathsf{U} \, b$

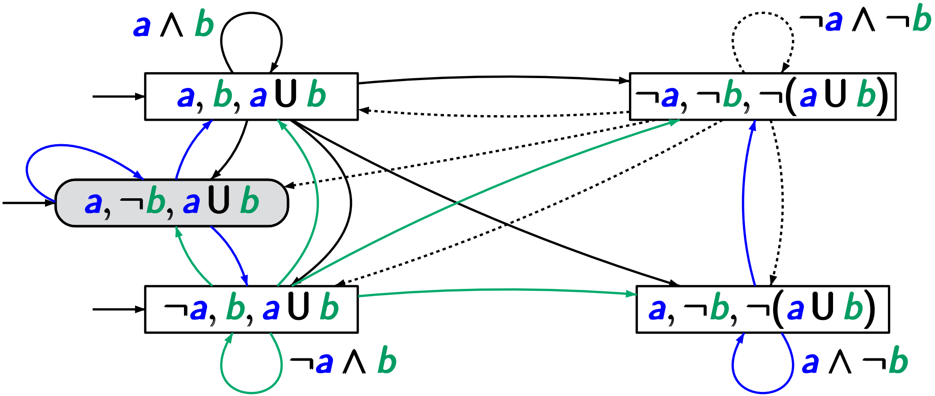


$$\{a\} \quad \{a\} \quad \{a, b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \, \mathsf{U} \, b$$

$\downarrow$

a
$\neg b$
φ

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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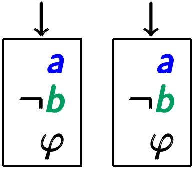

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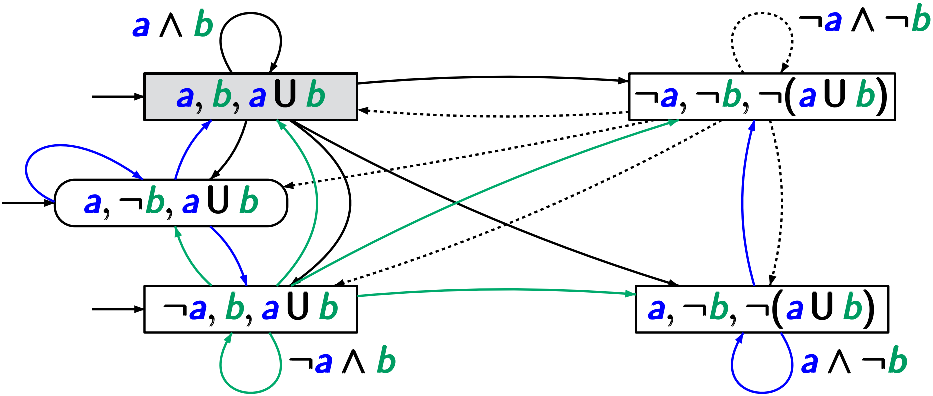
Example: (G)NBA for $\varphi = a \mathsf{U} b$



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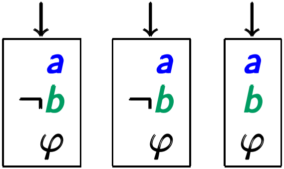

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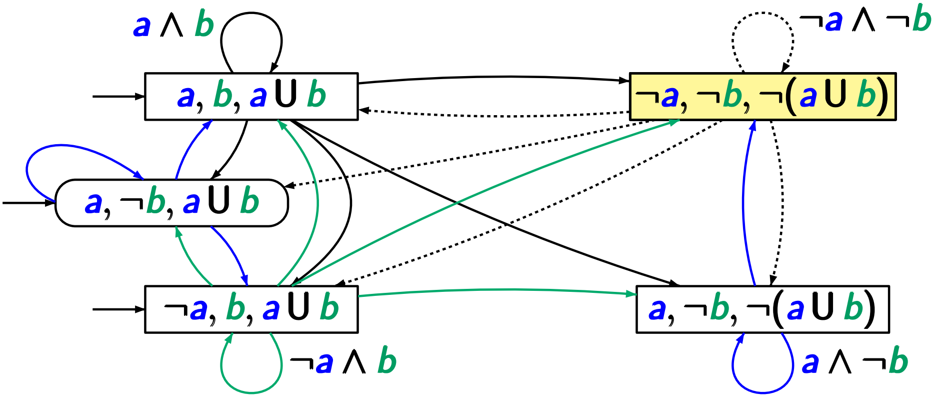
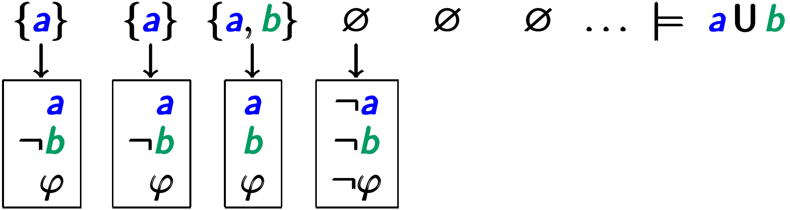
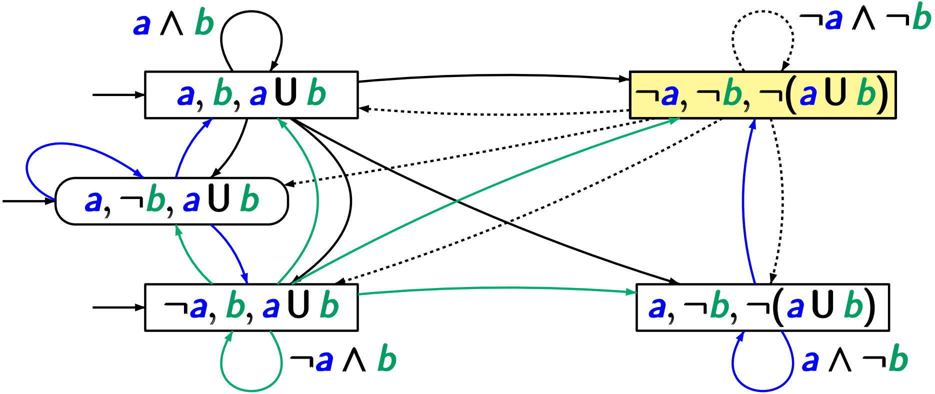




$\{a\} \quad \{a\} \quad \{a, b\} \quad \varnothing \quad \varnothing \quad \varnothing \quad \ldots \models a \,\mathsf{U}\, b$

a	a	a	$\neg a$
$\neg b$	$\neg b$	b	$\neg b$
φ	φ	φ	$\neg\varphi$

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
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φ	φ	φ	$\neg\varphi$	$\neg\varphi$

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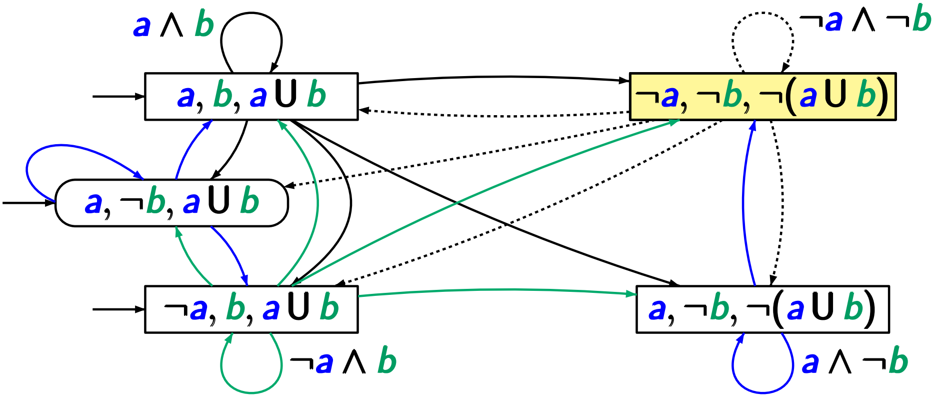

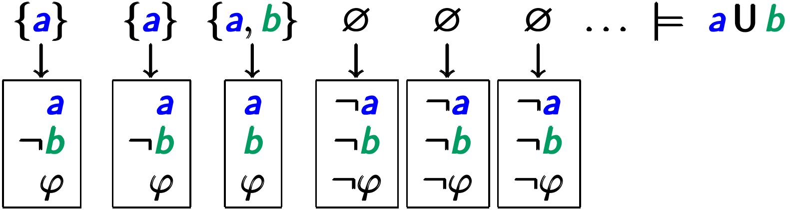

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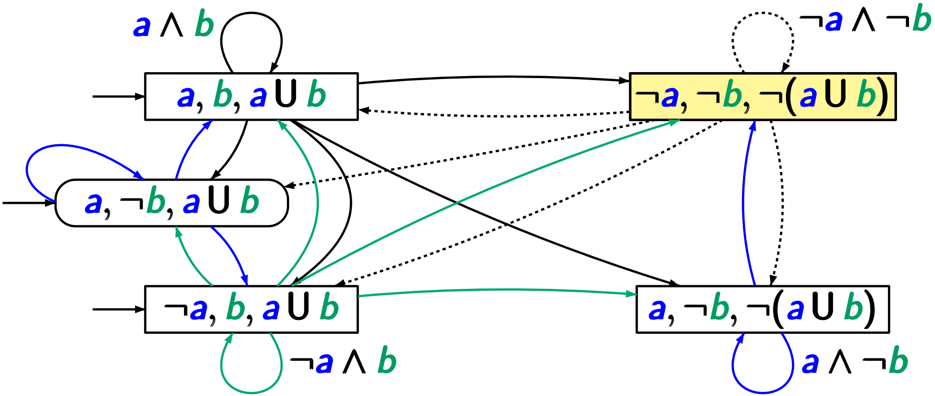
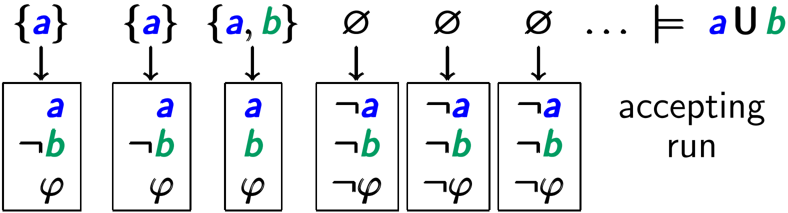
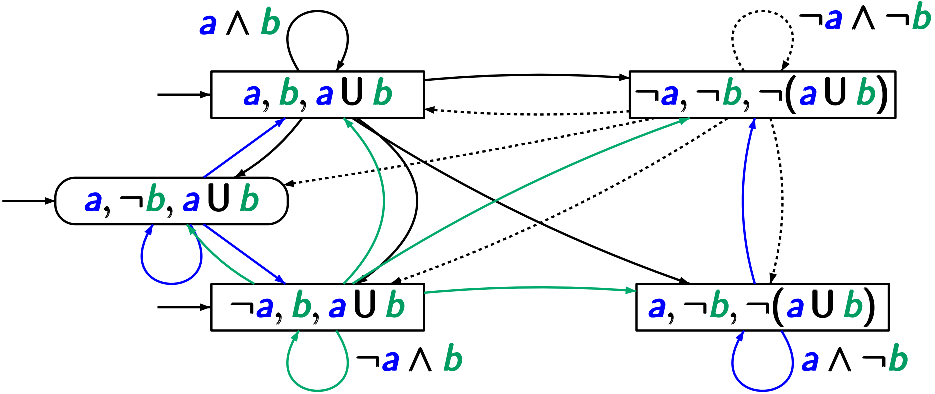




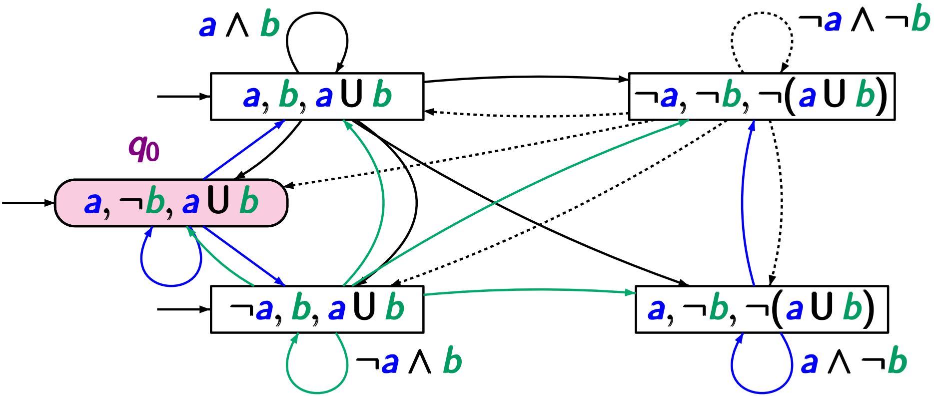
accepting
run

Example: (G)NBA for $\varphi = a \mathsf{U} b$



$$\{a\}\,\{a\}\,\{a\}\,\{a\}\,\ldots \not\models \varphi$$

Example: (G)NBA for $\varphi = a \,\mathsf{U}\, b$



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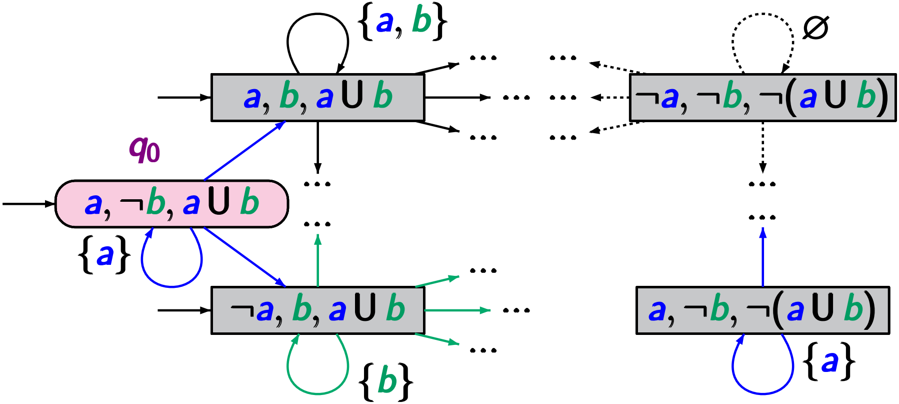

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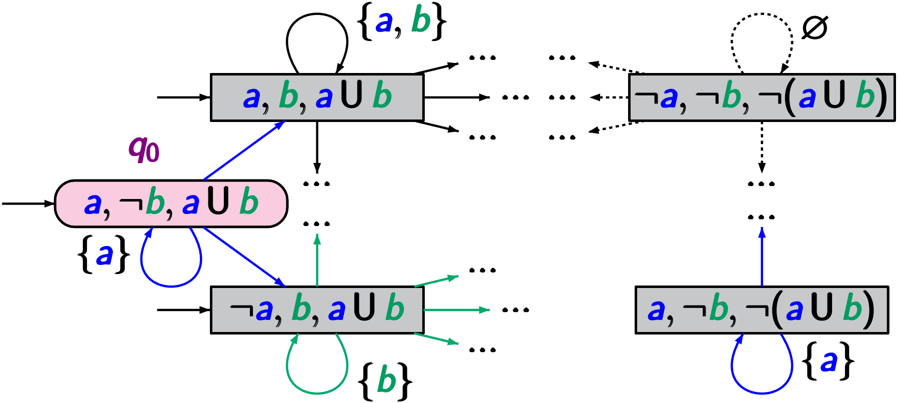

$\{a\}\,\{a\}\,\{a\}\,\{a\} \ldots \not\models \varphi$

only 1 infinite run: $q_0\, q_0\, q_0 \ldots$ not accepting

GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary }\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\boxed{\begin{array}{rll} \bigcirc\psi \in B & \text{iff} & \psi \in B' \\ \psi_1 \mathsf{U} \psi_2 \in B & \text{iff} & (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B') \end{array}}$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Soundness

.... of the construction LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$

Soundness

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $\mathit{Words}(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

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Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0\,A_1\,A_2 \ldots \in (2^{AP})^\omega$

with $A_0\,A_1\,A_2 \ldots \models \varphi$

has an accepting run in $\mathcal{G}$

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Accepting runs for the elements of $\mathit{Words}(\varphi)$

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

states of $\mathcal{G} \;\; \widehat{=} \;\;$ elementary formula-sets $B \subseteq cl(\varphi)$

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Accepting runs for the elements of *Words*(φ)

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$$\begin{tikzcd}
\{a\} \arrow[d] & \{a\} \arrow[d] & \{a, b\} \arrow[d] & \{b\} \arrow[d] & \varnothing \arrow[d] & \varnothing \arrow[d] & \ldots \models \varphi \\
B_0 & B_1 & B_2 & B_3 & B_4 & B_5 &
\end{tikzcd}$$

Accepting runs for the elements of $Words(\varphi)$

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

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Example: $\varphi \;=\; a\,\mathsf{U}(\neg a \wedge b) \qquad \psi = \neg a \wedge b$

$$\begin{array}{ccccccl} \{a\} & \{a\} & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots \models \varphi \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & \end{array}$$

where the B_i's are states in $\mathcal{G}$, i.e., elementary subsets of $\{a, \neg a, b, \neg b, \psi, \neg\psi, \varphi, \neg\varphi\}$

Accepting runs for the elements of $Words(\varphi)$

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $Words(\varphi)$

states of $\mathcal{G}$ $\widehat{=}$ elementary formula-sets $B \subseteq cl(\varphi)$

s.t. each word $\sigma = A_0\, A_1\, A_2 \ldots \in Words(\varphi)$ can be extended to an accepting run $B_0\, B_1\, B_2 \ldots$ in $\mathcal{G}$

Example: $\varphi \;=\; a\,\mathsf{U}(\neg a \wedge b)$ $\qquad \psi = \neg a \wedge b$

$$\{a\} \quad \{a\} \quad \{a, b\} \quad \{b\} \quad \varnothing \quad \varnothing \quad \ldots \models \varphi$$

$$\begin{array}{c} \downarrow \\ \boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} \end{array}$$

Accepting runs for the elements of $Words(\varphi)$

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\{a\} \arrow[d] & \{a\} \arrow[d] & \{a,b\} & \{b\} & \varnothing & \varnothing & \ldots & \models \varphi \\
\boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} & \boxed{\begin{array}{r} a \\ \neg b \\ \neg\psi \\ \varphi \end{array}} & & & & & &
\end{tikzcd}$$

Accepting runs for the elements of $Words(\varphi)$

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\end{tikzcd}$$

Accepting runs for the elements of $\mathit{Words}(\varphi)$

LTL formula $\varphi \rightsquigarrow$ GNBA $\mathcal{G}$ for $\mathit{Words}(\varphi)$

states of $\mathcal{G}$ $\quad\widehat{=}\quad$ elementary formula-sets $B \subseteq cl(\varphi)$

s.t. each word $\sigma = A_0\, A_1\, A_2 \ldots \in \mathit{Words}(\varphi)$ can be extended to an accepting run $B_0\, B_1\, B_2 \ldots$ in $\mathcal{G}$

Example: $\varphi \;=\; a\,\mathsf{U}(\neg a \wedge b) \qquad \psi = \neg a \wedge b$

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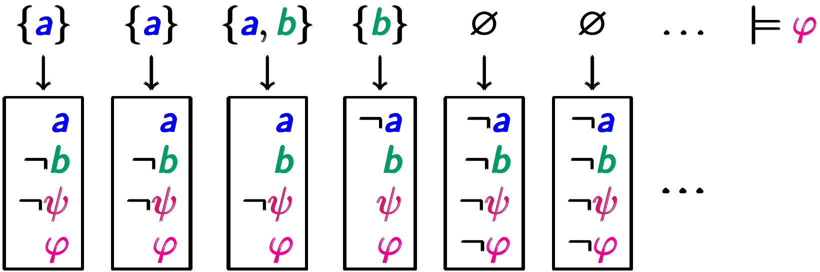

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\end{array}$$

GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic,
i.e., if $\psi,\ \psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\boxed{\begin{array}{rcl} \psi \notin B & \text{iff} & \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \text{iff} & \psi_1 \in B \text{ and } \psi_2 \in B \\ \mathit{true} \in cl(\varphi) & \text{implies} & \mathit{true} \in B \end{array}}$$

(ii) B is locally consistent with respect to until **U**,
i.e., if $\psi_1 \,\mathsf{U}\, \psi_2 \in cl(\varphi)$ then:

$$\boxed{\begin{array}{l} \text{if } \psi_1 \,\mathsf{U}\, \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B \\ \text{if } \psi_2 \in B \text{ then } \psi_1 \,\mathsf{U}\, \psi_2 \in B \end{array}}$$

Soundness

Let φ be an LTL-formula and $\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ be the constructed GNBA.

Claim: $Words(\varphi) = \mathcal{L}_\omega(\mathcal{G})$

"$\subseteq$" show: each infinite word $A_0\, A_1\, A_2 \ldots \in (2^{AP})^\omega$

with $A_0\, A_1\, A_2 \ldots \models \varphi$

has an accepting run in $\mathcal{G}$

"$\supseteq$" show: for all infinite words $A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$:

$A_0\, A_1\, A_2 \ldots \models \varphi$

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Proof of $\mathcal{L}_\omega(\mathcal{G}) \subseteq Words(\varphi)$

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

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$\boxed{\text{as } B_0 \in Q_0}$

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and $(*)$ holds

as $B_0 \in Q_0$

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$\uparrow$ as $B_0 \in Q_0$

$\Longrightarrow$ $\sigma = A_0\,A_1\,A_2 \ldots \models \varphi$

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Proof by structural induction on ψ

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Proof by structural induction on ψ

base of induction:

$\psi = \mathit{true}$

$\psi = a \in AP$

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Proof by structural induction on ψ

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induction step:

$\psi = \neg\psi'$

$\psi = \psi_1 \wedge \psi_2$

$\psi = \bigcirc\psi'$

$\psi = \psi_1 \mathsf{U} \psi_2$

Base of induction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Suppose $\psi = \boldsymbol{true} \in cl(\varphi)$.

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note: ***true*** is contained in all elementary formula-sets

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Base of induction:

Suppose $\psi = \mathit{true} \in cl(\varphi)$. Then $\mathit{true} \in B_0$ and

$$A_0\, A_1\, A_2 \ldots \models \mathit{true}$$

note: ***true*** is contained in all elementary formula-sets

true holds for all paths/traces

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Let $\psi = a \in AP$.

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Let $\psi = a \in AP$. Then:

$a \in B_0$

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Base of induction

Claim: If $\boxed{B_0 \xrightarrow{A_0} B_1} \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \; \overset{\infty}{\exists} j \geq 0.\, B_j \in F \qquad \boxed{A_0 = B_0 \cap AP}$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Base of induction:

Suppose $\psi = \textit{true} \in cl(\varphi)$. Then $\textit{true} \in B_0$ and

$$A_0\, A_1\, A_2 \ldots \models \textit{true}$$

Let $\psi = a \in AP$. Then:

$$a \in B_0 \iff a \in A_0 \iff A_0\, A_1\, A_2 \ldots \models a$$

Induction step: negation

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0. \ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

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Induction step: for $\psi = \neg\psi'$:

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \ldots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

$\psi \in B_0$

iff $\psi' \notin B_0$ (maximal consistency)

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

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Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \not\models \psi'$ (induction hypothesis)

Induction step: negation

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \neg\psi'$:

$$\psi \in B_0$$

iff $\psi' \notin B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \not\models \psi'$ (induction hypothesis)

iff $A_0\, A_1\, A_2 \ldots \models \psi$ (semantics of ¬)

Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic, i.e., if ψ, $\psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\boxed{\begin{array}{c} \psi \notin B \quad \text{iff} \quad \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B \\ \textit{true} \in cl(\varphi) \quad \text{implies} \quad \textit{true} \in B \end{array}}$$

(ii) B is locally consistent with respect to until **U**, i.e., if $\psi_1 \, \mathbf{U} \, \psi_2 \in cl(\varphi)$ then:

$$\boxed{\begin{array}{l} \text{if } \psi_1 \, \mathbf{U} \, \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B \\ \text{if } \psi_2 \in B \text{ then } \psi_1 \, \mathbf{U} \, \psi_2 \in B \end{array}}$$

Elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

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$$\psi \notin B \quad \text{iff} \quad \neg\psi \in B$$

$$\boxed{\psi_1 \wedge \psi_2 \in B \quad \text{iff} \quad \psi_1 \in B \text{ and } \psi_2 \in B}$$

$$\mathit{true} \in cl(\varphi) \quad \text{implies} \quad \mathit{true} \in B$$

(ii) B is locally consistent with respect to until **U**, i.e., if $\psi_1 \,\mathsf{U}\, \psi_2 \in cl(\varphi)$ then:

$$\text{if } \psi_1 \,\mathsf{U}\, \psi_2 \in B \text{ and } \psi_2 \notin B \text{ then } \psi_1 \in B$$

$$\text{if } \psi_2 \in B \text{ then } \psi_1 \,\mathsf{U}\, \psi_2 \in B$$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0. \, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 \, A_1 \, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in $\mathcal{G}$ s.t.

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$$\psi \in B_0 \quad \text{iff} \quad A_0 A_1 A_2 \dots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$\psi \in B_0$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\psi \in B_0$$

iff $\psi_1, \psi_2 \in B_0$ (maximal consistency)

iff $A_0\, A_1\, A_2 \ldots \models \psi_1$ and $A_0\, A_1\, A_2 \ldots \models \psi_2$ (IH)

Induction step: conjunction

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \psi_1 \wedge \psi_2$

$$\begin{aligned} & \psi \in B_0 \\ \text{iff} \quad & \psi_1, \psi_2 \in B_0 && \text{(maximal consistency)} \\ \text{iff} \quad & A_0\, A_1\, A_2 \ldots \models \psi_1 \text{ and } A_0\, A_1\, A_2 \ldots \models \psi_2 && \text{(IH)} \\ \text{iff} \quad & A_0\, A_1\, A_2 \ldots \models \psi && \text{(semantics of } \wedge) \end{aligned}$$

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{ B \subseteq cl(\varphi) : B \text{ is elementary} \}$

initial states: $Q_0 = \{ B \in Q : \varphi \in B \}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\boxed{\bigcirc \psi \in B \quad \text{iff} \quad \psi \in B'}$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{ F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi) \}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{ B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B \}$

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$$\psi \in B_0$$

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F \qquad \boxed{B_1 \in \delta(B_0, A_0)}$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc\psi'$:

$$\psi \in B_0$$

iff $\psi' \in B_1$ (definition of δ)

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \overset{\infty}{\exists} j \geq 0.\, B_j \in F \qquad \boxed{B_1 \in \delta(B_0, A_0)}$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0\, A_1\, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$\psi \in B_0$

iff $\psi' \in B_1$ (definition of δ)

iff $A_1\, A_2\, A_3 \ldots \models \psi'$ (induction hypothesis)

Induction step: next step

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

$$\boxed{B_1 \in \delta(B_0, A_0)}$$

then for all formulas $\psi \in cl(\varphi)$:

$$\psi \in B_0 \quad \text{iff} \quad A_0 \, A_1 \, A_2 \ldots \models \psi$$

Induction step: for $\psi = \bigcirc \psi'$:

$\psi \in B_0$

iff $\psi' \in B_1$ (definition of δ)

iff $A_1 \, A_2 \, A_3 \ldots \models \psi'$ (induction hypothesis)

iff $A_0 \, A_1 \, A_2 \, A_3 \ldots \models \psi$ (semantics of $\bigcirc$)

Induction step: until

Recall: elementary formula-sets

$B \subseteq cl(\varphi)$ is elementary iff:

(i) B is maximal consistent w.r.t. prop. logic, i.e., if ψ, $\psi_1 \wedge \psi_2 \in cl(\varphi)$ then:

$$\begin{array}{rcl} \psi \notin B & \text{iff} & \neg\psi \in B \\ \psi_1 \wedge \psi_2 \in B & \text{iff} & \psi_1 \in B \text{ and } \psi_2 \in B \\ \mathit{true} \in cl(\varphi) & \text{implies} & \mathit{true} \in B \end{array}$$

(ii) B is locally consistent with respect to until U, i.e., if $\psi_1 \mathsf{U} \psi_2 \in cl(\varphi)$ then:

if $\psi_1 \mathsf{U} \psi_2 \in B$ and $\psi_2 \notin B$ then $\psi_1 \in B$

if $\psi_2 \in B$ then $\psi_1 \mathsf{U} \psi_2 \in B$

Recall: GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary }\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

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if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\boxed{\psi_1 \,\mathsf{U}\, \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \,\mathsf{U}\, \psi_2 \in B')}$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \,\mathsf{U}\, \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \,\mathsf{U}\, \psi_2 \notin B \vee \psi_2 \in B\}$

Recall: GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

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$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \, \mathsf{U} \, \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \, \mathsf{U} \, \psi_2 \in B')$$

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Induction step: until

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Induction step: until

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \mathsf{U} \psi_2$:

Induction step: until (part "$\Longleftarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \; \overset{\infty}{\exists} j \geq 0.\, B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\,A_1\,A_2\,\ldots \models \psi$

Induction step for $\psi = \psi_1 \,\mathsf{U}\, \psi_2$:

"$\Longleftarrow$": Suppose $A_0\,A_1\,A_2 \ldots \models \psi$.

Induction step: until (part "$\Longleftarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, j \geq 0.\, B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \, \mathsf{U} \, \psi_2$:

"$\Longleftarrow$": Suppose $A_0\, A_1\, A_2 \ldots \models \psi$. Let $j \geq 0$ s.t.

$$A_j\, A_{j+1}\, A_{j+2} \ldots \models \psi_2$$
$$A_{j-1}\, A_j\, A_{j-1} \ldots \models \psi_1$$
$$A_{j-2}\, A_{j-1}\, A_j \ldots \models \psi_1$$
$$\vdots$$
$$A_0\, A_1\, A_2\, A_3 \ldots \models \psi_1$$

Induction step: until (part "$\Longleftarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

"$\Longleftarrow$": Suppose $A_0\, A_1\, A_2 \ldots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{lcll}
A_j\, A_{j+1}\, A_{j+2} \ldots & \models \psi_2 & \overset{\text{IH}}{\Rightarrow} & \psi_2 \in B_j \\
A_{j-1}\, A_j\, A_{j-1} \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} \\
A_{j-2}\, A_{j-1}\, A_j \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} \\
\vdots & & & \vdots \\
A_0\, A_1\, A_2\, A_3 \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_0
\end{array}$$

Induction step: until (part "$\Longleftarrow$")

Claim: If $B_0 \overset{A_0}{\rightarrow} B_1 \overset{A_1}{\rightarrow} B_2 \overset{A_2}{\rightarrow} \ldots$ is a path in $\mathcal{G}$ s.t.

$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$

$\boxed{B_j \text{ is elementary}}$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

"$\Longleftarrow$": Suppose $A_0\, A_1\, A_2 \ldots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{lclclcl}
A_j\, A_{j+1}\, A_{j+2} \ldots & \models \psi_2 & \overset{\text{IH}}{\Rightarrow} & \psi_2 \in B_j & \Rightarrow & \psi \in B_j \\
A_{j-1}\, A_j\, A_{j-1} \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & & \\
A_{j-2}\, A_{j-1}\, A_j \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & & \\
\vdots & & & \vdots & & \\
A_0\, A_1\, A_2\, A_3 \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_0 & &
\end{array}$$

Induction step: until (part “$\Longleftarrow$”)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\, B_j \in F \qquad \boxed{B_j \in \delta(B_{j-1}, A_{j-1})}$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \, \mathsf{U} \, \psi_2$:

“$\Longleftarrow$”: Suppose $A_0\, A_1\, A_2 \ldots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{lllll}
A_j\, A_{j+1}\, A_{j+2} \ldots & \models \psi_2 & \overset{\text{IH}}{\Rightarrow} & \psi_2 \in B_j & \Rightarrow & \psi \in B_j \\
A_{j-1}\, A_j\, A_{j-1} \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & \wedge & \psi \in B_{j-1} \\
A_{j-2}\, A_{j-1}\, A_j \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & & \\
\vdots & & & \vdots & & \\
A_0\, A_1\, A_2\, A_3 \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_0 & &
\end{array}$$

Induction step: until (part "$\Longleftarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$\forall F \in \mathcal{F}\ \overset{\infty}{\exists} j \geq 0.\ B_j \in F$ $\boxed{B_{j-1} \in \delta(B_{j-2}, A_{j-2})}$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

"$\Longleftarrow$": Suppose $A_0\, A_1\, A_2 \ldots \models \psi$. Let $j \geq 0$ s.t.

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A_j\, A_{j+1}\, A_{j+2} \ldots & \models \psi_2 & \overset{\text{IH}}{\Rightarrow} & \psi_2 \in B_j & \Rightarrow & \psi \in B_j \\
A_{j-1}\, A_j\, A_{j-1} \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & \wedge & \psi \in B_{j-1} \\
A_{j-2}\, A_{j-1}\, A_j \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & \wedge & \psi \in B_{j-2} \\
\vdots & & & \vdots & & \\
A_0\, A_1\, A_2\, A_3 \ldots & \models \psi_1 & \Rightarrow & \psi_1 \in B_0 & &
\end{array}$$

Induction step: until (part “$\Longleftarrow$”)

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \dots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0.\ B_j \in F \qquad \boxed{B_1 \in \delta(B_0, A_0)}$$

then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \dots \models \psi$

Induction step for $\psi = \psi_1 \,\mathsf{U}\, \psi_2$:

“$\Longleftarrow$”: Suppose $A_0\, A_1\, A_2 \dots \models \psi$. Let $j \geq 0$ s.t.

$$\begin{array}{lllllll}
A_j\, A_{j+1}\, A_{j+2} \dots & \models \psi_2 & \overset{\text{IH}}{\Rightarrow} & \psi_2 \in B_j & \Rightarrow & \psi \in B_j \\
A_{j-1}\, A_j\, A_{j-1} \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-1} & \wedge & \psi \in B_{j-1} \\
A_{j-2}\, A_{j-1}\, A_j \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_{j-2} & \wedge & \psi \in B_{j-2} \\
\vdots & & & \vdots & \vdots & \\
A_0\, A_1\, A_2\, A_3 \dots & \models \psi_1 & \Rightarrow & \psi_1 \in B_0 & \wedge & \psi \in B_0
\end{array}$$

Induction step: until (part "$\Longrightarrow$")

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Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} j \geq 0. \, B_j \in F$$

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$$\psi \in B_0 \ \wedge \ \psi_2 \notin B_0$$

Induction step: until (part “$\Longrightarrow$”)

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$$\forall F \in \mathcal{F}\ \overset{\infty}{\exists} j \geq 0.\, B_j \in F \qquad \boxed{B_{i+1} \in \delta(B_i, A_i)}$$

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$$\begin{aligned} & \psi \in B_0 \ \wedge\ \psi_2 \notin B_0 \\ \Rightarrow\ & \psi \in B_1 \end{aligned}$$

Induction step: until (part "$\Longrightarrow$")

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"$\Longrightarrow$" Suppose $\psi \in B_0$. There exists $j \geq 0$ with $\psi_2 \in B_j$, since otherwise $\forall j \geq 0.\ \psi_2 \notin B_j$ and therefore:

$$\begin{aligned} & \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow \ & \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow \ & \psi \in B_2 \end{aligned}$$

Induction step: until (part "$\Longrightarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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"$\Longrightarrow$" Suppose $\psi \in B_0$. There exists $j \geq 0$ with $\psi_2 \in B_j$, since otherwise $\forall j \geq 0.\, \psi_2 \notin B_j$ and therefore:

$$\begin{array}{rlll} & \psi \in B_0 & \wedge & \psi_2 \notin B_0 \\ \Rightarrow & \psi \in B_1 & \wedge & \psi_2 \notin B_1 \\ \Rightarrow & \psi \in B_2 & \wedge & \psi_2 \notin B_2 \\ & & \vdots & \end{array}$$

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Induction step for $\psi \ = \ \psi_1 \, \mathsf{U} \, \psi_2$:

"$\Longrightarrow$" Suppose $\psi \in B_0$. There exists $j \geq 0$ with $\psi_2 \in B_j$, since otherwise $\forall j \geq 0.\ \psi_2 \notin B_j$ and therefore:

$$\left.\begin{array}{rl} & \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow & \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow & \psi \in B_2 \ \wedge \ \psi_2 \notin B_2 \\ & \vdots \end{array}\right\} \quad \Longrightarrow \forall j \geq 0.\ B_j \notin F_\psi \text{ where}$$

$$F_\psi = \{B : \psi \notin B \text{ or } \psi_2 \in B\}$$

Induction step: until (part "$\Longrightarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step for $\psi \ = \ \psi_1 \, \mathsf{U} \, \psi_2$:

"$\Longrightarrow$" Suppose $\psi \in B_0$. There exists $j \geq 0$ with $\psi_2 \in B_j$, since otherwise $\forall j \geq 0.\ \psi_2 \notin B_j$ and therefore:

$$\left.\begin{array}{rl} & \psi \in B_0 \ \wedge \ \psi_2 \notin B_0 \\ \Rightarrow & \psi \in B_1 \ \wedge \ \psi_2 \notin B_1 \\ \Rightarrow & \psi \in B_2 \ \wedge \ \psi_2 \notin B_2 \\ & \vdots \end{array}\right\} \quad \Longrightarrow \forall j \geq 0.\ B_j \notin F_\psi \text{ where}$$

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Contradiction!

Induction step: until (part "$\Longrightarrow$") LTLMC3.2-65

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Induction step for $\psi = \psi_1 \mathsf{U} \psi_2$:

Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

Induction step: until (part "$\Longrightarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step for $\psi = \psi_1 \mathsf{U}\, \psi_2$:

Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \quad \models \psi_2$$

Induction step: until (part "$\Longrightarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step for $\psi = \psi_1 \, \mathsf{U} \, \psi_2$:

Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \quad \models \psi_2$$

$$\begin{array}{ll} \neg\psi_2 & \in B_{j-1} \\ \neg\psi_2 & \in B_{j-2} \\ \vdots & \\ \neg\psi_2 & \in B_1 \\ \neg\psi_2 & \in B_0 \end{array}$$

Induction step: until (part "$\Longrightarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step for $\psi = \psi_1 \, \mathbf{U} \, \psi_2$:

Let $\boxed{\psi \in B_0}$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

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$\neg\psi_2 \quad \in B_{j-1}$

$\neg\psi_2 \quad \in B_{j-2}$

$\vdots$

$\neg\psi_2 \quad \in B_1$

$\neg\psi_2, \quad \psi \in B_0$ $\longleftarrow$ by assumption

Induction step: until (part "$\Longrightarrow$")

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Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \quad \models \psi_2$$

$$\begin{array}{ll}
\neg\psi_2 & \in B_{j-1} \\
\neg\psi_2 & \in B_{j-2} \\
\quad \vdots & \\
\neg\psi_2 & \in B_1 \\
\neg\psi_2, \psi_1, \psi & \in B_0
\end{array}$$

$\leftarrow$ local consistency w.r.t. U

Induction step: until (part "$\Longrightarrow$")

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then for all $\psi \in cl(\varphi)$: $\psi \in B_0$ iff $A_0\, A_1\, A_2 \ldots \models \psi$

Induction step for $\psi = \psi_1 \mathbf{U}\, \psi_2$:

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$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \quad \models \psi_2$$

$$\begin{array}{ll} \neg\psi_2 & \in B_{j-1} \\ \neg\psi_2 & \in B_{j-2} \\ \quad\vdots & \\ \neg\psi_2, \psi_1, \psi & \in B_1 \\ \neg\psi_2, \psi_1, \psi & \in B_0 \end{array}$$

← local consistency w.r.t. $\mathbf{U}$

Induction step: until (part "$\Longrightarrow$")

Claim: If $B_0 \xrightarrow{A_0} B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} \ldots$ is a path in $\mathcal{G}$ s.t.

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Induction step for $\psi = \psi_1 \mathbf{U} \psi_2$:

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$$\overset{\text{IH}}{\Longrightarrow} \quad A_j\, A_{j+1} \ldots \quad \models \psi_2$$

$\neg\psi_2, \psi_1, \psi \in B_{j-1}$

$\neg\psi_2, \psi_1, \psi \in B_{j-2}$

$\vdots$

$\neg\psi_2, \psi_1, \psi \in B_1$

$\neg\psi_2, \psi_1, \psi \in B_0$ ← local consistency w.r.t. $\mathbf{U}$

Induction step: until (part "$\Longrightarrow$")

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Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

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 & \overset{\text{IH}}{\Longrightarrow} & A_j\, A_{j+1} \ldots & \models \psi_2 \\
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\neg\psi_2, \psi_1, \psi \in B_{j-2} & \Longrightarrow & A_{j-2}\, A_{j-1} \ldots & \models \psi_1 \\
\vdots & & & \\
\neg\psi_2, \psi_1, \psi \in B_1 & & & \\
\neg\psi_2, \psi_1, \psi \in B_0 & \leftarrow & \text{local consistency w.r.t. } \mathbf{U} &
\end{array}$$

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 & \overset{\text{IH}}{\Longrightarrow} & A_j\, A_{j+1} \ldots & \models \psi_2 \\
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\neg\psi_2, \psi_1, \psi \in B_{j-2} & \Longrightarrow & A_{j-2}\, A_{j-1} \ldots & \models \psi_1 \\
\vdots & \vdots & \vdots & \vdots \\
\neg\psi_2, \psi_1, \psi \in B_1 & \Longrightarrow & A_1\, A_2\, A_3 \ldots & \models \psi_1 \\
\neg\psi_2, \psi_1, \psi \in B_0 & & &
\end{array}$$

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\neg\psi_2, \psi_1, \psi \in B_{j-1} & \Longrightarrow & A_{j-1}\, A_j \ \ldots & \models \psi_1 \\
\neg\psi_2, \psi_1, \psi \in B_{j-2} & \Longrightarrow & A_{j-2}\, A_{j-1} \ldots & \models \psi_1 \\
\vdots & \vdots & \vdots & \vdots \\
\neg\psi_2, \psi_1, \psi \in B_1 & \Longrightarrow & A_1\, A_2\, A_3 \ldots & \models \psi_1 \\
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\end{array}$$

Induction step: until (part "$\Longrightarrow$")

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Let $\psi \in B_0$ and $j \geq 0$ minimal s.t. $\psi_2 \in B_j$

$$\begin{array}{rcll}
 & \overset{\mathrm{IH}}{\Longrightarrow} & A_j\, A_{j+1} \ldots & \models \psi_2 \\
\neg\psi_2, \psi_1, \psi \in B_{j-1} & \Longrightarrow & A_{j-1}\, A_j \ \ldots & \models \psi_1 \\
\vdots & \vdots & \vdots & \vdots \\
\neg\psi_2, \psi_1, \psi \in B_0 & \Longrightarrow & A_0\, A_1\, A_2 \ldots & \models \psi_1 \\
 & & \Downarrow &
\end{array}$$

Induction step: until (part “$\Longrightarrow$”)

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\vdots & \vdots & \vdots & \vdots \\
\neg\psi_2, \psi_1, \psi \in B_0 & \Longrightarrow & A_0\, A_1\, A_2 \ldots & \models \psi_1 \\
 & & \Downarrow & \\
 & & A_0\, A_1\, A_2 \ldots \models \psi = \psi_1 \,\mathbf{U}\, \psi_2 &
\end{array}$$

Complexity: LTL $\rightsquigarrow$ NBA

Complexity: LTL ⇝ NBA

For each **LTL** formula φ, there is an **NBA** $\mathcal{A}$ s.t.

$$\mathcal{L}_\omega(\mathcal{A}) \quad = \quad \mathit{Words}(\varphi)$$

Complexity: LTL $\rightsquigarrow$ NBA

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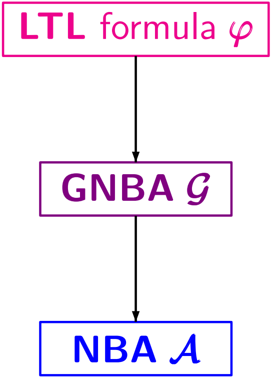

LTL formula φ

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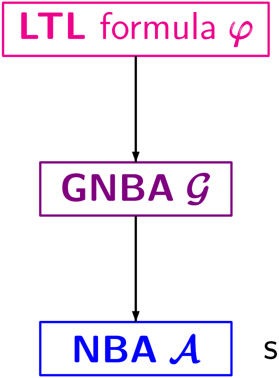

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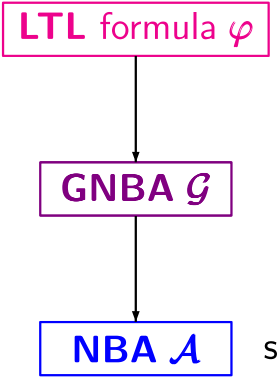

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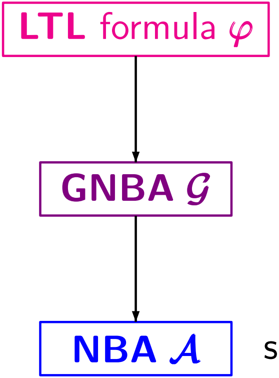

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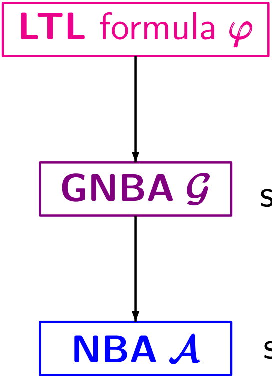

LTL formula φ

GNBA $\mathcal{G}$ size: $2^{|cl(\varphi)|}$

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Complexity: LTL $\rightsquigarrow$ NBA

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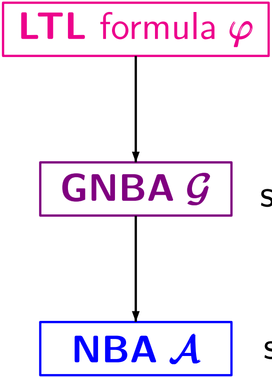
$$\begin{aligned} \mathcal{L}_\omega(\mathcal{A}) &= \mathit{Words}(\varphi) \text{ and} \\ \mathit{size}(\mathcal{A}) &\leq 2^{|cl(\varphi)|} \cdot |\varphi| \end{aligned}$$



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↓

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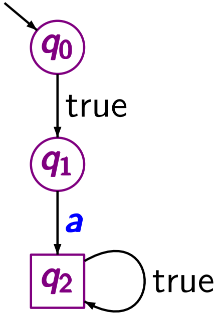
NBA for $\bigcirc a$



constructed GNBA has **4** states and **8** edges

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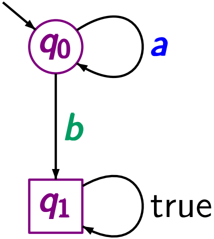
NBA for $a\,\mathsf{U}\,b$



constructed (G)NBA has
5 states and **20** edges

Size of NBA for LTL formulas

For the proposed transformation **LTL** $\rightsquigarrow$ **NBA**:

The constructed NBA for LTL formulas are often unnecessarily complicated

... but there exists LTL formulas φ_n such that

- $|\varphi_n| = \mathcal{O}(poly(n))$
- each NBA for φ_n has at least 2^n states

LT-properties that have no "small" NBA

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consider the following family of LT-properties $(E_n)_{n\geq 1}$:

$$E_n = \left\{ \begin{array}{l} \text{set of all infinite words over } 2^{AP} \text{ of the form} \\ \quad A_1\, A_2\, A_3 \ldots A_n\, A_1\, A_2\, A_3 \ldots A_n\, B_1\, B_2\, B_3\, B_4 \ldots \end{array} \right.$$

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LT-property E_n for n=1

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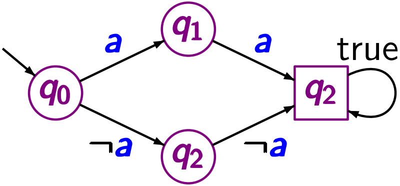

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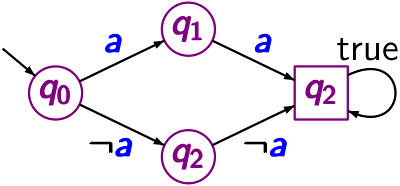

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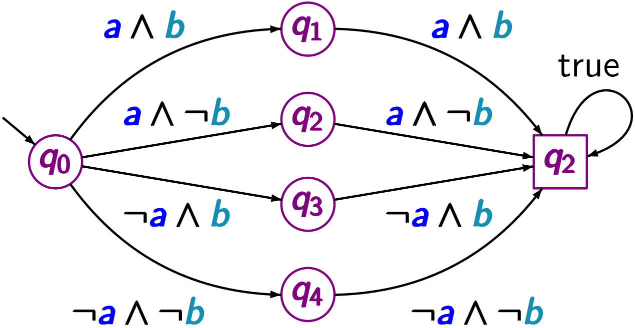

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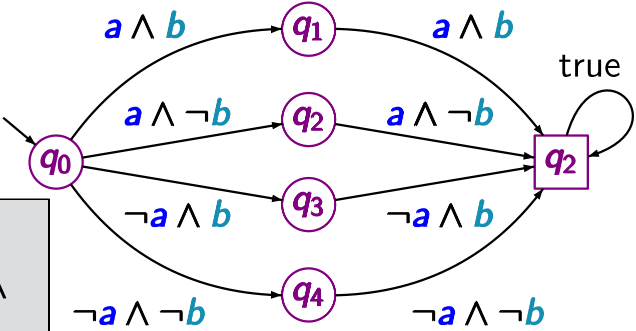

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$(b \leftrightarrow \bigcirc b)$

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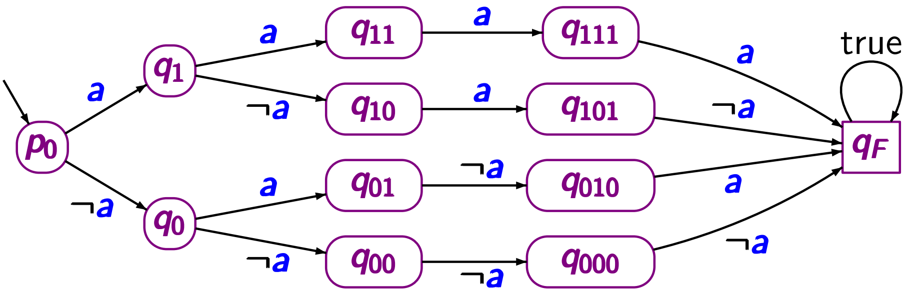

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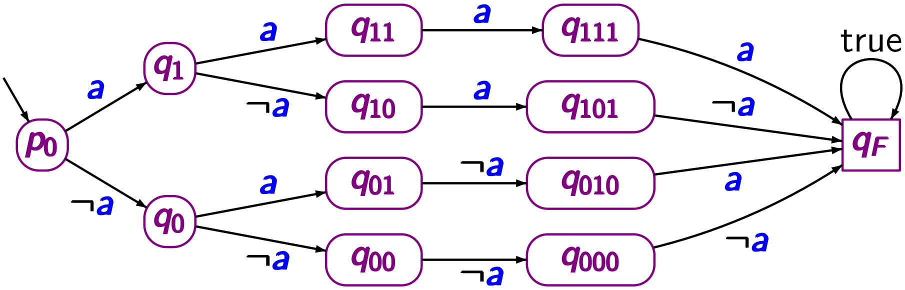

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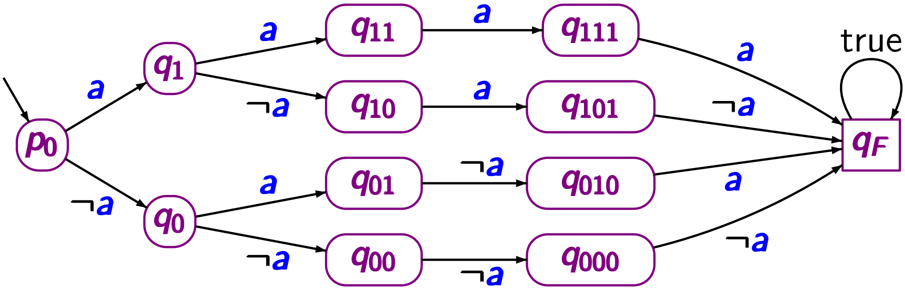

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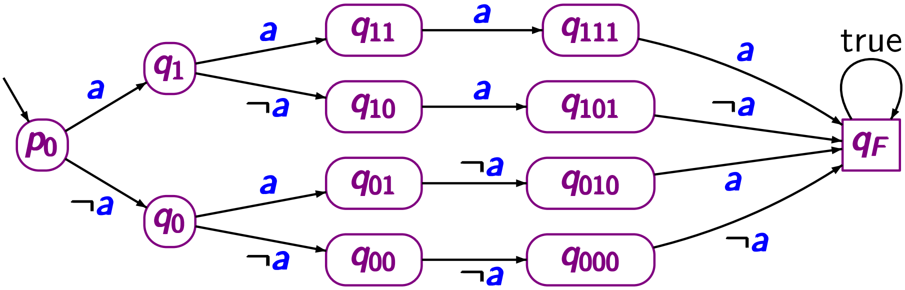

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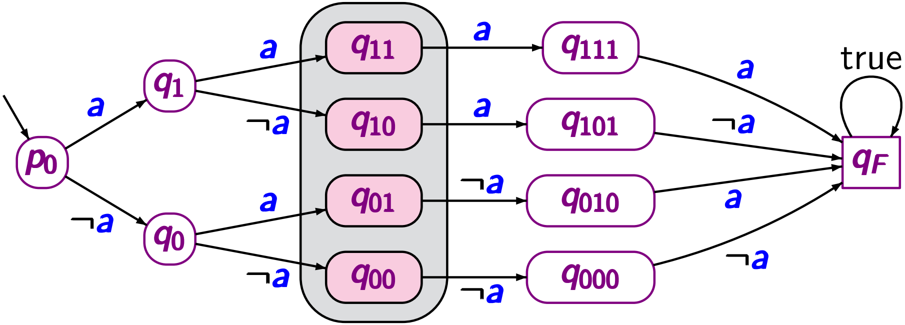

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