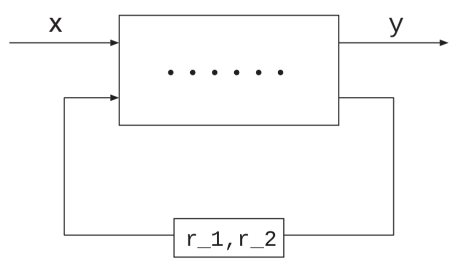
TD4 MVFA: LTL

Exercise 1

Consider the following sequential circuit:



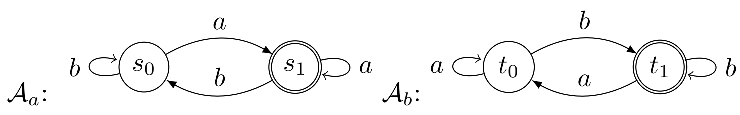
Let $AP = \{x, y, r_1, r_2\}$. Provide LTL formulae for the following properties:

1. "It is impossible that the circuit outputs two successive 1s."
2. "Whenever the input bit is 1, in at most two steps the output bit will be 1."
3. "Whenever the input bit is 1, the register bits do not change in the next step."
4. "Register r_1 has infinitely often the value 1."

Exercise 2

1. We consider the following construction: given two NBA $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0_1}, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0_2}, F_2)$, we let $\mathcal{G}_{\mathcal{A}_1 \sqcap \mathcal{A}_2} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ be the GNBA where:
   - $Q = Q_1 \times Q_2$
   - $Q_0 = Q_{0_1} \times Q_{0_2}$
   - $\mathcal{F} = \{F_1 \times Q_2, Q_1 \times F_2\}$
   - $\delta(\langle q_1, q_2 \rangle, a) = \{\langle p_1, p_2 \rangle \mid p_1 \in \delta(q_1, a), p_2 \in \delta(q_2, a)\}$

   Consider the following NBA $\mathcal{A}_a$ and $\mathcal{A}_b$ on the alphabet $\Sigma_{ab} = \{a, b\}$:



2. Define formally $\mathcal{G}_{\mathcal{A}_a \sqcap \mathcal{A}_b}$.
3. (*) Prove that for any NBA $\mathcal{A}_1$ and $\mathcal{A}_2$, $\mathcal{L}_\omega(\mathcal{G}_{\mathcal{A}_1 \sqcap \mathcal{A}_2}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$.

4. We now consider the following construction from GNBA to NBA: let $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ be an GNBA with $\mathcal{F} = \{F_0, \ldots, F_{k-1}\}$, we define the NBA $\mathcal{A}_{\mathcal{G}} = (Q', \Sigma, \delta', Q'_0, F')$ where:

   - $Q' = Q \times \{1, \ldots, n\}$
   - $Q'_0 = Q_0 \times \{1\}$
   - $F' = F_1 \times \{1\}$
   - $\delta'(\langle q, i\rangle, a) = \begin{cases} \{\langle q', i\rangle : q' \in \delta(q,a)\} \text{ if } q \notin F_i \\ \{\langle q', i+1 (\texttt{mod } k)\rangle : q' \in \delta(q,a)\} \text{ if } q \in F_i \\ \{\langle q', 0\rangle : q' \in \delta(q,a)\} \text{ if } q \in F_i, i = n \end{cases}$

   Define formally $\mathcal{A}_{\mathcal{G}_{\mathcal{A}_a \sqcap \mathcal{A}_b}}$ where $\mathcal{G}_{\mathcal{A}_a \sqcap \mathcal{A}_b}$ is the GNBA you constructed earlier.

5. (*) Prove that for any GNBA $\mathcal{G}$, $\mathcal{L}_\omega(\mathcal{A}_{\mathcal{G}}) = \mathcal{L}_\omega(\mathcal{G})$.

Exercise 3

Let $\varphi = (a \wedge \bigcirc a) \cup (a \wedge \neg \bigcirc a)$ be an LTL-formula over $AP = \{a\}$.

1. Compute all elementary sets with respect to φ.
2. Construct the GNBA $\mathcal{G}(\varphi)$ such that $\mathcal{L}_\omega(\mathcal{G}(\varphi)) = Words(\varphi)$.
3. Give an ω-regular expression E such that $\mathcal{L}_\omega(\mathcal{G}(\varphi)) = \mathcal{L}_\omega(E)$.

Exercise 4 (*)

Which of the following equivalences are correct? Prove the equivalence or provide a counterexample that illustrates that the formula on the left and the formula on the right are not equivalent.

1. $\Box\varphi \rightarrow \Diamond\psi \equiv \varphi \text{ U } (\psi \vee \neg\varphi)$
2. $\Diamond\Box\varphi \rightarrow \Box\Diamond\psi \equiv \Box(\varphi \text{ U } (\psi \vee \neg\varphi))$
3. $\Box\Box(\varphi \vee \neg\psi) \equiv \neg\Diamond(\neg\varphi \wedge \psi)$
4. $\Diamond(\varphi \wedge \psi) \equiv \Diamond\varphi \wedge \Diamond\psi$
5. $\Box\varphi \wedge \bigcirc\Diamond\varphi \equiv \Box\varphi$
6. $\Diamond\varphi \wedge \bigcirc\Box\varphi \equiv \Diamond\varphi$
7. $\Box\Diamond\varphi \rightarrow \Box\Diamond\psi \equiv \Box(\varphi \rightarrow \Diamond\psi)$
8. $\neg(\varphi_1 \text{ U } \varphi_2) \equiv \neg\varphi_2 \text{ W } (\neg\varphi_1 \wedge \neg\varphi_2)$
9. $\bigcirc\Diamond\varphi_1 \equiv \Diamond\bigcirc\varphi_2$
10. $(\Diamond\Box\varphi_1) \wedge (\Diamond\Box\varphi_2) \equiv \Diamond(\Box\varphi_1 \wedge \Box\varphi_2)$
11. $(\varphi_1 \text{ U } \varphi_2) \text{ U } \varphi_2 \equiv \varphi_1 \text{ U } \varphi_2$

Exercise 5

Let φ and ψ be LTL formulae. Consider the following new operators:

- "At next" $\varphi AX\psi$: $A_0A_1\ldots \models \varphi AX\psi \Leftrightarrow$ for all $i \geq 0$ where $A_iA_{i+1}\ldots \models \psi$, for which there exists no $0 \leq j < i$ where $A_jA_{j+1}\ldots \models \psi$, $A_iA_{i+1}\ldots \models \varphi$ holds.
- "While" $\varphi WH\psi$: $A_0A_1\ldots \models \varphi WH\psi \Leftrightarrow$ for all $i \geq 0$ where $A_jA_{j+1}\ldots \models \psi$ for all $0 \leq j < i$, $A_kA_{k+1}\ldots \models \varphi$ for all $0 \leq k < i$.
- "Before" $\varphi B\psi$: $A_0A_1\ldots \models \varphi B\psi \Leftrightarrow$ for all $i \geq 0$ where $A_iA_{i+1}\ldots \models \psi$, there exists some $0 \leq j < i$ where $A_jA_{j+1}\ldots \models \varphi$.

Show that these operators are LTL-definable by providing equivalent LTL formulae. You may use both the until and weak until operator.