Checking Presence Reachability Properties on Parameterized Shared-Memory Systems
Parameterized Shared-memory Systems

Arbitrary number of identical, anonymous agents communicating using a shared memory.
A model for shared-memory systems

Each register has a value from the finite set of symbols $\Sigma$

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<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>a</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>b</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$d_0$</td>
<td></td>
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</tbody>
</table>

Registers are *initialized* to value $d_0$

No atomic read/write combinations

Round-based algorithms

We want to model round-based distributed algorithms that look like this:

\[
\text{for } k = 0 \text{ to } \infty \text{ do} \\
\quad \text{read register 0 of round } k \\
\quad \text{if read value is } a \text{ then ... else ...} \\
\quad \text{write to register 1 of round } k \\
\quad \ldots \\
\quad \text{read register 0 of round } k - 1 \\
\text{end}
\]

Asynchronous rounds

Each round has its own set of shared registers \( \rightarrow \) unbounded memory!

read and write to registers of nearby rounds only

An example

\[
\begin{align*}
q_0 & \xrightarrow{\text{write}(a)} q_0 \\
q_0 & \xrightarrow{\text{read}(d_0)} q_f \\
q_0 & \xrightarrow{\text{Increment round}} q_0 \\
q_0 & \xrightarrow{\text{read}(d_0)} q_f \\
q_0 & \xrightarrow{\text{Increment round}} q_0 \\
q_f & \xrightarrow{\text{read}^{-1}(b)} q_f
\end{align*}
\]
An example

Write to register of current round of the process

Increment round

$q_0$

$write(a)$

Increase round

$read(d_0)$

Read from register one round below

Send process to next round

Increment round

$read(a)$

Read from register of current round

$q_f$

$read^{-1}(b)$

$read(d_0)$

Increment round

Write to register of current round of the process

Increment round

Read from register one round below

Send process to next round

Increment round

Read from register of current round
An example

Two processes, both on round 0

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>2</td>
<td>$d_0$</td>
</tr>
<tr>
<td>1</td>
<td>$d_0$</td>
</tr>
<tr>
<td>0</td>
<td>$d_0$</td>
</tr>
</tbody>
</table>
An example

\[ \begin{array}{c}
\text{write}(a) \\
\text{read}(d_0) \\
\text{read}(a) \\
\text{read}^{-1}(b) \\
\text{write}(b) \\
\text{Increment round} \\
\text{Increment round} \\
q_0 \\
0 \\
0 \\
0 \\
q_f
\end{array} \]

\[ \begin{array}{c}
\vdots \\
2 \quad d_0 \\
1 \quad d_0 \\
0 \quad d_0
\end{array} \]
An example

\[ \begin{align*}
    q_0 & \xrightarrow{\text{Increment round}} q_0 \\
    q_0 & \xrightarrow{\text{read} (d_0)} 1 \\
    1 & \xrightarrow{\text{Increment round}} 1 \\
    1 & \xrightarrow{\text{read} (d_0)} q_f \\
    q_f & \xrightarrow{\text{read}^{-1}(b)} q_f \\
    q_f & \xrightarrow{\text{read} (a)} q_f
\end{align*} \]
An example

\[
\begin{array}{c}
\text{write}(a) \\
q_0 \\
\text{read}(d_0) \\
1 \\
\text{read}(a) \\
\text{read}^{-1}(b) \\
q_f \\
\text{read}(d_0) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Increment round} \\
\text{Increment round} \\
\text{Increment round} \\
\text{Increment round} \\
\text{Increment round} \\
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
2 \\
1 \\
0 \\
\end{array}
\]

\[
\begin{array}{c}
d_0 \\
d_0 \\
d_0 \\
\end{array}
\]
An example

Increment round

write(a)

$q_0$

Increment round

write(b)

Increment round

read ($d_0$)

read ($d_0$)

Increment round

read (a)

read$^{-1}$(b)

$q_f$

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<tr>
<td>2</td>
<td>$d_0$</td>
</tr>
<tr>
<td>1</td>
<td>$d_0$</td>
</tr>
<tr>
<td>0</td>
<td>$d_0$</td>
</tr>
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</table>
An example

\[
\begin{align*}
\text{write}(a) & \quad \rightarrow \quad \text{Increment round} \\
q_0 & \quad \rightarrow \quad \text{write}(b) \\
\text{Increment round} & \quad \rightarrow \quad \text{read}(d_0) \\
\text{Increment round} & \quad \rightarrow \quad \text{read}^{-1}(b) \\
2 & \quad \rightarrow \quad \text{read}(a) \\
\text{Increment round} & \quad \rightarrow \quad \text{read}(d_0) \\
0 & \quad \rightarrow \quad q_f
\end{align*}
\]

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<tr>
<td>2</td>
<td>(d_0)</td>
</tr>
<tr>
<td>1</td>
<td>(d_0)</td>
</tr>
<tr>
<td>0</td>
<td>(d_0)</td>
</tr>
</tbody>
</table>
An example

\[
\begin{align*}
\text{write}(a) & \quad \text{Increment round} \\
\text{write}(b) & \\
\text{read} \ (d_0) & \quad \text{Increment round} \\
\text{read} \ (a) & \quad \text{read}^{-1}(b) \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>d_0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>a</td>
<td>0</td>
<td>1</td>
<td>d_0</td>
</tr>
<tr>
<td>1</td>
<td>d_0</td>
<td>d_0</td>
<td>d_0</td>
<td>d_0</td>
</tr>
<tr>
<td>2</td>
<td>d_0</td>
<td>d_0</td>
<td>d_0</td>
<td>d_0</td>
</tr>
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</table>
An example

\[
\begin{array}{c}
\text{write}(a) \\
\text{Increment round} \\
q_0 \\
\text{read} (d_0) \\
\text{Increment round} \\
1 \\
\text{write}(b) \\
\text{Increment round} \\
2 \\
\text{read} (d_0) \\
\text{read} (a) \\
\text{read}^{-1}(b) \\
q_f
\end{array}
\]
An example

Increment round
write(a)

Increment round
read(d₀)

Increment round
read(a)

Increment round
read⁻¹(b)

write(b)

$q₀$  
$q₁$  
$q₂$  
$q_f$

<table>
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<th>d₀</th>
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<tr>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>b</td>
</tr>
<tr>
<td>2</td>
<td>d₀</td>
</tr>
</tbody>
</table>
An example

\[
\begin{align*}
\text{Increment round} & \rightarrow \text{write(a)} \\
q_0 \quad 2 \quad \text{read} (d_0) & \rightarrow \text{Increment round} \\
& \rightarrow \text{read} (a) \\
2 \quad \text{read} (d_0) & \rightarrow \text{Increment round} \\
& \rightarrow \text{read}^{-1}(b) \\
\end{align*}
\]

\[
\begin{array}{c}
\vdots \\
2 \\
1 \\
0 \\
a
\end{array}
\]
An example

Increment round

write(a)

Increment round

read $d_0$

Increment round

read (a)

read$^{-1}$(b)

$q_f$
An example

\[ \begin{align*}
\text{write}(a) & \quad \text{Increment round} & \quad \text{write}(b) \\
q_0 & \quad \text{Increment round} & \\
\text{read} (d_0) & \\
\text{read} (d_0) & \quad \text{Increment round} & \quad \text{read} (a) & \quad \text{read}^{-1}(b)
\end{align*} \]

\[ q_0 \rightarrow 2 \rightarrow q_f \]

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<td>b</td>
</tr>
<tr>
<td>0</td>
<td>a</td>
</tr>
</tbody>
</table>
An example

Increment round

write(a)

$q_0$

Increment round

read $(d_0)$

Increment round

read $(a)$

read$^{-1}(b)$

$q_f$

2

write(b)

$q_0$

$q_f$

...
Reachability problems in round-based setting

COVER: \[ \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \exists k \gamma(q_f, k) > 0 \]

Parameterized by \( n \): arbitrarily large number of agents
Reachability problems in round-based setting

\[
\exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \exists k \gamma(q_f, k) > 0?
\]
Reachability problems in round-based setting

COVER: \( \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \exists k \gamma(q_f, k) > 0 \)
Reachability problems in round-based setting

COVER: \( \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \exists k \gamma(q_f, k) > 0 \)?

- Increment round
- Write (a)
- q_0
- Increment round
- Write (b)
- a is written to even rounds only

- Read (d_0)
- Increment round
- Read (a)
- q_f
- Increment round
- Read^{-1}(a)
- a must be read two rounds in a row
Reachability problems in round-based setting

**COVER:** \[ \exists n, \exists \rho: \gamma_0 \to^* \gamma, \exists k \gamma(q_f, k) > 0 \ ? \]

**TARGET:** \[ \exists n, \exists \rho: \gamma_0 \to^* \gamma, \forall k, \forall q \neq q_f, \gamma(q, k) = 0 \ ? \]
Reachability problems in round-based setting

COVER: \( \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \exists k \gamma(q_f, k) > 0 \) ?

TARGET: \( \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \forall k, \forall q \neq q_f, \gamma(q, k) = 0 \) ?

The last process will not be able to take this.
Reachability problems in round-based setting

COVER: \[ \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \exists k \, \gamma(q_f, k) > 0 \]?

TARGET: \[ \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \forall k, \forall q \neq q_f, \gamma(q, k) = 0 \]?

Presence Reachability Problem: \[ \exists n, \exists \rho: \gamma_0 \rightarrow^* \gamma, \gamma \models \psi \]?

with \( \psi \) a first-order formula on rounds with no nested quantifiers

Example: \( \psi = \exists k \, (\#(q_1, k + 1) > 0 \land \#(q_1, k) = 0) \lor \forall k \, \#(q_0, k) = 0'' \)

For some \( k \), \((q_1, k)\) is not populated but \((q_1, k + 1)\) is

no process is on \( q_0 \)
**Complexity results**

*Theorem* 5: COVER is PSPACE-hard.

*Theorem* 6: The Presence Reachability Problem is PSPACE-complete.


6. W: *Checking Presence Reachability Properties on Parameterized Shared-Memory Systems*, MFCS’22
Thanks for your attention!
Any questions?
A process is described by its state + round

\[ p \times q \]

\[ ... ... ... ... \]

\[ a \quad d_0 \quad d_0 \quad d_0 \]

\[ b \quad c \quad a \quad b \]

\[ a \quad a \quad c \quad b \]

\[ a \quad a \quad b \quad d_0 \]

Initial value

\[ \text{reg}_1[k] \quad \text{reg}_2[k] \quad \text{reg}_3[k] \quad \text{reg}_4[k] \]