On the Size of Refutation Kripke Models for Some Linear Modal and Tense Logics

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Abstract. Let $\boldsymbol{L}$ be any modal or tense logic with the finite model property. For each m, define $r_{\boldsymbol{L}}(m)$ to be the smallest number r such that for any formula A with m modal operators, A is provable in $\boldsymbol{L}$ if and only if A is valid in every $\boldsymbol{L}$-model with at most r worlds. Thus, the function $r_{\boldsymbol{L}}$ determines the size of refutation Kripke models for $\boldsymbol{L}$. In this paper, we will give an estimation of $r_{\boldsymbol{L}}(m)$ for some linear modal and tense logics $\boldsymbol{L}$.

1. Introduction

We will investigate the size of a Kripke model, in which a given unprovable formula is refutable. For example, as Rasiowa and Sikorski showed in [5], if A is a formula not provable in the modal logic ***S4***, then we need to check valuations into a Boolean algebra of size 2^{2^r}, where r is the number of subformulas of A. If we will use the filtration method (e.g, [6]) to find a refutation Kripke model for a given unprovable formula in ***S4***, we will need also Kripke models in exponential size. On the other hand, it is shown that if a formula with m modal operators is not provable in the modal logic ***S5***, then A is refutable in an ***S5***-model with at most $m+1$ worlds (see [2]). Moreover, this number $m+1$ is shown to be best.

In general, we can formulate this problem in the following way. Let $\boldsymbol{L}$ be any modal or tense logic with the finite model property. For each integer $m \geqq 1$, define $r_{\boldsymbol{L}}(m)$ to be the smallest number r (if there exists) which satisfies the following condition:

For any formula A with m modal (*or tense*) *operators, A is provable in* $\boldsymbol{L}$ *if and only if A is valid in every* $\boldsymbol{L}$*-model with at most r worlds.*

The above result says that $r_{\boldsymbol{S5}}(m) = m+1$. In [4], we proved that $r_{\boldsymbol{S4.3}}(m) \leqq (m+1)^2$. Using this fact, we have answered affirmatively Ladner's conjecture [2] on the computational complexity of the satisfiability problem of the modal logic ***S4.3***.

In this paper, we will give an estimation of $r_{\boldsymbol{L}}(m)$ for some linear modal and tense logics $\boldsymbol{L}$ by elaborating our method in [4]. More precisely, $r_{\boldsymbol{L}}(m)$ is shown to be bounded by some polynomial $p(m)$ of degree one in each case. Thus, our method can be considered as a refinement of the filtration method. As an application, we will show in section 4 that the satisfiability problems of these tense logics are also log space complete in nondeterministic polynomial time.

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In the following, we will mainly devote to linear tense logics. Linear modal logics can be treated quite similarly (and more easily). We will take $\wedge$, $\neg$, G and H for primitive logical connectives. Other connectives are defined in the usual way. In particular, unary operators F and P are defined as

$$FA \equiv \neg G \neg A \quad \text{and} \quad PA \equiv \neg H \neg A.$$

Axioms and rules of the tense logic $\boldsymbol{K_t}$ are as follows.

A1. Axioms for the classical propositional logic
A2. $G(A \supset B) \supset (GA \supset GB)$
A3. $H(A \supset B) \supset (HA \supset HB)$
A4. $A \supset HFA$
A5. $A \supset GPA$

R1. Modus ponens, i.e, from A and $A \supset B$ infer B
R2. from A infer GA
R3. from A infer HA

The tense logic $\boldsymbol{CL}$ (by N.B. Cocchiarella) is obtained from $\boldsymbol{K_t}$ by adding following three axiom schemata.

A6. $FFA \supset FA$
A7. $(FA \wedge FB) \supset \big(F(A \wedge B) \vee F(A \wedge FB) \vee F(FA \wedge B)\big)$
A8. $(PA \wedge PB) \supset \big(P(A \wedge B) \vee P(A \wedge PB) \vee P(PA \wedge B)\big)$

Remark that $PPA \supset PA$ is provable in $\boldsymbol{CL}$ (see [3]). The tense logic $\boldsymbol{SL}$ (by D. Scott) is obtained from $\boldsymbol{CL}$ by adding the following two axiom schemata.

A9. $GA \supset FA$
A10. $HA \supset PA$

The tense logic $\boldsymbol{PL}$ (by A.N. Prior) is obtained from $\boldsymbol{SL}$ by adding

A11. $FA \supset FFA$.

We remark also that $PA \supset PPA$ is provable in $\boldsymbol{PL}$ (see [3]).

A Kripke frame (M, R) is a pair of nonempty set M and a binary relation R on M. A valuation W on a Kripke frame (M, R) is a mapping from $M \times \Phi_0$ to $\{t, f\}$, where Φ_0 is the set of all propositional variables. Each valuation W can be uniquely extended to a mapping from $M \times \Phi$ to $\{t, f\}$ in the following way, where Φ is the set of all formulas;

for any $A, B \in \Phi$ and any $a \in M$,
$W(A \wedge B, a) = t$ iff $W(A, a) = t$ and $W(B, a) = t$,
$W(\neg A, a) = t$ iff $W(A, a) = f$,

$W(GA, a) = t$ iff for any b such that aRb, $W(A, b) = t$,
$W(HA, a) = t$ iff for any b such that bRa, $W(A, b) = t$.

A Kripke model is an ordered triple (M, R, W), where (M, R) is a Kripke frame and W is a valuation on it. A formula A is *refutable* (or *satisfiable*) in a Kripke model (M, R, W) if $W(A, a) = f$ (or t, respectively) for some $a \in M$. A Kripke model is a *refutation model* of a formula A if A is refutable in it. A formula A is *valid* in a Kripke frame (M, R) if for any valuation W and any $a \in M$, $W(A, a) = t$. The following theorem is well-known.

THEOREM 1. *For any formula A, A is provable in $\boldsymbol{K_t}$ if and only if A is valid in any Kripke frame.*

Let R be a binary relation on M. Then,

1) R is *transitive* if for any x, y, z, xRy and yRz imply xRz,

2) R is *linear* if for any x, y, xRy or $x = y$ or yRx,

3) R is *non-ending* if for any x, there exist y and z such that yRx and xRz,

4) R is *dense* if for any x, y, there exists z such that xRy implies both xRz and zRy.

Then, a Kripke frame (M, R) is;

1) a ***CL**-frame* if R is transitive and linear,

2) an ***SL**-frame* if it is a **CL**-frame and R is non-ending,

3) a ***PL**-frame* if it is an **SL**-frame and R is dense.

A Kripke model (M, R, W) is an **L**-model if (M, R) is an **L**-frame. We can get the completeness theorem of these linear tense logics. (See e.g. [3]. It is easily verified that the condition *left-* and *right-linearity* on **CL**-frames in [3] can be replaced by the condition *linearity*.)

THEOREM 2. *Let **L** be any one of **CL**, **SL** and **PL**. Then for any formula A, A is provable in **L** if and only if A is valid in any **L**-frame.*

2. Finite model property of linear tense logics

We will show the finite model property of **CL**, **SL** and **PL** in this section. Though the proof can be obtained in the standard way, it seems that the result has not been published anywhere. So we will give the outline of the proof. For the sake of brevity, we assume in the rest of this paper that unary operators F and H are primitives, instead of G and H. Note that

$W(FA, a) = t$ iff for some b such that aRb, $W(A, b) = t$.

THEOREM 3. *Let **L** be any one of **CL**, **SL** and **PL**. Then for any formula A, A is provable in **L** if and only if A is valid in any finite **L**-frame.*

PROOF. Only-if part follows immediately from Theorem 2. So, it suffices to show that if A is refutable in an **L**-model then A is refutable in a finite $\boldsymbol{L}$-model. We will use a filtration method, called *Lemmon filtration* in [6]. Suppose that $W(A, a_0) = f$ for an $\boldsymbol{L}$-model (M, R, W) and $a_0 \in M$. Define a binary relation $\sim$ on M by

$a \sim b$ iff $W(B, a) = W(B, b)$ for any $B \in \Phi_A$,

where Φ_A is the set of all subformulas of A. Clearly, $\sim$ is an equivalence relation on M. Let $M^* = M/\sim$. Since Φ_A is finite, M^* is also finite. The equivalence class containing an element x in M is denoted by $[x]$. Next, define a binary relation R^* on M^* by

$[a]R^*[b]$ iff 1) for any $FB \in \Phi_A$, $W(FB, a) = f$ implies both $W(FB, b) = f$ and $W(B, b) = f$, and
2) for any $HB \in \Phi_A$, $W(HB, b) = t$ implies both $W(HB, a) = t$ and $W(B, a) = t$.

It is easy to see that R^* is well-defined. We can confirm the following four propositions.

(1) aRb implies $[a]R^*[b]$,
(2) R^* is transitive and linear,
(3) R^* is non-ending if R is non-ending,
(4) R^* is dense if R is dense.

Propositions from (1) to (3) can be easily verified. We will show only the proof of (4). Suppose that $[a]R^*[b]$. Since R is linear, either $a = b$ or aRb or bRa holds. If $a = b$, then $[a]R^*[a]$ holds by the assumption. Thus, $[a]R^*[a]$ and $[a]R^*[b]$ hold. If aRb then aRc and cRb hold for some c, since R is dense. Hence $[a]R^*[c]$ and $[c]R^*[b]$ hold by (1). If bRa then $[b]R^*[a]$ holds. By (2), $[a]R^*[a]$ holds. So, $[a]R^*[a]$ and $[a]R^*[b]$ hold. Therefore for each case, there exists some $[x] \in M^*$ such that $[a]R^*[x]$ and $[x]R^*[b]$. Hence R^* is dense. Thus, we have a finite $\boldsymbol{L}$-frame (M^*, R^*) for any given $\boldsymbol{L}$-frame (M, R), where $\boldsymbol{L}$ is any one of $\boldsymbol{CL}$, $\boldsymbol{SL}$ and $\boldsymbol{PL}$. It remains to show that A is not valid in (M^*, R^*). We define a valuation W^* on (M^*, R^*) by

$$W^*(p, [a]) = W(p, a),$$

for any propositional variable p in Φ_A and any $a \in M$. Clearly, W^* is well-defined. Moreover, we can show by induction that for any $B \in \Phi_A$ and $a \in M$,

$$W^*(B, [a]) = W(B, a).$$

Taking A for B and a_0 for a, we have $W^*(A, [a_0]) = f$. Thus, A is refutable in a finite $\boldsymbol{L}$-model (M^*, R^*, W^*).

3. The size of refutation models

In this section, we will give some upper bounds of $r_{\boldsymbol{L}}(m)$, which is introduced in section 1, for each tense logic $\boldsymbol{CL}$, $\boldsymbol{SL}$ or $\boldsymbol{PL}$. $r_{\boldsymbol{L}}(m)$ can be defined also in the following manner. Let $\boldsymbol{L}$ be any one of tense logics

$\boldsymbol{CL}$, $\boldsymbol{SL}$ and $\boldsymbol{PL}$ and A be any formula with m ($\geqq 1$) tense operators not belonging to $\boldsymbol{L}$. By Theorem 3, there exists a finite refutation $\boldsymbol{L}$-model of A. Define $s_{\boldsymbol{L}}(A)$ to be the smallest number of worlds of refutation $\boldsymbol{L}$-models of A. Next, we define $r_{\boldsymbol{L}}(m)$ by $r_{\boldsymbol{L}}(m) = \sup\{s_{\boldsymbol{L}}(A);\ A$ contains m tense operators$\}$. Of course, $r_{\boldsymbol{L}}(m)$ can be defined similarly for modal logics.

LEMMA 4. *Let* $\boldsymbol{L}$ *be any one of tense logics* $\boldsymbol{CL}$, $\boldsymbol{SL}$ *and* $\boldsymbol{PL}$. *Then,* $r_{\boldsymbol{L}}(m) \geqq m+1$.

PROOF. Take the following formula A_m.

$$A_m:\quad \neg\Big(p_1\wedge p_2\wedge\ldots\wedge p_m\wedge F\Big(\neg p_1\wedge p_2\wedge\ldots\wedge p_m\wedge F\Big(\neg p_1\wedge\neg p_2\wedge\ldots\wedge p_m\wedge F\big(\ldots F(\neg p_1\wedge\neg p_2\wedge\ldots\wedge\neg p_m)\ldots\big)\Big)\Big)\Big).$$

Clearly, A_m contains m tense operators and $s_{\boldsymbol{L}}(A_m) = m+1$. Thus, $r_{\boldsymbol{L}}(m) \geqq m+1$.

We will make some preparations. Let (M, R) be any finite $\boldsymbol{CL}$-frame. We define two binary relations $\simeq$ and $<$ on M as follows.

$$a \simeq b \quad \text{iff} \quad a = b \text{ or both } aRb \text{ and } bRa,$$
$$a < b \quad \text{iff} \quad aRb \text{ and not } bRa.$$

It is easy to see that 1) $\simeq$ is an equivalence relation on M and 2) $<$ is a transitive relation such that $a < b$ implies $a \neq b$ for any a and b. The equivalence class determined by $\simeq$, which contains $x \in M$ is denoted by $\|x\|$. We call these equivalence classes *clusters*.

Next, define a binary relation $<^*$ on $M/\simeq$ by

$$\|a\| <^* \|b\| \text{ iff } a < b.$$

We can show that $<^*$ is well-defined. Moreover, $<^*$ is a strictly linear ordering on $M/\simeq$, i.e,

1) $<^*$ is transitive,

2) for any $\|a\|, \|b\| \in M/\simeq$, one and only one of relations $\|a\| = \|b\|$, $\|a\| <^* \|b\|$, $\|b\| <^* \|a\|$ holds.

We write $\|a\| \leqq^* \|b\|$ if either $\|a\| = \|b\|$ or $\|a\| <^* \|b\|$ holds. The following remarks will be useful in the succeeding discussions. We say that an element $x \in M$ is reflexive if xRx holds.

REMARK 1. An element x is reflexive if $\|x\|$ contains at least two elements. For, if $x \neq y$ and $x \simeq y$ then xRy and yRx by the definition. Thus, xRx holds by the transitivity of R. From this it follows that if xRy and if either x or y is contained in some cluster consisting of at least two elements, then for some z, both xRz and zRy hold.

REMARK 2. If R is non-ending and if x is either in the minimum or the maximum cluster, then x is reflexive. (Since M is finite and $<^*$ is a strictly linear ordering, there exist the minimum and the maximum clusters among $M/\simeq$.) For, suppose that x is in the maximum cluster. Since R is non-ending, xRy for some $y \in M$. If not yRx then $\|x\| <^* \|y\|$. But this contradicts the assumption. So yRx. Hence xRx holds. It can be verified similarly for the case of the minimum cluster.

REMARK 3. If R is dense and $\|x\| <^* \|y\|$ holds for non-reflexive elements x, y, then there exists a reflexive element z such that $\|x\| <^* \|z\| <^* \|y\|$. For, from the assumption it follows that xRy. By the denseness of R, there exists w such that xRw and wRy. Let $S = \{w \in M;\ xRw$ and $wRy\}$. Then, take an element $z \in S$ such that for any $w \in S \|z\| \leqq^* \|w\|$. Since both x any y are non-reflexive, $\|x\| <^* \|z\| <^* \|y\|$. Furthermore, there exists u such that xRu and uRz, since xRz holds. Of course, $\|u\| \geqq^* \|z\|$. But $\|z\| \geqq^* \|u\|$ holds too, since $u \in S$. Thus $\|z\| = \|u\|$. If $z \neq u$ then z is reflexive by Remark 1. If $z = u$ then uRz implies that z is reflexive.

Now, let A be a formula containing m ($\geqq 1$) tense operators such that $W(A, a_0) = f$ for some $a_0 \in M$ in a finite CL-model (M, R, W). We will show that A is also refutable in some CL-model (M', R', W') with at most $m+1$ worlds. Define the set F_A (or H_A) to be the set of all subformulas of A which are of the form FB (or HB, respectively). We enumerate elements of F_A and H_A as $FC_1, \ldots, FC_s$ and $HD_1, \ldots, HD_t$ (s, $t \geqq 0$). Of course, $s+t = m$. Suppose that FC_i is satisfiable in (M, R, W). Then, take such an element $u_i \in M$ that $W(C_i, u_i) = t$ and $W(C_i, w) = f$ for any w such that $\|u_i\| <^* \|w\|$. When FC_i is not satisfiable in (M, R, W), let $u_i = a_0$. Similarly, if HD_j is refutable in (M, R, W), then take such an element $v_j \in M$ that $W(D_j, v_j) = f$ and $W(D_j, w) = t$ for any w such that $\|w\| <^* \|v_j\|$. When HD_j is not refutable in (M, R, W), let $v_j = a_0$. Now, define a Kripke model (M', R', W') as follows.

1) $M' = \{a_0, u_1, \ldots, u_s, v_1, \ldots, v_t\}$,

2) R' and W' are restrictions of R and W to M', respectively.

Clearly, (M', R', W') is a CL-model and M' contains at most $m+1$ elements.

We can show by induction that for any subformula B of A,

(1) $\quad W'(B, x) = W(B, x)$

for every $x \in M'$. Here, we will give only a proof for the case where B is FC_i. If FC_i is not satisfiable in (M, R, W), then $W(C_i, w) = f$ for every $w \in M$ and hence $W'(C_i, w) = f$ for every $w \in M'$. Thus, $W'(FC_i, x) = f = W(FC_i, x)$ for every $x \in M'$. Next, suppose that FC_i is satisfiable in (M, R, W). If $\|u_i\| <^* \|x\|$ and xRw for $x, w \in M'$, then $\|u_i\| <^* \|w\|$. Thus, $W'(C_i, w) = W(C_i, w) = f$. Hence, $W'(FC_i, x) = f = W(FC_i, x)$. Next, suppose that xRu_i and $x \in M'$. Then $xR'u_i$ holds also, so $W'(FC_i, x) = t = W(FC_i, x)$. Finally, suppose that $x = u_i$ and u_i is not reflexive.

Then $u_i R w$ implies $\|u_i\| <^* \|w\|$. Therefore $W'(FC_i, u_i) = f = W(FC_i, u_i)$ can be shown similarly as the first case. Now, taking A for B and a_0 for x in (1), we have

$$W'(A, a_0) = W(A, a_0) = f.$$

Combining this fact with Lemma 4, we have the following theorem.

THEOREM 5. $r_{CL}(m) = m+1$.

Let ***S4.3*** be the modal logic obtained from ***S4*** by adding the axiom schema

$$\Box(\Box A \supset \Box B) \vee \Box(\Box B \supset \Box A).$$

S4.3Dum (or ***S4.3Grz***) is the modal logic obtained from ***S4.3*** by adding the axiom schema

$$\Box(\Box(A \supset \Box A) \supset A) \supset (\Diamond \Box A \supset A)$$

$$\big(\text{or } \Box\big(\Box(A \supset \Box A) \supset A\big) \supset A, \text{ respectively}\big).$$

See [6]. Quite similarly as the above, we have the following theorem, which is a strengthened version of theorems in [4].

THEOREM 6. *If* $\boldsymbol{L}$ *is any one of* ***S4.3***, ***S4.3Dum*** *and* ***S4.3 Grz***, *then* $r_L(m) = m+1$.

But in cases of ***SL*** and ***PL***, it is not so easy to estimate $r_L(m)$. For the Kripke model (M', R', W') may not be an ***SL***- (or a ***PL***-) model in general, even if the original model (M, R, W) is an ***SL***- (or a ***PL***-) model.

Suppose that $\|u\|'$ and $\|v\|'$ are the minimum and the maximum clusters among clusters in M', respectively. Moreover, assume that neither u nor v are reflexive. Then, R' is not non-ending by Remark 2. In such a case, we must take two elements u_0 and v_0 from the minimum and the maximum clusters in M, respectively and add them to M'. Of course, they are reflexive. So, the model thus obtained becomes an ***SL***-model with at most $m+3$ worlds. Similarly as the proof of Theorem 5, we can show that A is refutable in this model.

Next, we will consider the case for ***PL***-models. We suppose that no elements in M' are reflexive. Then, we must first add u_0 and v_0 to M' just as the above, to make the model non-ending. We remark here that R' is a strictly linear ordering on M', by our assumption. For each pair (z_1, z_2) such that z_2 is the immediate successor of z_1 with respect to R', take a reflexive element $w \in M$ such that $\|z_1\| <^* \|w\| <^* \|z_2\|$. The existence of such a element w is confirmed by Remark 3. We add also these reflexive elements to M'. Then, we can show that the model thus obtained becomes a ***PL***-model with at most $2m+3$ $(= (m+1)+2+m)$ worlds, in which A is refutable.

THEOREM 7.

1) $m+1 \leqq r_{SL}(m) \leqq m+3$,

2) $m+1 \leqq r_{PL}(m) \leqq 2m+3$.

It will be possible to improve these upper bounds. In the following, we will give some remarks about this matter.

REMARK 4. Consider the case where a given formula A contains only m F-operators (or m H-operators). Then we can show that $s_{SL}(A) \leqq m+2$. In particular, it can be verified that $s_{SL}(\neg p \vee Fp) = 3$. Thus, $r_{SL}(1) = 3 = m+2$. But we can show that $s_{SL}(A) \leqq m+1$ for $m = 2$. So, it seems plausible that $r_{SL}(m) = m+1$ for $m \geqq 2$.

REMARK 5. If m is odd then we can show that $r_{PL}(m) \geqq m+2$. For, let $m = 2n+1$ and consider the following formula B_n containing m tense operators;

$$B_n:\quad p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge \neg Fp_1 \wedge F\big(\neg p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge \neg Fp_2 \wedge F(\ldots \wedge F(\neg p_1 \wedge \neg p_2 \wedge \ldots \neg p_n \wedge q \wedge \neg Fq)\ldots)\big)$$

If $\neg B_n$ is refutable, or equivalently if B_n is satisfiable, then there must exist $n+1$ distinct, non-reflexive elements. To obtain a refutation $\boldsymbol{PL}$-model of $\neg B_n$, we must add the first and the last elements as before and n more reflexive elements between two successive, non-reflexive elements. Thus, the model thus obtained must contain at least $2n+3$ worlds. Thus $r_{PL}(m) \geqq 2n+3 = m+2$.

We do not know much about $r_{S4}(m)$ at present. It is interesting to know whether $r_{S4}(m)$ can be bounded by some polynomial $p(m)$.

4. An application to the computational complexity of satisfiability

Throughout this section, we suppose that $\boldsymbol{L}$ is any one of $\boldsymbol{CL}$, $\boldsymbol{SL}$, $\boldsymbol{PL}$, $\boldsymbol{S4.3}$, $\boldsymbol{S4.3Dum}$ and $\boldsymbol{S4.3Grz}$. For a given formula A, define $|A|$ to be the *length* of A, i.e. the number of all symbols appearing in A. By theorems in section 3, we have the following.

There exists a polynomial $p_L(x)$ of degree one such that if a formula A with $|A| = n$ is satisfiable in an $\boldsymbol{L}$-model then A is also satisfiable in an $\boldsymbol{L}$-model with at most $p_L(n)$ worlds.

Using this, we can get the following decision algorithm to check the satisfiability (in $\boldsymbol{L}$) of a given formula:

Let a formula A with $|A| = n$ be given. First, compute $p_L(n)$. Next, enumerate all $\boldsymbol{L}$-models with at most $p_L(n)$ worlds. It suffices to check the satisfiability of A only in these models. In each $\boldsymbol{L}$-model (M, R, W), the satisfiability of A can be checked within $q_L(n)$ steps, where $q_L(x)$ is some polynomial on x which is determined by $p_L(x)$ and does not depend on a given model.

But, if we can *guess* correctly an $\boldsymbol{L}$-model in which A is satisfiable from the beginning, then the whole decision procedure can be done within $q_{\boldsymbol{L}}(n)$ steps. In such a situation, it is said that the satisfiability problem of $\boldsymbol{L}$ is *computable in nondeterministic polynomial time* (abbreviated as *computable in NP-time*). There are a lot of problems which are known to be computable in *NP*-time. The satisfiability problem B_1 of the classical propositional logic is such an example.

If a problem P can be reduced to another problem Q and if this reduction can be simulated by a Turing machine, using at most $\log n$ distinct tape cells on the work tape for each input with length n, then P is said to be *log space reducible to* Q. In [1], it is shown that every problem computable in *NP*-time is log space reducible to B_1. In this case, B_1 is said to be *log space complete in NP-time.* (As for precise definition of notions in the theory of computational complexity, see e.g. [2].) Roughly speaking, B_1 is one of the most complicated problems among problems computable in *NP*-time.

It is clear that for each formula A without modal or tense operators, A is satisfiable in the two-valued Boolean algebra if and only if it is satisfiable in some $\boldsymbol{L}$-model. Thus, B_1 is also log space reducible to the satisfiability problem of $\boldsymbol{L}$. Hence, we have the following theorem.

THEOREM 8. *Let* **L** *be any one of* ***CL, SL, PL, S4.3, S4.3Dum*** *and* ***S4.3Grz***. *Then, the satisfiability problem of* **L** *is log space complete in NP-time.*

References

[1] S.A. COOK, *The complexity of theorem proving procedures*, Proceedings of Third Annual ACM Symposium on Theory of Computing (1971) 151-158.

[2] R.E. LADNER, *The computational complexity of provability in systems of modal propositional logic*, SIAM J. on Computing, 6 (1977) 467-480.

[3] R.P. MCARTHUR, ***Tense logic*** D. Reidel, 1976.

[4] H. ONO and A. NAKAMURA, *The computational complexity of satisfiability of modal propositional logic* ***S4.3***. Tech. Rep. No.C-5, Dept. of Applied Math., Hiroshima Univ. (1979).

[5] H. RASIOWA and R. SIKORSKI, *The mathematics of metamathematics*, ***Monografie Matematyczne*** 41, PWN, 1963.

[6] K. SEGERBERG, *An essay in classical modal logic*, ***Filosofiska Studier*** 13, Uppsala Univ. (1971).

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