TREES, AUTOMATA, AND GAMES

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ABSTRACT. In 1969 Rabin introduced tree automata and proved one of the deepest decidability results. If you worked on decision problems you did most probably use Rabin's result. But did you make your way through Rabin's cumbersome proof with its induction on countable ordinals? Building on ideas of our predecessors--and especially those of Büchi--we give here an alternative and transparent proof of Rabin's result. Generalizations and further results will be published elsewhere.

§1. INTRODUCTION. Here Σ is an alphabet. All our alphabets are finite and not empty. Recall that a non-deterministic Σ-automaton is a quadruple (S,T,s_{in},F) where S is an alphabet (of states), $T \subseteq S \times \Sigma \times S$ is the transition table, $s_{in} \in S$ is the initial state, and $F \subseteq S$ is the set of final states. The automaton is said to accept a string $\sigma_1 \ldots \sigma_n$ of letters in Σ if there is a string $s_0 s_1 \ldots s_n$ of states such that $s_0 = s_{in}$ and every $(s_i, \sigma_{i+1}, s_{i+1}) \in T$ and $s_n \in F$. The theory of automata working on finite strings is well-known. It was generalized in the 1960s for a theory of automata on finite trees; an algebraic treatment of automata on finite trees, a survey of results and further references can be found in Thatcher & Wright 1968. (The game technique, developed in this paper, gives an alternative and simple way to handle automata on finite trees.)

The idea to use automata for recognizing infinite sequences is due to Büchi 1962. A *Büchi Σ-automaton* is a usual non-deterministic Σ-automaton (S,T,s_{in},F) working on infinite sequences of letters of Σ. It *accepts* a sequence $\sigma_1\sigma_2\ldots$ if there is a sequence $s_0 s_1 s_2 \ldots$ of states such that $s_0 = s_{in}$, every $s_n \sigma_{n+1} s_{n+1} \in T$ and $\{n: s_n \in F\}$ is infinite.

Büchi used sequential automata to prove decidability of the monadic second-order theory of natural numbers with the successor relation which is called, for short, the second-order theory of one successor, or S1S. The variables of S1S range over sets of natural numbers. S1S atomic formulas have a form $X \subseteq Y$ or $Suc(X,Y)$. The latter means that there is a natural number n with $X = \{n\}$, $Y = \{n + 1\}$. Other S1S formulas are built from SIS atomic formulas using conjunction, disjunction, negation and the existential quantifier. Every set X of natural numbers can be identified with its characteristic function, i.e. $X(n) = 1$ if $n \in X$, and $X(n) = 0$ otherwise. For any natural number m, let Σ_m be the direct product of m copies of the set $\{0,1\}$.

THEOREM 1 (Büchi 1962). *For every S1S formula ϕ with m variables there is a Büchi Σ_m-automaton M such that for all sets $X_1,\ldots,X_m$ of natural numbers, $\phi(X_1,\ldots,X_m)$ holds iff M accepts the Σ_m-sequence*

$$X_1(0)\ldots X_m(0), X_1(1)\ldots X_m(1), X_1(2)\ldots X_m(2), \ldots .$$

The desired automaton M is constructed by induction on ϕ. The atomic case and the cases of conjunction, disjunction and the existential quantifier are easy. A natural way to handle the negation case would be to show that every Büchi automaton is equivalent to a deterministic Büchi automaton. This is not true however, and Büchi used Ramsey's theorem to solve the complementation problem.

THEOREM 2 (Büchi 1962). *The emptiness problem for Büchi automata is decidable.*

Theorem 2 is easy. Theorems 1 and 2 give decidability of S1S.

Muller 1969 entered the field through studying a problem in asynchronous switching theory. A *deterministic Muller automaton* is a quadruple (S,T,s_{in},F) where S is the alphabet of states, $T: S \times \Sigma \to S$, $s_{in} \in S$, and F is a set of subsets of S. It *accepts* a Σ-sequence $\sigma_1\sigma_2\ldots$ if F contains the set of states appearing infinitely often in the sequence

$$s_0 = s_{in}, \quad s_1 = T(s_0,\sigma_1), \quad s_2 = T(s_1,\sigma_2), \ldots .$$

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Given a deterministic Muller Σ-automaton, it is easy to construct a Büchi Σ-automaton accepting the same Σ-sequences.

THEOREM 3 (McNaughton 1966). *For every Büchi Σ-automaton there is a deterministic Muller Σ-automaton accepting the same Σ-sequences.*

McNaughton's proof is constructive and sophisticated. Theorem 3 gives another solution for the complementation problem for Büchi automata.

Then Rabin 1969 introduced automata working on infinite trees and proved decidability of the monadic second-order theory of the infinite binary tree which is called, for short, the second-order theory of two successors, or S2S. The second-order theories of 3,4 and even ω successors reduce easily to S2S.

The infinite binary tree can be seen as the set $\{\ell,r\}^*$ of all strings in the alphabet $\{\ell,r\}$. Variables of S2S range over subsets of the infinite binary tree. S2S formulas are defined in the same way as S1S formulas but instead of $\mathrm{Suc}(X,Y)$, atomic formulas $\mathrm{Suc}_\ell(X,Y)$, $\mathrm{Suc}_r(X,Y)$ are used. They mean that there is a string $w \in \{\ell,r\}^*$ such that $X = \{w\}$ and, $Y = \{w\ell\}$ or $Y = \{wr\}$ respectively.

Rabin proved the analogues of Theorems 1 and 2 for S2S. Once again the atomic case and the cases of conjunction, disjunction and existential quantifier were easy. The difficult parts of Rabin 1969 were the complementation and--to a lesser extent--the emptiness problem. Rackoff 1972 found a simple reduction of the emptiness problem for Rabin automata to the emptiness problem for automata on finite trees. Also he simplified to an extent Rabin's solution for the complementation problem. Using games we give in the sequel a transparent solution for all these problems. Our exposition is essentially self-contained.

The idea to use games is not new. It was aired by McNaughton and exploited in Landweber 1967, Büchi & Landweber 1969 and especially in Büchi 1977 where the complementation problem was reduced (for an able reader) to a certain determinancy result. Our §2 gives such a reduction too. Our §3 provides the necessary determinancy result. When this solution had been reported in several places including Purdue Büchi kindly sent us a manuscript, Büchi 1981. To be sure Büchi proved the determinancy result, and he certainly was the first to do so. His proof still is, however, a very complicated induction on countable ordinals, much more difficult than our §3.

Our games form a special case of games studied in set theory. The most relevant set-theoretic paper is Davis 1964. However the determinancy results of Davis 1964 and other set-theoretic papers do not suffice for our purposes because we are interested only in very special memory-restricted strategies.

Let us mention that David E. Muller and Paul E. Schupp are developing an alternative approach to handle S2S.

An impressive generalization of Rabin's decidability result was formulated in Shelah 1975 and proved in details in Stupp 1975. The proof used Rabin's technique. The game technique, developed in the sequel, gives the generalized result fairly easily.

A few words on negative results. Solving Rabin's uniformization problem, Gurevich & Shelah 198? prove that no tree automaton picks a unique element from any nonempty subset of the infinite binary tree. Using automata Büchi 1973 proved decidability of monadic second-order theory of ω_1. Gurevich & Magidor & Shelah 198? prove that the corresponding theory of ω_2 can be of any given Turing degree (in different set-theoretic worlds).

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§2. TREE AUTOMATA. The *infinite binary tree* is here the set $\{\ell,r\}^*$ of words in the alphabet $\{\ell,r\}$. Its *root* is the empty word e. The nodes $x\ell$ and xr are respectively the *left* and the *right successors* of a node $x \in \{\ell,r\}^*$. A mapping V from the infinite binary tree to an alphabet Σ will be called a *Σ-valuation* or a *Σ-tree*.

Rabin 1969 defined automata working on Σ-trees. They are somewhat inconvenient to play our games. Here is an alternative definition of tree automata.

A *tree Σ-automaton* is a quadruple (S,T,T_{in},F) where S is an alphabet (of *states*), and $T \subseteq S \times \{\ell,r\} \times \Sigma \times S$ is the *transition table*, and $T_{in} \subseteq \Sigma \times S$ is the *initial state table*, and F is a family of subsets of S.

Given a tree Σ-automaton $M = (S,T,T_{in},F)$ and a Σ-tree V consider the following game $\Gamma(M,V)$ between the automaton M and another player called Pathfinder:

The automaton chooses:	Pathfinder chooses:
s_0	
	d_1
s_1	
	d_2
s_2	
	d_3
s_3	
	...

Here $(V(e),s_0) \in T_{in}$, and every $d_n \in \{\ell,r\}$, and every $(s_n,d_{n+1},V(d_1\ldots d_{n+1}),s_{n+1}) \in T$.

The automaton *wins* a play $s_0d_1s_1d_2\ldots$ if F contains $\{s \in S\colon s = s_n \text{ for infinitely many } n\}$, otherwise Pathfinder wins the play. The automaton *accepts* V if it has a winning strategy in the game $\Gamma(M,V)$.

We clarify the notion of a strategy. Any finite prefix of any play $s_0d_1s_1d_2\ldots$ will be called a *position*. Note that the automaton makes a move in a position p iff the length $|p|$ is even. A (*deterministic*) *strategy* for the automaton is a function f assigning a state $s = f(p)$ to each position p of even length in such a way that ps is a position. A (*deterministic*) *strategy* for Pathfinder is a function assigning a letter ℓ or r to each position of odd length.

Exercises:

1. Let Σ_2 be the alphabet $\{0,1\} \times \{0,1\}$. Construct tree Σ_2-automata M that accepts a Σ_2-tree V iff the range of V avoids the letter 10. Construct a tree Σ_2-automaton N_ℓ (respectively N_r) that accepts a Σ_2-tree V iff there is $u \in \{\ell,r\}^*$ such that $V(u) = 10$ and $V(u\ell) = 01$ (respectively $V(ur) = 01$) and $V(w) = 00$ for any other node w.

2. Given tree Σ-automata M_1, M_2 construct tree Σ-automata M_3, M_4 such that M_3 accepts a Σ-tree V iff either M_1 or M_2 accepts V, and M_4 accepts a Σ-tree V iff both M_1 and M_2 accept V.

3. Given a tree $\Sigma_1 \times \Sigma_2$-automaton N construct a tree Σ_1-automaton N_1 such that N_1 accepts a Σ_1-tree V_1 iff there is a Σ_2-tree V_2 such that N accepts the $\Sigma_1 \times \Sigma_2$-tree given by V_1 and V_2. N_1 will be called a Σ_1-*projection* of N.

4. Given a Rabin Σ-automaton construct a tree Σ-automaton accepting exactly the same Σ-trees.

In the rest of this section M is a tree Σ-automaton (S,T,T_{in},F), V ranges over Σ-trees, and p,q range over positions in $\Gamma(M,V)$.

The *node* of a position p is the string $\text{Node}(p)$ of even letters in p, so that

$$\text{Node}(s_0 d_1 s_1 \dots d_n) = \text{Node}(s_0 d_1 s_1 \dots d_n s_n) = d_1 \dots d_n.$$

Given a node x of the infinite binary tree, we define the x-*residue* of V. It is the Σ-tree $V_x: \{\ell,r\}^* \to \Sigma$ given by $V_x(y) = V(xy)$.

The following definition will be used in this section and later.

DEFINITION. Let A,B be alphabets and $\{C^a: a \in A\}$ be a family of disjoint subsets of B^*. A word $x \in A^*$ is an A-*display* if each letter appears in x at most once. For every letter $a \in A$, *Exposure*$_a$ is a unary operation on the set of A-displays: if $x = x_1 a x_2$ then $\text{Exposure}_a(x) = x_1 x_2 a$, if a does not appear in x then $\text{Exposure}_a(x) = xa$. By induction on $y \in B^*$ we define the *later appearance record* $R(y)$ with respect to sets C^a, $a \in A$. If $e \in C^a$ for some a then $R(e) = a$, otherwise $R(e) = e$. Suppose $x = R(y)$ and $b \in B$. If $yb \in C^a$ for some a then $R(yb) = \text{Exposure}_a(x)$, otherwise $R(yb) = x$.

To define the *later appearance record* of a position p (shortly $\text{LAR}(p)$) apply the preceding definition with $A = S$, $B = S \cup \{\ell,r\}$, and $C^s = \{y \in B^*: s$ is the last letter in $y\}$ for $s \in S$.

THEOREM 1 (Forgetful Determinancy). *One of the players has a strategy f winning $\Gamma(M,V)$ and satisfying the following condition. If p,q are positions where the winner makes moves, and the Node(p)-residue of V coincides with the Node(q)-residue of V, and $LAR(p) = LAR(q)$ then $f(p) = f(q)$.*

NOTE. Suppose that the same player makes moves in positions p,q, the Node(p)-residue of V coincides with the Node(q)-residue of V and $\text{LAR}(p) = \text{LAR}(q)$. Then p and q have the same future, i.e. the naturally defined p-residue and q-residue of $\Gamma(M,V)$ coincide. However p,q may have different histories in the game (i.e. p,q may be different) except for the later appearance records. This explains the adjective forgetful in the name of Theorem 1.

Theorem 1 will be proved in the next section.

COROLLARY 2. *Suppose that Σ is a one-letter alphabet. Then one of the players has a strategy f winning $\Gamma(M,V)$ such that if p,q are positions where the winner makes moves and $LAR(p) = LAR(q)$ then $f(p) = f(q)$.*

Corollary 2 solves the emptiness problem for tree automata.

COROLLARY 3. *One of the players has a strategy f winning $\Gamma(M,V)$ such that if p,q are positions where the winner makes moves, $Node(p) = Node(q)$ and $LAR(p) = LAR(q)$, then $f(p) = f(q)$.*

THEOREM 4 (Complementation). *There is a tree Σ-automaton accepting exactly those Σ-trees that are not accepted by M.*

In the rest of this section we deduce Theorem 4 from Corollary 3.

Let Σ' be the set of mappings

$$\sigma': (\text{the set of } S\text{-displays}) \to \{\ell,r\}.$$

A Σ'-valuation V' *yields* the following strategy for Pathfinder: if x is the node of a position p, $\sigma' = V'(x)$ and r is the later appearance record of p, then make the move $\sigma'(r)$.

We break the proof of Theorem 4 into several lemmas.

LEMMA 5. *The following statements are equivalent for every V:*

(1) M does not accept V,

(2) There is a Σ'-tree V' that yields a winning strategy for Pathfinder in $\Gamma(M,V)$.

Proof: The implication (2) → (1) is clear. To prove the implication (1) → (2) use Corollary 3. □

LEMMA 6. *For every Σ'-tree V', the following statements are equivalent:*

(3) V' yields a winning strategy for Pathfinder in $\Gamma(M,V)$, and

(4) Every path $(e, d_1, d_1 d_2, \dots)$ through $\{\ell,r\}^*$ satisfies the following condition. Let $x_n = d_1 \dots d_n$,

$\sigma_n = V(x_n)$, $\sigma'_n = V'(x_n)$ *for all* n.
Then

(*) *For every sequence* $s_0, s_1, \ldots$ *of states and every sequence* $r_0, r_1, \ldots$ *of S-displays,*

if $r_0 = s_0$ *and* $(\sigma_0, s_0) \in T_{in}$,
and $d_{n+1} = \sigma'_n(r_n)$,
$(s_n, d_{n+1}, \sigma_{n+1}, s_{n+1}) \in T$,
$r_{n+1} = Exposure_{s_{n+1}}(r_n)$ *for all* n

then $\{s: s = s_n$ *for infinitely many* $n\} \notin F$.

Proof is obvious. □

It is easy to see that (*) is expressible in S1S. Use the Büchi-McNaughton technique, described in the introduction, to construct a deterministic Muller $\{e,\ell,r\}\times\Sigma\times\Sigma'$-automaton $M' = (S', T', s_{in}, F')$ that accepts a sequence

$$e\sigma_0\sigma'_0,\ d_1\sigma_1\sigma'_1,\ d_2\sigma_2\sigma'_2, \ldots$$

iff (*) holds. Let M'' be a deterministic tree $\Sigma\times\Sigma'$-automaton (S', T'', T''_{in}, F') where

$$T''_{in}(\sigma\sigma') = T'(s_{in}, e\sigma\sigma') \text{ and}$$

$$T''(s, d, \sigma\sigma') = T'(s, d\sigma\sigma').$$

LEMMA 7. *For every* Σ'*-tree* V', *the statement (4) is equivalent to the statement*

(5) M'' *accepts the* $\Sigma\times\Sigma'$*-tree given by* V *and* V'.

Proof is obvious. □

Thus M does not accept an arbitrary Σ-tree V iff the Σ-projection of M'' accepts V. Theorem 4 is now proved.

§3. FORGETFUL DETERMINANCY.

In order to prove the Forgetful Determinancy Theorem we generalize our games and make them more symmetrical.

Here MOVE is an alphabet, and μ ranges over MOVE, and p, q, r range over MOVE*. A subset A of MOVE* will be called an *arena* if it contains the empty word e, and it is closed under prefixes (i.e. $pq \in A$ implies $p \in A$), and for every $p \in A$ there is μ with $p\mu \in A$. An arena A can be considered as a tree. The empty word is the root, and every $p\mu \in A$ is an *A-successor* of p. A subset P of an arena A will be called an *A-path* if $e \in P$ and for every $p \in P$ there is a unique μ with $p\mu \in P$.

Given an arena A, $\varepsilon \in \{0,1\}$ and a set W of A-paths consider the following game $\Gamma = (A, \varepsilon, W)$ between Mr. 0 and Mr. 1.

Mr. ε chooses	Mr. $2-\varepsilon$ chooses
$\mu_0 \in \{\mu: \mu \in A\}$	
	$\mu_1 \in \{\mu: \mu_0\mu_1 \in A\}$
$\mu_2 \in \{\mu: \mu_0\mu_1\mu \in A\}$	
	$\mu_3 \in \{\mu: \mu_0\mu_1\mu_2\mu \in A\}$
...	

Mr. ε *wins* a play $\mu_0, \mu_1, \ldots$ if the corresponding A-path $e, \mu_0, \mu_0\mu_1, \ldots$ belongs to W, otherwise Mr. $2-\varepsilon$ wins the play. W is called the *winning set* for Mr. ε, the complementing set of A-paths is called the *winning set* for Mr. $2-\varepsilon$. Elements of A are *positions* of Γ.

In the rest of this section A is an arena, $\varepsilon \in \{0,1\}$, $\delta = 2-\varepsilon$, and Γ is a game of the described type on A (i.e. $\Gamma = (A, \varepsilon', W)$ for some ε', W).

DEFINITION (Residual arenas and games). Suppose that $p \in A$. Then $A_p = \{q: pq \in A\}$. Evidently A_p is also an arena. If $C \subseteq A$ let $C_p = \{q: pq \in C\}$, so that $C_p \subseteq A_p$. If E is an equivalence relation on A let $E_p = \{(q,r): (pq, pr) \in E\}$, so that E_p is an equivalence relation on A_p. If f is a partial function from A to the set of natural numbers or to MOVE let $f_p(q) = f(pq)$ for $pq \in \mathrm{dom}(f)$, so that f_p is a partial function on A_p and $\mathrm{dom}(f_p) = (\mathrm{dom}\ f)_p$. If W is a set of A-paths let $W_p = \{X_p: X \in W\}$, so that W_p is a set of A_p-paths. Let $\varepsilon_p = \varepsilon$ if the length of p is even, and $\varepsilon_p = \delta$ otherwise. If Γ is a game (A, ε, W), let $\Gamma_p = (A_p, \varepsilon_p, W)$, so that Γ_p is a game on A_p. It is easy to see that Γ_p is a residue of Γ.

DEFINITION. Let $C \subseteq A$. The Γ-C-ε-*rank* is a partial function f from A into the set ω of natural numbers such that

$$f(p) = 0 \text{ iff } p \in C, \text{ and}$$

$$f(p) = n+1 \text{ iff there is no } m \le n \text{ with}$$

$f(p) = m$, and if Mr. ε makes a move in the position p then there is an A-successor $p\mu$ of p with $f(p\mu) \le n$, and if Mr. δ makes a move in p then $f(p\mu) \le n$ for every A-successor $p\mu$ of p.

LEMMA 1. *If* $C \subseteq A$, f *is the* Γ-C-ε*-rank and* $p \in A$ *then* f_p *is the* Γ_p-C_p-ε*-rank.*

Proof is easy.

A (*non-deterministic*) *strategy* for Mr. ε in a game Γ is a function F assigning a subset of MOVE to each position where Mr. ε makes a move. It *respects* an equivalence relation E on A if $(p,q) \in E$ and $p \in \mathrm{dom}(F)$ imply $q \in \mathrm{dom}(F)$ and $F(p) = F(q)$.

THEOREM 2. *Suppose that* $C \subseteq A$ *and the winning set for Mr.* ε *in* Γ *is the set of all those A-paths that meet* C. *Then one of the players has a strategy that wins* Γ *and respects the relation*

$$E = \{(p,q): p,q \in A,\ |p| \equiv |q| \text{ modulo } 2,\ A_p = A_q, \text{ and } C_p = C_q\}.$$

Proof: Let f be the Γ-C-ε-rank. If $e \in \mathrm{dom}(f)$ then the following strategy (called "Decrease the rank") for Mr. ε is winning:

$\{(p,\mu)$: $p\mu \in A$, and Mr. ε makes a move in p, and if $p \in \mathrm{dom}(f)$ and $f(p) > 0$ then $p\mu \in \mathrm{dom}(f)$ and $f(p\mu) < f(p)\}$.

If $e \notin \mathrm{dom}(f)$ then the following strategy (called "Keep out of dom(f)") for Mr. δ is winning:

$\{(p,\mu): p\mu \in A,$ and Mr. δ makes a move in p, and if $p \notin \mathrm{dom}(f)$ then $p\mu \notin \mathrm{dom}(f)\}$.

Use Lemma 1 to check that both strategies respect E. □

DEFINITION. An equivalence relation E on A is *right-invariant* if $(p,q) \in E$ and $pr \in A$ imply $qr \in A$ and $(pr,qr) \in E$.

If E is a right-invariant equivalence relation on A and $(p,q) \in E$ then $E_p = E_q$. For, suppose that $(r,r') \in E_p$. Then $(qr,pr) \in E$, $(pr,pr') \in E$, $(pr',qr') \in E$. Hence $(qr,qr') \in E$ and $(r,r') \in E_q$.

DEFINITION. A right-invariant equivalence relation E on A is a *congruence* for Γ if $(p,q) \in E$ implies $\Gamma_p = \Gamma_q$.

EXAMPLE. In Theorem 2 the equivalence relation E is a congruence for Γ.

LEMMA 3 (The Sewing Lemma). *Suppose that E is a congruence for Γ, and F is a Γ-strategy for Mr. δ respecting E, and D is the set of positions $p \in A$ such that Mr. ε has a Γ_p-strategy respecting E_p and winning against any refinement of F_p. Then there is a strategy G for Mr. ε in Γ such that G respects E and for every $p \in \mathrm{dom}(\Gamma\text{-}D\text{-}\varepsilon\text{-rank})$, G_p wins against any refinement of F_p.*

Proof: The idea of the desired strategy G is simple. First try to reach D. When in D pick up a winning strategy for the residual game. There is however a problem. G should respect E. Picking up a winning strategy for the residual game should respect E. Here is the formal proof.

Order the alphabet MOVE in an arbitrary way. Let $R = \{(p,q): |p| < |q|$ or else $|p| = |q|$ and p precedes q lexicographically$\}$, so that R is a linear order on MOVE* and every p has only a finite number of R-preceding positions.

Let f be the Γ-D-ε-rank. Check that $f_p = f_q$ if $(p,q) \in E$.

For every $p \in D$ let G^p be a Γ_p-strategy for Mr. ε that respects E and wins against any refinement of F_p. We are ready to construct the desired strategy G.

Suppose that Mr. ε makes a move in a position p. If $p \notin \mathrm{dom}(f)$ set $G(p) = \{\mu: p\mu \in A\}$. Suppose $p \in \mathrm{dom}(f)$. If $f(p) > 0$ set $G(p) = \{\mu: p\mu \in \mathrm{dom}(f)$ and $f(p\ ) < f(p)\}$. Suppose that $f(p) = 0$ i.e. $p \in D$. Pick the R-minimal $q \in D$ such that some $r \in A_q$ satisfies the following condition: r is a position in the subgame of Γ_q imposed by both F_q and G^q, and $(p,qr) \in E$. Set $G(p) = G^q(r)$. (Note that $G(p)$ does not depend on the choice of r.) □

For $C \subseteq A$ let $[C]$ be the set of A-paths P such that $C \cap P$ is infinite.

THEOREM 4. *Suppose that $C \subseteq A$ and $[C]$ is the Γ-winning set for one of the players. Then one of the players has a strategy winning Γ and respecting the relation*

$$E = \{(p,q): p,q \in A,\ |p| = |q| \text{ modulo } 2,\ A_p = A_q, \text{ and } C_p = C_q\}.$$

Proof: Evidently E is a congruence for Γ. Without loss of generality $[C]$ is the winning set for Mr. 0.

Let $D = \{p \in A$: Mr. 1 has a strategy winning Γ_p and respecting $E_p\}$. If $e \in D$ there is nothing to prove. Suppose $e \notin D$. By the Sewing Lemma D is the domain of the Γ-D-1-rank. Let F be the strategy "Keep out of D" for Mr. 0.

For every $p \in A$ let f^p be the Γ_p-$(C_p - \{p\})$-0-rank. It is easy to see that if $p \notin D$ then $e \in \mathrm{dom}(f^p)$ and $f^p(e) > 0$. The strategy

$\{(p,\mu) \in F$: if $p \notin D$ then $\mu \in \mathrm{dom}(f^p)$ and $f^p(\mu) < f^p(e)\}$

for Mr. 0 wins Γ and respects E. □

THEOREM 5. *Suppose S is an alphabet and $\{C^s: s \in S\}$ is a collection of disjoint subsets of A. For $p \in A$ let $LAR(p)$ be the later appearance record of p with respect to $\{C^s: s \in S\}$. Suppose that the Γ-winning sets are Boolean combinations of sets $[C^s]$, $s \in S$. Then one of the players has a strategy winning Γ and respecting*

$$E = \{(p,q): p,q \in A, \text{ and } |p| \equiv |q| \text{ modulo } 2, \text{ and } A_p = A_q, \text{ and } C^s_p = C^s_q \text{ for } s \in S, \text{ and } LAR(p) = LAR(q)\}.$$

Proof by induction on $|S|$. We can suppose that $|S| \geq 2$ because Theorem 4 takes care about the case $|S| = 1$. We can suppose also that the Γ-winning set of Mr. 0 includes the intersection of all sets $[C^s]$. Evidently E is a congruence for Γ.

Let $D = \{p \in A$: Mr. 1 has a strategy winning Γ_p and respecting $E_p\}$. If $e \in D$ there is nothing to prove. Suppose $e \notin D$. By the Sewing Lemma D is the domain of the Γ-D-1-rank. We seek a winning strategy for Mr. 0 among refinements of the strategy "Keep out of D". Thus we can suppose that D is empty.

For every $s \in S$ let $D^s = A - \mathrm{dom}(\Gamma\text{-}C^s\text{-0-rank})$, and F^s be the strategy "Keep inside D^s" for Mr. 1, and Γ^s be the subgame of Γ imposed by F^s. If $p \in D^s$ then--by the induction hypothesis--one of the players has a strategy winning Γ^s_p and respecting E_p. It can be only Mr. 0. By the Sewing Lemma Mr. 0 has a Γ-strategy G^s such that G^s respects E and for every $p \in D^s$, G^s_p wins Γ^s_p.

We are now ready to construct the desired winning strategy H for Mr. 0. Here is an idea. Mr. 0 tries to hit every C^s. (Remember that his winning set includes the intersection of sets $[C^s]$. The latest appearance records tell him which C^s is to be hit. If Mr. 1 can avoid hitting a certain C^s then Mr. 0 plays G^s. Here is a more formal definition of H.

Suppose that p is a position where Mr. 0 makes a move, and x = LAR(p). If every letter of S appears in x let s be the leftmost letter in x, otherwise let s be the first letter of S that does not appear in x. (We suppose that S was ordered a priori.) If $p \notin D^s$ decrease the Γ-C^s-D-rank. If $p \in D^s$ set $H(p) = G^s(p)$. □

To deduce the Forgetful Determinancy Theorem we describe a game $\Gamma(M,V)$ in terms of this section. The alphabet MOVE is $S \cup \{\ell,r\}$ where S is the alphabet of states of M. The arena A is the set of positions in $\Gamma(M,V)$. Mr. 0 is the automaton M, Mr. 1 is Pathfinder. The alphabet S of Theorem 5 is the alphabet of states of M. Every $C^s = \{p \in A$: s is the rightmost letter in p$\}$. It is easy to see that $\Gamma(M,V)$ is a game (A,0,W) where W is a Boolean combination of sets $[C^s]$. By Theorem 5 one of the players has a strategy winning $\Gamma(M,V)$ and respecting the equivalence relation E of Theorem 5. It respects also a finer equivalence relation

$\{(p,q)$: $p,q \in A$, and $|p| \equiv |q|$ modulo 2,
and the Node(p)-residue of V coincides
with the Node(q)-residue of V, and
LAR(p) = LAR(q)$\}$

and it can be refined to a deterministic strategy respecting this equivalence relation.

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