REMARKS ON BERGER'S PAPER ON THE DOMINO PROBLEM

Yu. Sh. Gurevich and I. O. Koryakov

UDC 518.5

This paper will strengthen the result of R. Berger, presented in his doctoral dissertation [1].

The domino concept was introduced by Hao Wang [2] and is related to the decision problem for the formulas of the predicate calculus with the prefix AEA (i.e., an existential quantifier between two universal quantifiers). (Also see [3] which, with adequate completeness, presents the results on dominos prior to Berger's paper. A popularized discussion of this question can be found in [4].)

We consider squares (domino types) of a single size with colored sides. We assume that the colors are chosen from some fixed countable set $\mathfrak{A}$. A collection of dominos (over $\mathfrak{A}$) is the name given to a finite set of types. We shall also assume that, for each type, there are infinitely many copies of it, called pieces.

We shall say that a collection is D-realizable if the pieces (perhaps not all of them) of types of D can cover an infinite plane in such fashion that the following conditions are met:

a) it is impossible either to rotate the pieces or to turn them over (to make mirror images of them);

b) contiguous sides of neighboring pieces must be colored identically.

A solution of a collection is any concrete covering of a plane by pieces of the types of this collection. (Initially, the problem was considered for a quadrant, and not for the plane. But this distinction is not essential, since it is easy to prove (cf., for example, [1]) that the problems of covering the plane and of covering a quadrant are equivalent.)

We shall say that collection D covers a torus (has a periodic solution) if, from pieces of the types of D one can, while observing conditions a) and b), build a rectangular block whose opposite sides have identical sequences of symbols, so that this block can be considered as a unit (but rectilinear) piece, capable of covering the plane. (One obtains an "actual" torus by splicing the opposite sides in the aforementioned block.)

Let $\mathfrak{D}$ be the class of all collections of dominos over a fixed countable set of symbols (pieces) $\mathfrak{A}$.

The question was posed in [2] of the existence of an algorithm for recognizing realized collections in class $\mathfrak{D}$. It was proven in [1] that the problem of recognizing such collections (the domino problem) is not decidable.

Let $\nu\colon \mathfrak{D} \to N = \{0, 1, 2, \ldots\}$ be an effective enumeration of the collections of class $\mathfrak{D}$; let $\mathfrak{J}$ be the class of collections of $\mathfrak{D}$ having solutions, but only nonperiodic ones; let $\mathfrak{F}$ be the class of collections covering tori; finally, let $\mathfrak{N}$ be the class of collections having no solution. The basic result of this note is the following

THEOREM 1. The sets $\nu(\mathfrak{J})$, $\nu(\mathfrak{F})$, $\nu(\mathfrak{N})$ are pairwise effectively indistinguishable.

We now formulate exact equivalents of the concepts introduced.

A finite set D of ordered quadruples of natural numbers is called a domino. The elements of D are called types of dominos.

Let Q be the set of all integers; $\varkappa_i$ ($i = 1, 2, 3, 4$) are the coordinate functions defined on Q^4. We shall say that set D is realizable (covers the plane) if there exists a mapping $\varphi\colon Q^2 \to D$ satisfying the conditions

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 13, No. 2, pp. 459-463, March-April, 1972. Original article submitted February 1, 1971.

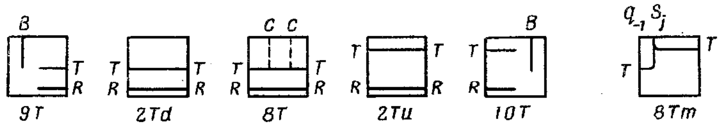

Fig. 1

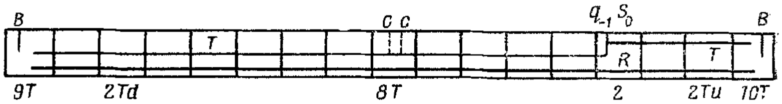

Fig. 2

$$\varkappa_1\varphi(x, y+1) = \varkappa_3\varphi(x, y), \qquad \varkappa_2\varphi(x+1, y) = \varkappa_4\varphi(x, y). \tag{1}$$

Each such φ is called a solution of set D.

We say that collection D covers a torus if there exists a periodic solution to collection D, i.e., a mapping φ satisfying (1) for which one can find natural numbers p_1, $p_2 > 0$ such that

$$\forall xy[\varphi(x+p_1, y) = \varphi(x, y) = \varphi(x, y+p_2)]. \tag{2}$$

In this case, if p_1 and p_2 are the smallest positive numbers satisfying (2), the solution φ is called (p_1, p_2)-periodic.

To prove the undecidability of the domino problem, R. Berger used Turing machines. We adopt the following variant of the Turing machine, more appropriate for our aim.

These machines must have two tape symbols S_0 and S_1, and the set of internal states of each machine will be a finite subset of some fixed (effectively generated) countable set $\mathfrak{L}$ (machines over $\mathfrak{L}$) having three special symbols: q_1 (the initial state), q_{-1} and q_0 (the halting states).

We assume that the machines always begin operation in state q_1 while scanning a cell with the symbol S_0, while, when it is convenient, we can always "splice" to the tape (only) on the right a new cell containing S_0. We further assume that no machine, finding itself at the leftmost cell, can receive a command to move left. The machines can halt only in states q_{-1} and q_0, while there are no programs which begin with these symbols.

We consider the class of all machines over $\mathfrak{L}$ (class $\mathfrak{M}$). Let μ: $\mathfrak{M} \to N$ be an effective enumeration of machines of class $\mathfrak{M}$; let $\mathfrak{I}'$ be the class of perpetually operating machines of $\mathfrak{M}$; $\mathfrak{F}'$ the class of machines stopped in state q_{-1}; $\mathfrak{N}'$ the class of machines halted in state q_0. The basis for the proof of Theorem 1 is the well-known (cf. [3])

LEMMA 1. Sets $\mu(\mathfrak{I}')$, $\mu(\mathfrak{F}')$, $\mu(\mathfrak{N}')$ are pairwise effectively indistinguishable.

Adding to the construction of Berger, we specify an algorithm corresponding to each machine Z of $\mathfrak{M}$ a set D_Z of $\mathfrak{D}$ in the following way: if $Z \in \mathfrak{I}'$ then $D_Z \in \mathfrak{I}$; if $Z \in \mathfrak{F}'$ then $D_Z \in \mathfrak{F}$; if $Z \in \mathfrak{N}'$, then $D_Z \in \mathfrak{N}$. From the existence of this algorithm and from Lemma 1, Theorem 1 follows immediately.

Proof of Theorem 1. To the set of skeletal prototypes ([1], Table 2), we add the prototypes 2Td, 2Tu, 8T, 9T, 10T, and to the collection of forms of machine prototypes ([1], Table 6), we add form 8Tm (cf. Fig. 1), where the T signals are new, being common to these, and only these, prototypes. The T signals occur on different levels: in the skeletal prototypes, 2Tu and 10T are higher than 2Td, 8T, and 9T. The corresponding channels, which are specially derived, are common with the analogous channels of the machine prototypes of form 8Tm.

To the definition of skeletal set K ([1], Table 4) at points (c) and (d), it is necessary to add the following constraints;

1) skeletal prototype 8T forms a product with selecting prototypes 10p-14p, and only with them;

2) skeletal prototype 9T forms a product with selecting prototype V13, and only with it;

3) skeletal prototype 10T forms a product with selecting prototype H12, and only with it;

The corresponding constraints for prototypes 2Td and 2Tu are obtained automatically.

The definition of collection D_Z ([1], Table 7) in paragraph (b) is augmented by the constraint: the machine prototypes of form 8Tm form products only with K-prototypes having skeletal prototype 2 and selecting prototypes 10p-14p.

LEMMA 2. 1) If machine Z operates perpetually, collection D_Z has a solution, but only a nonperiodic one.

2) If Z arrives at state q_{-1}, D_Z has a periodic solution.

3) If Z arrives at state q_0, D_Z has no solution.

Proof. 1) Let Z never halt. Then, for no $n \geq 0$ does its n-configuration contain symbol q_0. Therefore, as in [1], signals leaving from the lower part of any growing n-register enter into the upper part of its receiving (n + 1)-register. The machine configurations do not contain the symbol q_{-1} either, from which it follows (analogously to the proof of R. Berger in [1], section 4.2.3) that in the plane solution of collection D_Z there cannot appear a piece having machine prototype 8Tm. Since signals of this, and only this, prototype can connect T-signals occurring on different levels in skeletal (registor) prototypes 8T, on one hand, and 2Tu (or 10T) on the other, in no register does a T-signal appear.

For a T-signal which occurs in skeletal prototypes necessarily in conjunction with R-signals, there remains (by virtue of Lemmas 3-4 and addition 1 of [1]) a unique possibility of appearing in (some) solutions of collection D_Z, existing by virtue of Theorem 3-3 of [1]: in the form of an infinite signal in conjunction with an infinite R-signal in the series comprised of the pieces having skeletal prototypes only 2Td or only 2Tu. But such a situation, admissible in [1], does not enter into the proofs of Theorems 3-6 of [1], from which follows the necessity of nonperiodic solutions for collection D_Z.

2) At the moment n, let machine Z halt at state q_{-1}. The $q_{-1}S_j$-signal from the lower part of a growing (n + 1)-register either terminates at a piece with machine prototype 8 in a decaying (n + 2)-register, having selection symbol 12, 21, or 22, or is changed to a horizontal T-signal in a piece with machine prototype of form 8Tm in an (n + 2)-register with selection symbol 11. This last register must be constructed from skeletal prototypes with T-signals, i.e., must again be decaying. Figure 2 shows such a 4-register appearing in any solution of collection D_Z, where Z has the program (omitting "nonworking" commands): $q_1S_0Rq_2$, $q_2S_0Rq_{-1}$. (Under the register are the numbers of the corresponding skeletal prototypes; the selecting signals and basis numbers are omitted.)

Thus, there will be no successors for any of the (n + 2)-registers. It is not hard to see that if all, besides the T, skeletal and selecting signals belong to some iterative construction ([1], section 3.2.5), then this solution is $(2^{n+5}, 2^{n+4})$-periodic.

3) Let Z halt at state q_0 at moment n. In the lower parts of the growing (n + 1)-registers there must appear a q_0 S_j-signal which will add the leading prototypes to the upper parts of (n + 2)-registers. But, among the machine prototypes, there is no prototype capable of forming a product with the K-prototypes of a growing register and having the symbol q_0S_j on the upper level. Lemma 2 is proven.

As already mentioned, Theorem 1 follows from Lemmas 1 and 2.

The authors wish to thank R. Berger for his generous provision of a copy of his dissertation.

LITERATURE CITED

1. R. Berger, "The undecidability of the domino problem," Mem. Amer. Math. Soc., 66, 1-72 (1966).
2. Hao Wang, "Proving theorems by pattern recognition, II," Bell System Tech. J., 40, 1-41 (1961).
3. Hao Wang, "Dominoes and the AEA case of the decision problem," in: Symposium on the Mathematical Theory of Automata, Polytechnic Institute of Brooklyn, 1962, Polytechnic Press, New York (1963).
4. Hao Wang, "Games, logic, and computers," in: Kibernet. Sb., Novaya Ser., 5, 195-207 (1967).