Congestion Games with Linearly Independent Paths: Convergence Time and Price of Anarchy

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Abstract. We investigate the effect of linear independence in the strategies of congestion games on the convergence time of best response dynamics and on the pure Price of Anarchy. In particular, we consider symmetric congestion games on extension-parallel networks, an interesting class of networks with linearly independent paths, and establish two remarkable properties previously known only for parallel-link games. More precisely, we show that for arbitrary non-negative and non-decreasing latency functions, any best improvement sequence converges to a pure Nash equilibrium in at most n steps, and that for latency functions in class $\mathcal{D}$, the pure Price of Anarchy is at most $\rho(\mathcal{D})$.

1 Introduction

Congestion games provide a natural model for non-cooperative resource allocation in large-scale communication networks and have been the subject of intensive research in algorithmic game theory. In a *congestion game*, a finite set of non-cooperative players, each controlling an unsplittable unit of load, compete over a finite set of resources. All players using a resource experience a latency (or cost) given by a non-negative and non-decreasing function of the resource's load (or congestion). Among a given set of resource subsets (or strategies), each player selects one selfishly trying to minimize her *individual cost*, that is the sum of the latencies on the resources in the chosen strategy. A natural solution concept is that of a *pure Nash equilibrium*, a configuration where no player can decrease her individual cost by unilaterally changing her strategy.

The prevailing questions in recent work on congestion games have to do with quantifying the inefficiency due to the players' selfish behaviour (see e.g. [19,20,14,5,7,4,6]), and bounding the convergence time to pure Nash equilibria if the players select their strategies in a selfish and decentralized fashion (see e.g. [11,18,1]). In this work, we investigate the effect of linear independence in the strategies of congestion games on the convergence time of best improvement sequences and on the inefficiency of pure Nash equilibria. In particular, we consider symmetric congestion games on extension-parallel networks, an interesting class of networks whose paths are linearly independent, in the sense that every path contains an edge not included in any other path. For this class of congestion games, which comprises a natural and non-trivial generalization of the extensively studied class of parallel-link games (see e.g. [19,20,14,11,18,6]), we provide best possible answers to both research questions above.

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Convergence Time to Pure Nash Equilibria. Rosenthal [23] proved that the pure Nash equilibria of congestion games correspond to the local optima of a natural potential function. Hence Rosenthal established that every congestion game admits at least one pure Nash equilibrium (PNE) reached in a natural way when players iteratively select strategies that minimize their individual cost given the strategies of other players. Nevertheless, this may take an exponential number of steps, since computing a PNE is PLS-complete even for asymmetric network congestion games as shown by Fabricant *et al.* [12]. In fact, the proof of Fabricant *et al.* establishes the existence of instances where any best improvement sequence is exponentially long. Even for symmetric network congestion games, where a PNE can be found efficiently by a min-cost flow computation [12], Ackermann *et al.* [1] presented instances where any best improvement sequence is exponentially long.

A natural approach to circumvent the negative results of [12,1] is to identify large classes of congestion games for which best improvement sequences reach a PNE in a polynomial number of steps. For instance, it is well known that for symmetric singleton congestion games (aka parallel-link games), any best improvement sequence converges to a PNE in at most n steps, where n denotes the number of players. Ieong *et al.* [18] proved that even for asymmetric singleton games with non-monotonic latencies, best improvement sequences reach a PNE in polynomial time. Subsequently, Ackermann *et al.* [1] generalized this result to *matroid* congestion games, where the strategy space of each player consists of the bases of a matroid over the set of resources. Furthermore, Ackermann *et al.* proved that the matroid property on the players' strategy spaces is necessary for guaranteeing polynomial-time convergence of best improvement sequences if one does not take into account the global structure of the game.

Contribution. The negative results of [12,1] leave open the possibility that some particular classes of symmetric network congestion games can guarantee fast convergence of best improvement sequences. We prove that for symmetric congestion games on extension-parallel networks with arbitrary non-negative and non-decreasing latency functions, any best improvement sequence converges to a PNE in at most n steps[1]. In particular, we show that in a best improvement sequence, every player moves at most once. This result is best possible, since there are instances where reaching a PNE requires that every player moves at least once.

Price of Anarchy. Having reached a PNE, selfish players enjoy a minimum individual cost given the strategies of other players. However, the public benefit is usually measured by the *total cost* incurred by all players. Since a PNE does not need to minimize the total cost, one seeks to quantify the inefficiency due to the players' non-cooperative and selfish behaviour. The *Price of Anarchy* was introduced by Koutsoupias and Papadimitriou [19] and has become a widely accepted measure of the performance degra-

[1] We highlight that matroid games and games on extension-parallel networks have a different combinatorial structure and may have quite different properties. For example, a network consisting of two pairs of parallel links connected in series is not extension-parallel, but the corresponding network congestion game is a symmetric matroid game. For another example, Milchtaich [22, Example 4] proved that weighted congestion games on extension-parallel networks may not admit a PNE. On the other hand, Ackermann *et al.* [2, Theorem 2] proved that every weighted matroid congestion game admits a PNE.

dation due to the players' selfish behaviour. The (pure) Price of Anarchy is the worst-case ratio of the total cost of a (pure) Nash equilibrium to the optimal total cost. Many recent contributions have provided strong upper and lower bounds on the pure Price of Anarchy (PoA) for several classes of congestion games, mostly congestion games with affine and polynomial latency functions and congestion games on parallel links[2].

For the special case of parallel links with linear latency functions, Lücking *et al.* [20] proved that the PoA is $4/3$. For parallel links with polynomial latency functions of degree d, Gairing *et al.* [14] proved the PoA is at most $d + 1$. Awerbuch *et al.* [5] and Christodoulou and Koutsoupias [7] proved independently that the PoA of congestion games is $5/2$ for affine latency functions and $d^{\Theta(d)}$ for polynomial latency functions of degree d. Subsequently, Aland *et al.* [4] obtained exact bounds on the PoA of congestion games with polynomial latency functions. In the non-atomic setting, where the number of players is infinite and each player controls an infinitesimal amount of load, Roughgarden [24] proved that the PoA is independent of the strategy space and equal to $\rho(\mathcal{D})$, where ρ depends on the class of latency functions $\mathcal{D}$ only (e.g. ρ is equal to $4/3$ for affine and 1.626 for quadratic functions). Subsequently, Correa *et al.* [8] introduced $\beta(\mathcal{D}) = 1 - \frac{1}{\rho(\mathcal{D})}$ and gave a simple proof of the same bound. Recently Fotakis [13] and independently Caragiannis *et al.* [6, Theorem 23] proved that the PoA of (atomic) congestion games on parallel links with latency functions in class $\mathcal{D}$ is also $\rho(\mathcal{D})$.

Contribution. Despite the considerable interest in the PoA of congestion games, it remains open whether some better upper bounds close to ρ are possible for symmetric congestion games on simple networks other than parallel links (e.g. extension-parallel networks, series-parallel networks), or strong lower bounds similar to the lower bounds of [5,7,4] also apply to them. As a first step in this direction, we prove that the PoA of symmetric congestion games on extension-parallel networks with latency functions in class $\mathcal{D}$ is at most $\rho(\mathcal{D})$. On the negative side, we show that this result cannot be further generalized to series-parallel networks.

Related Work on Congestion Games with Linearly Independent Strategies. There has been a significant volume of previous work investigating the impact of linear independent strategies on properties of congestion games. Holzman and Law-Yone [16] proved that a symmetric strategy space admits a strong equilibrium[3] for any selection of non-negative and non-decreasing latency functions iff it consists of linearly independent strategies. Furthermore, Holzman and Law-Yone showed that for symmetric congestion games with linearly independent strategies, every PNE is a strong equilibrium and also a minimizer of Rosenthal's potential function. Subsequently, Holzman and Law-Yone [17] proved that the class of congestion games on extension-parallel networks is the network equivalent of congestion games with linearly independent strategies.

Milchtaich [21] was the first to consider networks with linearly independent paths (under this name). Milchtaich proved that an undirected network has linearly independent paths iff it is extension-parallel. Furthermore, Milchtaich showed that extension-parallel networks is the only class of networks where for any selection of non-negative

[2] Here we cite only the most relevant results on the pure PoA for the objective of total cost. For a survey on the PoA of congestion games , see e.g. [15].

[3] A configuration is a *strong equilibrium* if no coalition of players can deviate in a way profitable for all its members.

and increasing (resp. non-decreasing) latency functions, all equilibria in the non-atomic setting are (resp. weakly) Pareto efficient.

Recently Epstein *et al.* [10,9] considered fair connection games and congestion games on extension-parallel networks. In [10], they proved that fair connection games on extension-parallel networks admit a strong equilibrium. In [9], they showed that extension-parallel networks is the only class of networks where for all non-negative and non-decreasing latencies, any PNE minimize the maximum players' cost.

2 Model and Preliminaries

For any integer $k \geq 1$, we let $[k] \equiv \{1, \dots, k\}$. For a vector $x = (x_1, \dots, x_n)$, we let $x_{-i} \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $(x_{-i}, x'_i) \equiv (x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$.

Congestion Games. A *congestion game* is a tuple $\Gamma(N, E, (\Sigma_i)_{i \in N}, (d_e)_{e \in E})$, where N denotes the set of players, E denotes the set of resources, $\Sigma_i \subseteq 2^E \setminus \{\emptyset\}$ denotes the strategy space of each player i, and $d_e : \mathbb{N} \mapsto \mathbb{R}_{\geq 0}$ is a non-negative and non-decreasing latency function associated with each resource e. A congestion game is *symmetric* if all players have a common strategy space.

A *configuration* is a vector $\sigma = (\sigma_1, \dots, \sigma_n)$ consisting of a strategy $\sigma_i \in \Sigma_i$ for each player i. For every resource e, we let $\sigma_e = |\{i \in N : e \in \sigma_i\}|$ denote the congestion induced on e by σ. The individual cost of player i in the configuration σ is $c_i(\sigma) = \sum_{e \in \sigma_i} d_e(\sigma_e)$. A configuration σ is a *pure Nash equilibrium* (PNE) if no player can improve her individual cost by unilaterally changing her strategy. Formally, σ is a PNE if for every player i and every strategy $s_i \in \Sigma_i$, $c_i(\sigma) \leq c_i(\sigma_{-i}, s_i)$.

In the following, we let n denote the number of players. We focus on *symmetric network* congestion games, where the players' strategies are determined by a directed network $G(V, E)$ with a distinguished source s and sink t (aka $s - t$ network). The network edges play the role of resources and the common strategy space of the players is the set of (simple) $s - t$ paths in G, denoted $\mathcal{P}$. For any $s - t$ path p and any pair of vertices v_1, v_2 appearing in p, we let $p[v_1, v_2]$ denote the segment of p between v_1 and v_2 ($p[v_1, v_2]$ is empty if v_1 appears after v_2 in p). For consistency with the definition of strategies as resource subsets, we usually regard paths as sets of edges.

Flows and Configurations. Let $G(V, E)$ be a $s - t$ network. A $s - t$ *flow* f is a vector $(f_e)_{e \in E} \in \mathbb{R}^m_{\geq 0}$ that satisfies the flow conservation at all vertices other than s and t. The *volume* of f is the total flow leaving s. A flow is *acyclic* if there is no directed cycle in G with positive flow on all its edges. For a flow f and a path p, we let $f^{\min}_p = \min_{e \in p}\{f_e\}$.

Given a configuration σ for a symmetric network congestion game Γ, we refer to the congestion vector $(\sigma_e)_{e \in E}$ as the *flow* induced by σ. We say that a flow σ is *feasible* if there is a configuration inducing congestion σ_e on every edge e. Hence any configuration of Γ corresponds to a feasible flow. We always let the same symbol denote both a configuration and the feasible flow induced by it.

Best Improvement Sequences. A strategy $s_i \in \Sigma_i$ is a *best response* of player i to a configuration σ (or equivalently to σ_{-i}) if for every strategy $s'_i \in \Sigma_i$, $c_i(\sigma_{-i}, s_i) \leq c_i(\sigma_{-i}, s'_i)$. If i's current strategy σ_i is not a best response to the current configuration σ, a best response of i to σ is a *best improvement* of i. We consider best improvement sequences, where in each step, a player i whose strategy σ_i is not a best response to the current configuration σ switches to her best improvement. Using a potential function, Rosenthal [23] proved that any such sequence reaches a PNE in a finite number of steps.

Social Cost and the Price of Anarchy. To quantify the inefficiency of PNE, we evaluate configurations using the objective of *total cost*. The total cost $C(\sigma)$ of a configuration σ is the sum of players' costs in σ : $C(\sigma) = \sum_{i=1}^{n} c_i(\sigma) = \sum_{e \in E} \sigma_e d_e(\sigma_e)$. The *optimal configuration*, usually denoted o, minimizes the total cost among all configurations in $\mathcal{P}^n$. The pure *Price of Anarchy* (PoA) of a congestion game Γ is the maximum ratio $C(\sigma)/C(o)$ over all PNE σ of Γ.

Extension-Parallel Networks. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two networks with sources s_1 and s_2 and sinks t_1 and t_2 respectively, and let $G'(V_1 \cup V_2, E_1 \cup E_2)$ be the union network of G_1 and G_2. The *parallel composition* of G_1 and G_2 results in a $s - t$ network obtained from G' by identifying s_1 and s_2 to the source s and t_1 and t_2 to the sink t. The *series composition* of G_1 and G_2 results in a $s - t$ network obtained from G' by letting s_1 be the source s, letting t_2 be the sink t, and identifying t_1 with s_2.

A directed $s-t$ network is *series-parallel* if it consists of either a single edge (s, t) or two series-parallel networks composed either in series or in parallel. A directed $s-t$ network is *extension-parallel* if it consists of either: (i) a single edge (s, t), (ii) a single edge and an extension-parallel network composed in series, or (iii) two extension-parallel networks composed in parallel. Every extension-parallel network is series-parallel, but the converse is true only if in every series composition, at least one component is a single edge.

A $s-t$ network has *linearly independent paths* if every $s-t$ path contains at least one edge not belonging to any other $s - t$ path[4]. Milchtaich [21, Proposition 5] proved that an undirected $s - t$ network has linearly independent paths iff it is extension-parallel. Therefore, every (directed) extension-parallel network has linearly independent paths (see also [17, Theorem 1]). Furthermore, [21, Propositions 3, 5] imply that a (directed) series-parallel network has linearly independent paths iff it is extension-parallel.

An interesting property of extension-parallel networks is that for any two $s - t$ paths p, p', the segments $p \setminus p'$ and $p' \setminus p$ where p and p' deviate from each other form two internally disjoint paths with common endpoints (see also [21, Proposition 4]). In addition, every $s-t$ path having an edge in common with $p \setminus p'$ does not intersect $p' \setminus p$ at any vertex other than its endpoints. The following proposition gives another interesting property of networks with linearly independent paths (and thus of extension-parallel networks).

Proposition 1. *Let Γ be a symmetric congestion game on a $s - t$ network G with linearly independent paths, let f be any configuration of Γ, and let π be any (simple) path with $f_\pi^{\min} > 0$. Then there exists a player i whose strategy in f includes π.*

Every configuration of a symmetric congestion game on a series-parallel (and thus on an extension-parallel) network corresponds to a feasible acyclic flow of volume n. Proposition 1 implies that for any congestion game Γ on an extension-parallel network, every

[4] The name is motivated by the fact that in such a network, it is not possible to express any path as the symmetric difference of some other paths [21, Proposition 6].

feasible acyclic $s-t$ flow corresponds to a unique Γ's configuration (uniqueness is up to players' permutation, see also [16, Section 6]). Therefore, for symmetric congestion games on extension-parallel networks, there is a correspondence between configurations and feasible acyclic flows.

3 Convergence Time to Pure Nash Equilibria

Next we show that for symmetric congestion games on extension-parallel networks, any best improvement sequence reaches a PNE after each player moves at most once.

Lemma 1. *Let Γ be a congestion game on an extension-parallel network, let σ be the current configuration, and let i be a player switching from her current strategy σ_i to her best improvement σ'_i. Then for every player j whose current strategy σ_j is a best response to σ, σ_j remains a best response of j to the new configuration $\sigma' = (\sigma_{-i}, \sigma'_i)$.*

Proof. For sake of contradiction, we assume that there is a player j whose current strategy σ_j is a best response to σ but not to σ'. Let σ'_j be a best response of j to σ', and let $p = \sigma_j \setminus \sigma'_j$ and $p' = \sigma'_j \setminus \sigma_j$ be the segments where σ_j and σ'_j deviate from each other. Due to the extension-parallel structure of the network, p and p' are internally disjoint paths with common endpoints, denoted u and w. Since p and p' are edge-disjoint and player j improves her individual cost in σ' by switching from p to p',

$$\sum_{e\in p} d_e(\sigma'_e) > \sum_{e\in p'} d_e(\sigma'_e + 1) \tag{1}$$

Using (1) and the fact that σ'_i is a best improvement of player i to σ, and exploiting the extension-parallel structure of the network, we establish that if player j prefers σ'_j to σ_j in the new configuration σ', then σ_j is not a best response of j to σ. In particular, we show that player j can also improve her individual cost in σ by switching from an appropriate segment of σ_j to the corresponding segment of σ'_i. Clearly, this contradicts the hypothesis that σ_j is a best response of j to σ and implies the lemma. The technical part of the proof proceeds by case analysis.

Case I, $u, w \in \sigma'_i$: We first consider the case where σ'_i contains u and w and thus $\sigma'_i[u, w]$ can serve as an alternative to p. We further distinguish between two subcases:

Case I.a, $p \cap \sigma'_i = \emptyset$: We start with the case where σ'_i and p are edge-disjoint. We first consider the case where $\sigma'_i[u, w] \setminus p'$ does not contain any edges of σ_i (Fig. 1.a). Then,

$$\begin{aligned}\sum_{e\in p'} d_e(\sigma'_e + 1) &\geq \sum_{e\in p'\cap\sigma'_i} d_e(\sigma_e + 1) + \sum_{e\in(p'\cap\sigma_i)\setminus\sigma'_i} d_e(\sigma_e) + \sum_{e\in(p'\setminus\sigma_i)\setminus\sigma'_i} d_e(\sigma_e + 1)\\ &\geq \sum_{e\in p'\cap\sigma'_i} d_e(\sigma_e + 1) + \sum_{e\in\sigma'_i[u,w]\setminus p'} d_e(\sigma_e + 1)\end{aligned} \tag{2}$$

For the first inequality, we use that when player i switches from σ_i to σ'_i: (i) the congestion of any edge e in σ'_i does not decrease (i.e. $\sigma'_e \geq \sigma_e$), (ii) the congestion of any edge e decreases by at most 1 (i.e. $\sigma'_e \geq \sigma_e - 1$), and (iii) the congestion of any edge

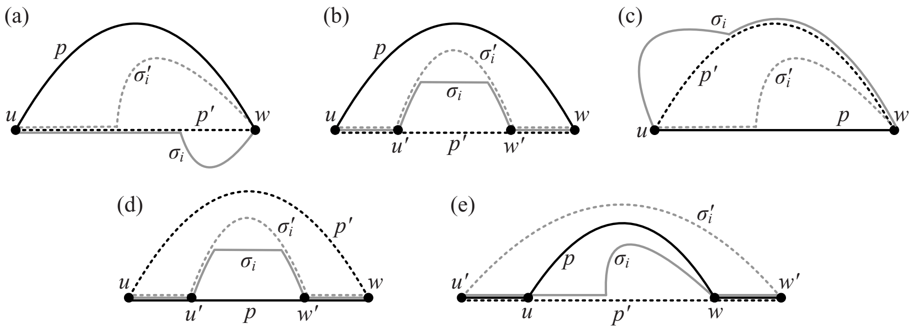

Fig. 1. The different cases considered in the proof of Lemma 1. In each case, the solid black path labeled p represents the best response of player j to σ between vertices u and v, the solid grey path labeled σ_i represents the strategy of player i in σ, and the dotted grey path labeled σ_i' represents the best improvement of player i. We assume that the best response of player j changes from p to the dotted black path labeled p' when player i switches from σ_i to σ_i' and establish a contradiction in all cases.

e not in $\sigma_i \cup \sigma_i'$ does not change (i.e. $\sigma_e' = \sigma_e$). For the second inequality, we observe that $\sum_{e \in (p' \cap \sigma_i) \setminus \sigma_i'} d_e(\sigma_e) + \sum_{e \in (p' \setminus \sigma_i) \setminus \sigma_i'} d_e(\sigma_e + 1)$ is the individual cost of player i on $p' \setminus \sigma_i'$ in σ (i.e. when the configuration of the remaining players is σ_{-i}) and that $\sum_{e \in \sigma_i'[u,w] \setminus p'} d_e(\sigma_e + 1)$ is the individual cost of i on $\sigma_i'[u,w] \setminus p'$ in σ (recall that $\sigma_i'[u,w] \setminus p'$ does not contain any edges of σ_i). Since σ_i' is a best response of i to σ_{-i}, the former cost is no less than the latter.

Using (2), we conclude that player j can improve her individual cost in σ by changing her path between u and w from p to $\sigma_i'[u,w]$, which contradicts the hypothesis that σ_j is a best response of player j to σ. Formally,

$$\sum_{e \in p} d_e(\sigma_e) \geq \sum_{e \in p} d_e(\sigma_e') > \sum_{e \in p'} d_e(\sigma_e' + 1) \geq \sum_{e \in \sigma_i'[u,w]} d_e(\sigma_e + 1)$$

The first inequality holds because $p \cap \sigma_i' = \emptyset$, and the congestion of the edges in p does not increase when player i switches from σ_i to σ_i'. The second inequality is (1) and the third inequality follows from (2).

If $\sigma_i'[u,w] \setminus p'$ contains some edges of σ_i, we can show that due to the extension-parallel structure of the network, the congestion of the edges in $p \cup p'$ does not change when player i switches from σ_i to σ_i' (see Fig. 1.b). This contradicts the hypothesis that the best response of player j changes from σ_j to σ_j' when player i moves from σ_i to σ_i'.

Case I.b, $p \cap \sigma_i' \neq \emptyset$: We proceed with the case where σ_i' and p are not edge-disjoint. Then, due to the extension-parallel structure of the network, σ_i' does not have any edges in common with p' and does not intersect p' at any vertex other than u and w. We first consider the case where $\sigma_i'[u,w] \setminus p$ does not contain any edges of σ_i (Fig. 1.c). Then,

$$\begin{aligned}\sum_{e\in p\cap\sigma'_i} d_e(\sigma'_e) + \sum_{e\in p\setminus\sigma'_i} d_e(\sigma_e) \geq \sum_{e\in p} d_e(\sigma'_e) > \sum_{e\in p'} d_e(\sigma'_e+1) \\ \geq \sum_{e\in p'\cap\sigma_i} d_e(\sigma_e) + \sum_{e\in p'\setminus\sigma_i} d_e(\sigma_e+1) \\ \geq \sum_{e\in\sigma'_i[u,w]} d_e(\sigma'_e) \\ = \sum_{e\in p\cap\sigma'_i} d_e(\sigma'_e) + \sum_{e\in\sigma'_i[u,w]\setminus p} d_e(\sigma_e+1)\end{aligned}$$

The first inequality holds because the congestion of any edge e not in σ'_i does not increase when player i switches from σ_i to σ'_i (i.e. $\sigma_e \geq \sigma'_e$). The second inequality is (1). The third inequality holds because when player i switches from σ_i to σ'_i: (i) the congestion of any edge e decreases by at most 1 (i.e. $\sigma'_e \geq \sigma_e - 1$), and (ii) the congestion of any edge e not in σ_i does not decrease (i.e. $\sigma'_e \geq \sigma_e$). For the fourth inequality, we observe that the left-hand side is equal to the individual cost of player i on p' in σ, and that the right-hand side is equal to the cost of player i on $\sigma'_i[u,w]$ in σ. Since σ'_i is a best response of player i to σ_{-i}, the former cost is not less than the latter. The equality holds because $\sigma'_i[u,w]\setminus p$ does not contain any edges of σ_i and thus the congestion of every edge $e\in\sigma'_i[u,w]\setminus p$ increases by 1 when player i switches from σ_i to σ'_i.

Therefore, $\sum_{e\in p\setminus\sigma'_i} d_e(\sigma_e) > \sum_{e\in\sigma'_i[u,w]\setminus p} d_e(\sigma_e+1)$, and player j can improve her individual cost in σ by switching from $p\setminus\sigma'_i$ to $\sigma'_i[u,w]\setminus p$. This contradicts the hypothesis that σ_j is a best response of player j to σ.

If $\sigma'_i[u,w]\setminus p$ contains some edges of σ_i, we can show that due to the extension-parallel structure of the network, the congestion of the edges in $p\cup p'$ does not change when player i switches from σ_i to σ'_i (see Fig. 1.d). This contradicts the hypothesis that the best response of player j changes from σ_j to σ'_j when player i moves from σ_i to σ'_i.

Case II, either $u\notin\sigma'_i$ or $w\notin\sigma'_i$: We proceed with the case where σ'_i does not contain either u or w. Then, σ'_i does not have any edges in common with p and p'.

If σ_i too does not contain either u or w, then σ_i does not have any edges in common with p and p'. Since $(\sigma_i\cup\sigma'_i)\cap(p\cup p')=\emptyset$, the congestion of the edges in $p\cup p'$ does not change when player i switches from σ_i to σ'_i. This contradicts the hypothesis that the best response of player j changes from σ_j to σ'_j when player i moves from σ_i to σ'_i.

Therefore, we can restrict our attention to the case where σ_i contains both u and w. Let $\sigma'_i\setminus\sigma_i$ and $\sigma_i\setminus\sigma'_i$ be the segments where σ_i and σ'_i deviate from each other. Due to the extension-parallel structure of the network, and since σ'_i does not contain either u or w and σ_i contains both u and w, $\sigma'_i\setminus\sigma_i$ and $\sigma_i\setminus\sigma'_i$ are (non-empty) internally disjoint paths with common endpoints, denoted u' and w'. Their first endpoint u' appears no later than u and their last endpoint w' appears no sooner than w in σ_i. Furthermore, either u is different from u' or w is different from w' (or both). Due to the extension-parallel structure of the network, and since σ_i deviates from at least one of p and p' between u and w, there is a unique path $\sigma_i[u',u]$ between u and u' and a unique path $\sigma_i[w,w']$ between w and w' (see Fig. 1.e). Let $z=\sigma_i[u',u]\cup\sigma_i[w,w']$. We highlight that both $\sigma_i[u', u]$ and $\sigma_i[w, w']$ are included in σ_j and σ'_j. In particular, $\sigma_j[u', w'] = z \cup p$. Using the previous observations, we obtain that:

$$\begin{aligned}\sum_{e \in \sigma_j[u',w']} d_e(\sigma_e) &\geq \sum_{e \in z} d_e(\sigma_e) + \sum_{e \in p} d_e(\sigma'_e) \\ &> \sum_{e \in z} d_e(\sigma_e) + \sum_{e \in p'} d_e(\sigma'_e + 1) \\ &\geq \sum_{e \in z} d_e(\sigma_e) + \sum_{e \in p' \cap \sigma_i} d_e(\sigma_e) + \sum_{e \in p' \setminus \sigma_i} d_e(\sigma_e + 1) \\ &\geq \sum_{e \in \sigma'_i[u',w']} d_e(\sigma_e + 1)\end{aligned}$$

The first inequality holds because the edges of p do not belong to σ'_i and the congestion of any edge $e \notin \sigma'_i$ does not increase when player i moves from σ_i to σ'_i (i.e. $\sigma_e \geq \sigma'_e$). The second inequality follows from (1). The third inequality holds because when player i switches from σ_i to σ'_i: (i) the congestion of any edge e decreases by at most 1 (i.e. $\sigma'_e \geq \sigma_e - 1$), and (ii) the congestion of any edge e not in σ_i does not decrease (i.e. $\sigma'_e \geq \sigma_e$). For the fourth inequality, we observe that the left-hand side is equal to the individual cost of player i on $\sigma_i[u', u] \cup p' \cup \sigma_i[w, w']$ in σ, and that the right-hand side is equal to the individual cost of player i on $\sigma'_i[u', w']$ in σ (recall that $\sigma'_i[u', w']$ and $\sigma_i[u', w']$ are edge disjoint). Since σ'_i is a best response of player i to σ_{-i}, the former cost is not less than the latter.

Therefore, player j can decrease her individual cost in σ by switching from $\sigma_j[u', w']$ to $\sigma'_i[u', w']$. This contradicts the hypothesis that σ_j is a best response of player j to σ. Since we have reached a contradiction in all different cases, this concludes the proof of the lemma. □

By Lemma 1, once a player moves to her best improvement strategy, she will not have an incentive to deviate as long as the subsequent players switch to their best improvement strategies. Hence we obtain the main result of this section:

Theorem 1. *For any n-player symmetric congestion game on an extension-parallel network, every best improvement sequence reaches a PNE in at most n steps.*

4 Bounding the Price of Anarchy

For a latency function $d(x)$, let $\rho(d) = \sup_{x \geq y \geq 0} \frac{x d(x)}{y d(y) + (x-y) d(x)}$, and let $\beta(d) = \sup_{x \geq y \geq 0} \frac{y(d(x) - d(y))}{x d(x)}$. For a class of latency functions $\mathcal{D}$, let $\rho(\mathcal{D}) = \sup_{d \in \mathcal{D}} \rho(d)$ and $\beta(\mathcal{D}) = \sup_{d \in \mathcal{D}} \beta(d)$. We note that $(1 - \beta(\mathcal{D}))^{-1} = \rho(\mathcal{D})$. In [24,8], it was shown that the PoA of non-atomic congestion games with latencies in class $\mathcal{D}$ is $\rho(\mathcal{D})$. Next we establish the same upper bound on the PoA of symmetric congestion games on extension-parallel networks. The proof is based on the following lemma.

Lemma 2. *Let Γ be a symmetric congestion game on an extension-parallel network $G(V, E)$, and let f be a PNE and g be any configuration of Γ. Then,*

$$\Delta(f, g) \equiv \sum_{e: f_e > g_e} (f_e - g_e) d_e(f_e) - \sum_{e: f_e < g_e} (g_e - f_e) d_e(f_e + 1) \leq 0$$

Proof. We assume wlog. that the configurations f and g are not identical and consider the corresponding feasible flows f and g. Let $\hat{G}(V, \hat{E})$ be the graph of the flow $f - g$. In particular, for each edge $(u, w) \in E$, $\hat{E}$ contains a *forward* edge (u, w) with flow $f_{(u,w)} - g_{(u,w)}$ if $f_{(u,w)} > g_{(u,w)}$, a *backward* edge (w, u) with flow $g_{(u,w)} - f_{(u,w)}$ if $f_{(u,w)} < g_{(u,w)}$, and no edge between u and w if $f_{(u,w)} = g_{(u,w)}$. For every cycle C of $\hat{G}$, let $C^+ = \{(u, w) \in E : (u, w) \in C \text{ and } f_{(u,w)} > g_{(u,w)}\}$ be the set of forward edges in C, and let $C^- = \{(u, w) \in E : (w, u) \in C \text{ and } f_{(u,w)} < g_{(u,w)}\}$ be the set of backward edges in C with their directions reversed (i.e. their directions are as in E).

Since f and g are feasible acyclic $s - t$ flows of the same volume, a flow decomposition of $f - g$ yields only cycles and no paths of $\hat{G}$. Let $\{C_1, \ldots, C_k\}$ be the set of (simple) cycles of $\hat{G}$ produced by the standard flow decomposition of $f - g$ (see e.g. the algorithm described in [3, Theorem 3.5]), and let s_i denote the amount of flow carried by each cycle C_i in that decomposition of $f - g$. Since f and g are feasible acyclic $s - t$ flows, every cycle C_i contains at least one forward and at least one backward edge.

By the properties of the standard flow decomposition algorithm, $\cup_{i \in [k]} C_i^+$ is equal to $\{e \in E : f_e > g_e\}$, and $\cup_{i \in [k]} C_i^-$ is equal to $\{e \in E : f_e < g_e\}$. Moreover, for every forward edge $(u, w) \in \hat{E}$, $\sum_{i:(u,w) \in C_i^+} s_i = f_{(u,w)} - g_{(u,w)}$, and for every backward edge $(w, u) \in \hat{E}$, $\sum_{i:(u,w) \in C_i^-} s_i = g_{(u,w)} - f_{(u,w)}$. Therefore,

$$\Delta(f, g) = \sum_{i=1}^{k} s_i \left(\sum_{e \in C_i^+} d_e(f_e) - \sum_{e \in C_i^-} d_e(f_e + 1) \right) \tag{3}$$

The following proposition shows that for every cycle C_i in the decomposition of $f - g$ (in fact, for every simple cycle of $\hat{G}$), $\sum_{e \in C_i^+} d_e(f_e) - \sum_{e \in C_i^-} d_e(f_e + 1) \leq 0$.

Proposition 2. *Let Γ be a symmetric congestion game on an extension-parallel network G, let f be a PNE and g be any configuration of Γ, and let $\hat{G}$ be the graph of the flow $f - g$. For every simple cycle C of $\hat{G}$,*

$$\sum_{e \in C^+} d_e(f_e) - \sum_{e \in C^-} d_e(f_e + 1) \leq 0$$

Proof sketch. Using induction on the extension-parallel structure of G, we prove that for every simple cycle C of $\hat{G}$, there are vertices u, w on C such that C^+ and C^- are two internally disjoint $u - w$ paths in G. Since C^+ consists of forward edges only, for every $e \in C^+$, $f_e > 0$. Hence by Proposition 1, there is a player i whose strategy in f includes C^+. Therefore, $\sum_{e \in C^+} d_e(f_e) \leq \sum_{e \in C^-} d_e(f_e + 1)$, since otherwise player i could switch from C^+ to C^- between u and w and improve her individual cost, which contradicts the hypothesis that f is a PNE. □

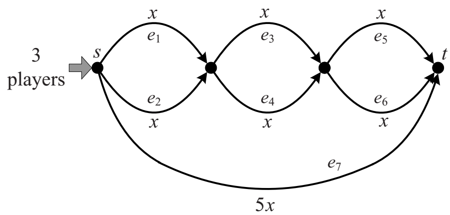

Fig. 2. A symmetric congestion game on a series-parallel network with linear latencies and PoA greater than $4/3$

Combining (3) and Proposition 2, we obtain that $\Delta(f, g) \leq 0$. □

Now we are ready to establish the main result of this section. The following theorem follows easily from Lemma 2 and the definition of $\rho(\mathcal{D})$.

Theorem 2. *For any symmetric congestion game on an extension-parallel network with latency functions in class $\mathcal{D}$, the PoA is at most $\rho(\mathcal{D})$.*

Proof. We consider a symmetric congestion game Γ on an extension-parallel network $G(V, E)$. The latency functions of Γ are such that $\{d_e(x)\}_{e \in E} \subseteq \mathcal{D}$. Let o be the optimal configuration, and let f be Γ's PNE of maximum total cost.

For every edge e with $f_e > o_e$,

$$\begin{aligned} f_e d_e(f_e) &= o_e d_e(f_e) + (f_e - o_e) d_e(f_e) \\ &\leq o_e d_e(o_e) + \beta(\mathcal{D}) f_e d_e(f_e) + (f_e - o_e) d_e(f_e)\,, \end{aligned} \tag{4}$$

where the inequality follows by applying the definition of $\beta(\mathcal{D})$ to the term $o_e d_e(f_e)$.

On the other hand, for every edge e with $f_e < o_e$,

$$\begin{aligned} f_e d_e(f_e) &= o_e d_e(o_e) - o_e d_e(o_e) + f_e d_e(f_e) \\ &\leq o_e d_e(o_e) - (o_e - f_e) d_e(f_e + 1) \end{aligned} \tag{5}$$

The inequality follows from $d_e(f_e) \leq d_e(f_e+1)$ and $d_e(f_e+1) \leq d_e(o_e)$, because the latency functions are non-decreasing and $f_e+1 \leq o_e$ (recall that o_e and f_e are integral).

Using (4), (5), and Lemma 2, we obtain that:

$$\begin{aligned} C(f) &\leq \sum_{e \in E} o_e d_e(o_e) + \beta(\mathcal{D}) \sum_{e: f_e > o_e} f_e d_e(f_e) + \Delta(f, o) \\ &\leq C(o) + \beta(\mathcal{D}) C(f)\,, \end{aligned}$$

which implies that $C(f) \leq (1 - \beta(\mathcal{D}))^{-1} C(o) = \rho(\mathcal{D}) C(o)$. For the first inequality, we apply (4) to every edge e with $f_e > o_e$ and (5) to every edge e with $f_e < o_e$. The last inequality follows from Lemma 2, which implies that $\Delta(f, o) \leq 0$. □

Remark 1. The PoA may be greater than $\rho(\mathcal{D})$ even for series-parallel networks with linear latencies. For example, let us consider the 3-player game in Fig 2. Since the latency functions are linear, $\rho = 4/3$. In the optimal configuration o, every edge has congestion 1 and the total cost is $C(o) = 11$. On the other hand, there is a PNE f where the first player is assigned to (e_1, e_3, e_6), the second player to (e_1, e_4, e_5), and the third player to (e_2, e_3, e_5). Each player incurs an individual cost of 5 and does not have an incentive to deviate to e_7. The total cost is $C(f) = 15$ and the PoA is $15/11 > 4/3$. In this example, Lemma 2 fails because Proposition 1 does not hold.

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