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The Existence of Equilibrium in Discontinuous Economic Games, II: Applications

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1. INTRODUCTION

In part I of this study we established two existence theorems for mixed-strategy equilibrium in games with discontinuous payoff functions (see Theorems 4(4*) and 5(5*) in Dasgupta and Maskin (1986); hereafter D-M (1986)). In this second part we use the second of these two theorems to prove the existence of mixed-strategy equilibrium in some well-known discontinuous economic games that fail to have pure-strategy equilibria. The models we analyse are the Bertrand-Edgeworth example of price setting duopolists with capacity constraints (see Section 2.2); price competition among firms producing differentiated products (Hotelling (1929) and d'Aspremont, Gabszewicz and Thisse (1979); see Section 2.3); spatial competition (Eaton and Lipsey (1975) and Shaked (1975); see Section 3); and models of market dependent information (Rothschild and Stiglitz (1976) and Wilson (1977); Section 4). The paper concludes (Section 5) with a classification of these models.

2. PRICE COMPETITION

2.1. *Introduction*

Consider a market for N (possibly differentiated) products, where commodity i is supplied by firm i. Firms choose prices as strategic variables, and a given consumer purchases from that firm which charges the lowest price, corrected for his perception of product quality. In this section we present two well-known special cases of such a market, the Bertrand-Edgeworth and Hotelling duopoly models which are known for failing, in general, to have pure strategy equilibria. We show, however, that these models satisfy the hypotheses of Theorem 1, and so have equilibria where firms choose prices randomly.

2.2. *The Bertrand-Edgeworth duopoly model*[1]

We consider a market for a single commodity with a continuum of consumers represented by the unit interval [0, 1]. Consumers are identical, and the representative consumer's demand for the commodity is a continuous, monotonically decreasing function, $Q(a)$, where a ($\geqq 0$) is the price. We assume that the demand curve cuts both axes and denote by $\bar{\bar{a}}$ (>0) the choke-off price for the representative consumer (i.e. $Q(a)=0$ for $a \geqq \bar{\bar{a}}$, and $Q(a)>0$ for $0 \leqq a < \bar{\bar{a}}$).

There are two firms in the industry $i=1, 2$. (It is easy to generalize all the results that follow to any number of firms). Firm i has an endowment of S_i units of the commodity; (alternatively we can think of S_i as the capacity of a zero-cost technology). Firms choose prices and play non-cooperatively. It is assumed that the firm quoting the lower price serves the entire market up to its capacity, and that the residual demand is met by the other firm. The residual demand depends, of course, on which consumers purchase from which firm. We assume that all consumers are identical and that rationing at the lower price is on a first-come-first-serve basis. On the other hand, if the duopolists set the same price they share the market demand in proportion to their capacities, so long as their capacities are not met.

Formally, let $\bar{a}$ be the *competitive* price of the commodity; that is, $\bar{a}$ solves the market clearing equation $Q(a)=S_1+S_2$ if $S_1+S_2<Q(0)$, and equals zero otherwise. Let a_i be the price chosen by firm i. We may as well suppose that $a_i \in A_i = [\bar{a}, \bar{\bar{a}}]$. We define the profit functions, $U_1(a_1, a_2)$ and $U_2(a_1, a_2)$ of the duopolists to be:[2]

$$U_1(a_1, a_2) = \begin{cases} \min\{a_1 S_1, a_1 Q(a_1)\} & \text{if } a_1 < a_2 \\ \min\{a_1 S_1, a_1 Q(a_1) S_1/(S_1+S_2)\} & \text{if } a_1 = a_2 \text{ and } S_2 \geqq \dfrac{Q(a_1)S_2}{(S_1+S_2)} \\ a_1 Q(a_1) - S_2 & \text{if } a_1 = a_2 \text{ and } S_2 < \dfrac{Q(a_2)S_2}{(S_1+S_2)} \\ \max\{0, a_1 Q(a_1)[Q(a_2)-S_2]/Q(a_2)\} & \text{if } a_1 > a_2, \end{cases} \tag{1}$$

and

$$U_2(a_1, a_2) = \begin{cases} \min\{a_2 S_2, a_2 Q(a_2)\} & \text{if } a_2 < a_1 \\ \min\{a_2 S_2, a_2 Q(a_2) S_2/(S_1+S_2)\} & \text{if } a_2 = a_1 \text{ and } S_1 \geqq \dfrac{Q(a_2)S_1}{(S_1+S_2)} \\ a_2 Q(a_2) - S_1 & \text{if } a_2 = a_1 \text{ and } S_1 < \dfrac{Q(a_1)S_1}{(S_1+S_2)} \\ \max\{0, a_2 Q(a_2)[Q(a_1)-S_1]/Q(a_1)\} & \text{if } a_2 > a_1. \end{cases} \tag{2}$$

It is well known that this duopoly market may not possess a Nash equilibrium in pure strategies; (see e.g. Chamberlin (1956) and d'Aspremont and Gabszewicz (1980)).[3] We now confirm that the market always possesses an equilibrium in mixed strategies.

Define the diagonal of the product of the strategy sets:

$$A^*(1) = A^*(2) = \{(a_1, a_2) \in [\bar{a}, \bar{\bar{a}}]^2 \mid a_1 = a_2\}. \tag{3}$$

From (1) and (2) it is immediate that the discontinuities in $U_i(\boldsymbol{a})$ are restricted to $A^*(i) - \{(\bar{a}, \bar{a}), (\bar{\bar{a}}, \bar{\bar{a}})\}$. Furthermore, it is simple to confirm that by lowering its price from a position where $\bar{\bar{a}} > a_1 = a_2 > \bar{a}$, a firm discontinuously *increases* its profit. Therefore $U_i(a_1, a_2)$ is everywhere left lower semi-continuous in a_i, and hence weakly lower semi-continuous. Obviously U_i is bounded. Finally, $U_1 + U_2$ is continuous, because discontinuous shifts in clientele from one firm to another occur only where both firms derive the same profit per customer. We may therefore use Theorem 5 (D–M (1986)) to conclude:

Theorem 1. *Consider the price-setting duopoly game* $[(A_i, U_i): i=1,2]$, *where* $A_i = [\underline{a}, \bar{a}]$, *with* $\underline{a} \geqq 0$, *and* $U_i: A_1 \times A_2 \to R^1$ *is defined by* (1) *and* (2). *The game has a mixed-strategy equilibrium.*

There has been some interest in the symmetric version of the Bertrand-Edgeworth model (see e.g. Beckmann (1965)), to which we now turn. Thus, let $S_1 = S_2$. It is well known that under those parametric conditions (e.g. $S_i > Q(0)$ for $i=1,2$) where the duopoly game does have a pure-strategy equilibrium, the equilibrium is unique, and consists of firms choosing the competitive price $\underline{a}$. We now proceed to confirm that under all parametric conditions the symmetric version of the game possesses a symmetric mixed-strategy equilibrium such that the equilibrium probability measure is atomless at all prices in excess of $\underline{a}$.

We have already noted that $A^{**}(i) = A^*(i) - \{(\underline{a}, \underline{a}), (\bar{a}, \bar{a})\}$. Thus $A_i^{**}(i) = (\underline{a}, \bar{a})$. We have also noted above that for all $\tilde{a} \in (\underline{a}, \bar{a})$, $\lim_{a_1 \overset{-}{\to} \tilde{a}} \inf U_1(a_1, \tilde{a}) > U_1(\tilde{a}, \tilde{a})$ and $\lim_{a_2 \overset{-}{\to} \tilde{a}} \inf U_2(\tilde{a}, a_2) > U_2(\tilde{a}, \tilde{a})$. Thus the game satisfies property (α^*). We may therefore appeal to Theorem 6 in D-M (1986) to assert

Theorem 2. *The symmetric version of the Bertrand-Edgeworth duopoly game possesses a symmetric mixed-strategy Nash equilibrium* (μ^*, μ^*), *such that* μ^* *is atomless in the open interval* $(\underline{a}, \bar{a})$.[4]

We can easily establish a bit more about the nature of symmetric equilibrium in the Bertrand-Edgeworth model. Define

$$R(a) = aQ(a)$$
$$R^*(a) = \min\{aS_1, R(a)\}$$
$$R^{**}(a) = \max_{\tilde{a} \leqq a} R^*(\tilde{a}).$$

Corollary.[5] *If* μ^* *is an equilibrium strategy of a symmetric equilibrium in the symmetric Bertrand-Edgeworth game, there exist* a' *and* a'' *with* $\underline{a} \leqq a' < a'' \leqq \bar{a}$ *such that the support of* μ^* *is*

$$T = [a', a''] \cap \{a \mid R^{**}(a) = R^*(a) \text{ and if } a \in (\underline{a}, \bar{a}), \; R^{**} \text{ is increasing either from the left or right at } a\}.$$

Proof. Let $a' = \inf \operatorname{supp} \mu^*$ and $a'' = \sup \operatorname{supp} \mu^*$. Define T as in the statement of the corollary. Consider a such that $R^*(a) < R^{**}(a)$. Choose $\tilde{a} < a$ such that $R^{**}(\tilde{a}) = R^*(\tilde{a})$. Then, $R^*(\tilde{a}) > R^*(a)$. Because $\tilde{a}S_1 < aS_1$ we know that $R(\tilde{a}) > R(a)$ and $\tilde{a}S_1 > R(a)$. We have, therefore,

$$\int U_1(\tilde{a}, a_2)\, d\mu^*(a_2) > \int U_1(a, a_2)\, d\mu^*(a_2).$$

We conclude that $a \notin \operatorname{supp} \mu^*$.

Next consider a such that $a \in (\underline{a}, \bar{a})$, $R^{**}(a) = R^*(a)$, and R^{**} is constant at a. If $a \in \operatorname{supp} \mu^*$, then because μ^* has no atoms in $(\underline{a}, \bar{a})$ there exist a^0 and a^{00} such that $a^0 < a^{00}$, $a \in [a^0, a^{00}] \subseteq \operatorname{supp} \mu^*$. Because R^{**} is constant at a, we may assume that R^{**} is constant on $[a^0, a^{00}]$. Furthermore, from the above argument, $R^{**} = R^*$ on $[a^0, a^{00}]$.

Therefore, in particular $R^*(a^0)=R^*(a^{00})$. Then $R(a^0)>R(a^{00})$, and so

$$\int U_1(a^0, a_2)d\mu^*(a_2) > \int U_1(a^{00}, a_2)d\mu^*(a_2).$$

Thus $a \notin \operatorname{supp} \mu^*$, after all.

Finally, consider $a \in T$. If $a \notin \operatorname{supp} \mu^*$ there exist a^0 and a^{00} such that $a \in (a^0, a^{00})$ and $[a^0, a^{00}] \cap \operatorname{supp} \mu^* = \{a^0, a^{00}\}$. Because R^{**} is non-decreasing everywhere, and, in particular strictly increasing either from the left or right at a, $R^{**}(a^0) < R^{**}(a^{00})$. Therefore, because $\{a^0, a^{00}\} \subseteq \operatorname{supp} \mu^*$, $R^*(a^0) < R^*(a^{00})$. Furthermore, because $(a^0, a^{00}) \cap \operatorname{supp} \mu^* = \phi$ the probability firm 2's price is greater than a^0 is the same as that it is greater than a^{00}. Therefore,

$$\int U_1(a^0, a_2)d\mu^*(a_2) < \int U_1(a^{00}, a_2)d\mu^*(a_2),$$

a contradiction. We conclude that $a \in \operatorname{supp} \mu^*$. ||

We may now confirm that $\bar{a} < a'$ when a pure strategy equilibrium does not exist. Now a pure strategy equilibrium does not exist if market demand is inelastic at $\bar{a}$. For in this case, if say, firm 2 chooses $\bar{a}$, firm 1 gains by raising its price marginally above $\bar{a}$. This immediately implies that at a symmetric mixed strategy equilibrium $\bar{a}$ is not in the support of the equilibrium strategy. Finally, one notes that a firm makes zero profit at $\bar{\bar{a}}$. An argument similar to the one above then demonstrates that $a'' < \bar{\bar{a}}$.

2.3. *The Hotelling model of price competition*

The model concerns a market for differentiated products. Consumers are distributed uniformly along the unit interval $[0, 1]$.[6] There are two firms $(i=1, 2)$ located at the points x_1 and x_2 (with $x_2 > x_1$). They costlessly produce products that are identical except for their location. The firms choose mill prices, and the cost of transporting a unit of the commodity is c (>0) per unit distance. Each consumer purchases precisely one unit of the commodity from the cheapest source (i.e. the firm minimizing mill price plus transport cost), so long as his payment does not exceed his reservation price V (>0). Otherwise he does without the product. We may then restrict each of the two firms to choose its (mill) price from the range $[0, V]$. Let $a_i \in A_i = [0, V]$ denote the i-th firm's (mill) price. Then the profit functions of the two firms can be expressed as:

$$U_1(a_1, a_2) = \begin{cases} 0 \quad \text{if } a_1 > a_2 + c(x_2 - x_1) \\ a_1 \min\{x_1, (V-a_1)/c\} + a_1 \min\{1-x_1, (V-a_1)/c\} \\ \quad \text{if } a_1 < a_2 - c(x_2 - x_1) \\ a_1 \min\{x_1, (V-a_1)/c\} + (a_1/2c) \min\{(a_2 - a_1) + c(x_2 - x_1), 2(V - a_1)\} \\ \quad \text{if } |a_1 - a_2| \leqq c(x_2 - x_1) \end{cases} \tag{4}$$

and

$$U_2(a_1, a_2) = \begin{cases} 0 \quad \text{if } a_2 > a_1 + c(x_2 - x_1) \\ a_2 \min\{x_2, (V-a_2)/c\} + a_2 \min\{1-x_2, (V-a_2)/c\} \\ \quad \text{if } a_2 < a_1 - c(x_2 - x_1) \\ a_2 \min\{1-x_2, (V-a_2)/c\} + (a_2/2c) \min\{c(x_2 - x_1) - (a_2 - a_1), 2(V - a_2)\} \\ \quad \text{if } |a_1 - a_2| \leqq c(x_2 - x_1). \end{cases} \tag{5}$$

For the problem to be interesting we must suppose that $V > c(x_2 - x_1)/2$. Otherwise the potential market areas of the two firms will not intersect, and there will be no room for competition—the essence of the investigation. In fact, for simplicity of exposition we assume the stronger condition that $V > c$. Define

$$A^*(1) = A^*(2) = \{(a_1, a_2) \in [0, V]^2 \,|\, |a_1 - a_2| = c(x_2 - x_1)\}. \tag{6}$$

In Figure 1 $A^*(1)$ $(= A^*(2))$ is depicted by the two straight lines in the square $[0, V]^2$. Let $A^{**}(i)$ denote the discontinuity set of $U_i(\boldsymbol{a})$. From (4) and (5) we conclude that $A^{**}(i) \subseteq A^*(i) = A^*(2)$ for $i = 1, 2$.

One can show that this game does not possess pure-strategy equilibria at all location points, (x_1, x_2).[7] We demonstrate, however, that an equilibrium exists for any pair of locations if firms choose price distributions.

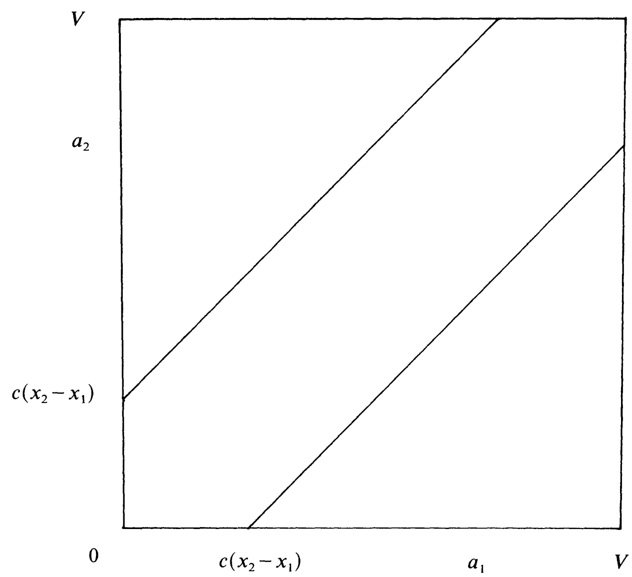

FIGURE 1

One may first verify from (4) and (5) that although $U_1 + U_2$ is not continuous, it is upper-semi-continuous. (For $a_2 > a_1 + c(x_2 - x_1)$, all customers buy at price a_1, whereas at $a_2 = a_1 + c(x_2 - x_1)$, some customers buy at a_2. Thus, total profit jumps up. Similarly, profit jumps up moving from $a_1 > a_2 + c(x_2 - x_1)$ to $a_1 = a_2 + c(x_2 - x_1)$). Note as well that U_1 is bounded and that for all $a_1 \in A_1^{**}(1)$,

$$\lim_{a_1 \to \bar{a}_1} \inf U_1(a_1, a_2) \geqq U_1(\bar{a}_1, a_2) \quad \text{for all } a_2 \in A_{-1}^{**}(\bar{a}_1)$$

and likewise for $i = 2$. That is, $U_i(a_i, \boldsymbol{a}_{-i})$ is weakly lower semi-continuous in a_i. We may therefore appeal to Theorem 5 in D–M (1986) and assert

Theorem 3. *The two-firm Hotelling model of price competition has a mixed strategy equilibrium for any pair of product qualities.*

Remark 1. The Hotelling model is a symmetric game if either $x_1 = x_2$ or $x_1 = 1 - x_2$. For the first case—i.e. when firms are located at the same point—there is a unique pure-strategy equilibrium, and it is given by $a_1 = a_2 = 0$. This is the competitive outcome. Notice that with $x_1 = x_2$, $A^*(1)$ $(= A^*(2))$ in (6) is in fact the diagonal. But from (4) and (5) it is clear that $(0, 0)$ is *not* a point of discontinuity of U_i. Thus $0 \notin A_1^{**}(1) = A_2^{**}(2)$. It follows that the competitive outcome is consistent with Theorem 6 in D-M (1986).

Remark 2. It should be clear that our analysis and results extend immediately to a market with three or more firms. They also extend to the case where firms' "locations" are points in spaces of two or more dimensions. This second extension, however, is less interesting from the point of view of this paper, since in two or more dimensions, the discontinuities of the one-dimensional model vanish. Therefore, in more than one dimension the standard Glicksberg and Fan existence theorems apply (see Theorem 1 in D-M (1986)).

3. PRODUCT COMPETITION

In the previous section we assumed that firms competed in prices and that their locations were fixed. In this section we look at the opposite case, where the product price is fixed and firms compete in the characteristics of the product they offer.

Consider a product consisting of m characteristics. The feasible set of characteristics is assumed to be a non-empty, compact, subset of R^m, which we denote by $\bar{\bar{A}}$. Consumers differ in their preferences over characteristics and it is supposed that there is a continuum of consumer types. A consumer whose favourite vector of characteristics is $x \in \bar{\bar{A}}$ is labelled as type x, and we assume a non-atomic distribution of consumers over $\bar{\bar{A}}$, which we denote by the (Borel) measure ρ.

There are N firms $(i, j = 1, \dots, N)$, which produce the commodity costlessly. The product price is fixed in advance and is the same for all firms. A firm chooses the vector of characteristics it will offer. It can offer at most one vector. Each consumer purchases at most one unit of the product from the firm nearest the consumer's favourite vector. To be precise, the realized utility level (net of payment for the product) of consumer of type $x \in \bar{\bar{A}}$ from consuming a unit of the product with characteristic $a \in \bar{\bar{A}}$ is:

$$W(x, a) = V - c\|a - x\|, \quad \text{where } V, c > 0. \tag{7}$$

For simplicity we assume that $V > c \sup_{x \in \bar{\bar{A}}} \sup_{a \in \bar{\bar{A}}} \|a - x\|$, so that each consumer purchases precisely one unit.

Let a_i be firm i's location in the feasible set of characteristics $\bar{\bar{A}}$. Now define

$$B_i(\boldsymbol{a}) \equiv B_i(a_i, \boldsymbol{a}_{-i}) \equiv \{x \in \bar{\bar{A}} \mid \|x - a_i\| \leqq \|x - a_j\| \ \forall j \neq i\}. \tag{8}$$

If no other firm coincides with i in its choice of location, then $B_i(\boldsymbol{a})$, firm i's market region, is the set of consumers who purchase from i. However, if n firms $(0 \leqq n \leqq N-1)$, other than i, are located at a_i, then each of these $(n+1)$ congruent firms has a $1/(n+1)$ share of $B_i(\boldsymbol{a})$. Therefore, the profit function of firm i is:

$$U_i(a_i, \boldsymbol{a}_{-i}) = (1+n)^{-1} \int_{B_i(\boldsymbol{a})} d\rho(x),$$

where $(1+n)$ is the number of firms located at a_i. (9)

Firms are profit maximizing, and their strategies are locations. The market is thus represented by the game $[(\bar{\bar{A}}, U_i){:}\ i = 1, \dots, N]$ where U_i is given by (9). Notice that the game is *symmetric*. Eaton and Lipsey (1975) noted that if $m = 1$, $\bar{\bar{A}} = [0, 1]$, $N = 3$ and ρ

is uniform, then the game does not possess a Nash equilibrium in pure strategies.[8] In an accompanying paper, Shaked (1975) proved an identical result for the case where $m=2$, $\bar{\bar{A}}$ is the unit circle, $N=3$ and ρ is uniform.

The payoff function (9) violates two hypotheses of the classical pure-strategy existence theorems (see Theorem 1 in D–M (1986)): namely continuity, and quasi-concavity in a_i. A slight modification of the payoff function suggests, however, that the decisive factor in the non-existence of equilibrium is the failure of the *second* rather than the first hypothesis. In particular, suppose that for given $\varepsilon > 0$ we redefine a firm's profit to be zero if it is within ε of any other firm but suppose that otherwise the profit function is (9). It is a straightforward matter to check that such a modified payoff function satisfies all the hypotheses of Theorem 2 in D–M (1986), except the required quasi-concavity. Therefore, the "blame" for the non-existence of equilibrium in pure strategies in this modified model—and therefore by extension, in the unmodified model—can be assigned to the violation of quasi-concavity.

We now prove that the location game $[(\bar{\bar{A}}, U_i): i=1, \dots, N]$ we have defined in (7)–(9) possesses a symmetric mixed-strategy equilibrium where, for $N \geqq 3$ the equilibrium mixed strategy is atomless. To do this we merely confirm that the game satisfies the hypotheses of Theorem 6* in D–M (1986). Define

$$A^*(i) \equiv \{(a_i, \boldsymbol{a}_{-i}) \in \bar{\bar{A}}^N \mid \exists j \neq i, a_i = a_j\}. \tag{10}$$

It is immediate from (9) that the discontinuities of U_i are confined to a subset of $A^*(i)$. Notice further that U_i is bounded and that the game is constant – sum. If ν is the uniform distribution on the unit circle (for $m=1$, ν places probability one-half on each of 1 and -1), then for all $(a_i, \boldsymbol{a}_{-i})$

$$\int_{B^m} \liminf_{\theta\to 0} U_i(a_i+\theta e, \boldsymbol{a}_{-i})\, d\nu(e) = \int_{B^m} \liminf_{\theta\to 0}\left[\int_{B_i(a_i+\theta e, \boldsymbol{a}_{-i})} d\rho(x)\right] d\nu(e)$$

$$= \begin{cases} \displaystyle\int_{B_i(\boldsymbol{a})} d\rho(x), & \text{if } n=0 \\ \displaystyle\tfrac{1}{2}\int_{B_i(\boldsymbol{a})} d\rho(x), & \text{if } n>0 \end{cases}$$

$$\geqq \frac{1}{n+1}\int_{B_i(\boldsymbol{a})} d\rho(x) = U_i(a_i, \boldsymbol{a}_{-i}),$$

where n is the number of firms other than i located at a_i and the inequality is strict if $n \geqq 2$. Therefore U_i satisfies property (α^*). (The preceding argument applies to a_i in the interior of $\bar{\bar{A}}$. If a_i lies on boundary, the distribution ν must be modified accordingly.)

Let $A^{**}(i)$ ($\subseteq A^*(i)$) denote the set of discontinuities of U_i. For $N>2$ note that $A_i^{**}(i) = \bar{\bar{A}}$; that is, any location by a firm is a potential point of discontinuity. (This is not true in the case $N=2$, since for example, if firms locate along a one-dimensional line segment ($m=1$), the mid-point is not an element of $A_i^{**}(i)$). We may now state:

Theorem 4. *For $N>2$, let $\bar{\bar{A}} \subseteq R^m$ ($m \geqq 1$) be non-empty, and compact, and let $U_i: \bar{\bar{A}}^N \to R^1$ satisfy (9). Then the game $[(\bar{\bar{A}}, U_i): i=1, \dots, N]$ possesses a symmetric mixed-strategy Nash equilibrium*

$$\underbrace{(\mu^*, \dots, \mu^*)}_{N \text{ times}}$$

where μ^* is atomless on $A_i^{**}(i) = \bar{\bar{A}}$.

Remark 1. Shaked (1982) has computed a symmetric mixed-strategy equilibrium for the case $\bar{\bar{m}}=1$, $\bar{A},=[0, 1]$, $N=3$ and ν uniform (the example discussed in footnote 8). It consists of each firm choosing a uniform distribution on the interval $[\frac{1}{4}, \frac{3}{4}]$ and zero weight on the outer quartiles. Osborne and Pitchik (1982) show that, in this case, there are other equilibria which involve a mixture of pure and mixed strategies.

Remark 2. We have studied the choices of location and price separately, but our analysis carries over to some cases where firms choose both. Suppose that a strategy consists of choosing a price and location simultaneously. Then our arguments can easily be modified to establish the existence of a mixed strategy equilibrium in this more elaborate model. The model originally considered by Hotelling (1929), however, was one where firms first choose locations and then prices. Because the strategy spaces of this two-stage model are infinite dimensional, our theorems do not immediately apply.

4. INSURANCE MARKETS

We next consider a model of the market for insurance due to Rothschild and Stiglitz (1976) and Wilson (1977). Our formulation relies substantially on Hahn's (1978) analysis of the Rothschild-Stiglitz model.

There is one commodity (money) in this model, and, for each consumer, there are two states of nature: that of having an accident and that of not. Let goods 1 and 2 be money in the "no accident" and "accident" states, respectively. Each consumer has a strictly positive initial endowment $w \in R^2_+$ representing his initial allocation of money in the two states. His preferences are represented by a strictly concave von Neumann-Morgenstern utility function u. We normalize u so that $u(w)=0$. For convenience we suppose that w and u are the same for all individuals.

Consumers fall into two classes according to their accident proneness. High risk consumers have accidents with probability p_H and low risks with p_L, where $p_L < p_H$. A consumer knows which risk class he belongs to. Therefore, u and p_J $(J=L$ or $H)$ determine his preferences over goods 1 and 2. Clearly, the preferences of a high risk consumer differ from those of a low risk. Indeed, at any consumption pair, the marginal rate of substitution between goods 2 and 1 is greater for the low-risk consumer (see Figure 2). Let us assume a (large) fixed population n of consumers, of whom n_L are low risks and n_H are high risks.

There are two firms (insurance companies).[9] Firms sell insurance contracts which are vectors $c=(c_1, c_2) \in R^2$. One interprets c_1 as the insurance premium and c_2 as the accident benefit net of premium. Each consumer can purchase at most one insurance contract. Consumers of a given risk class buy from the firm offering the most desirable contract for them. (Of course, they will buy that contract only if they prefer it to their initial endowment.) If the two firms offer equally desirable contracts for a given risk class, the consumers in that class divide themselves equally between them.

Firms are expected profit maximizers—their revenues are premia and costs are claim payments—and they regard different consumers' chances of having an accident as independent. They know u, p_H, p_L, n_H and n_L, but cannot tell to which class any given consumer belongs.

A strategy for a firm is to offer a set of contracts. Since there are only two risk classes, it is never necessary for a firm to offer more than two contracts.[10] Therefore we shall only consider strategies consisting of pairs of contracts (c^H, c^L), where without loss of generality we adopt the convention that high risk consumers find c^H at least as desirable as c^L, and low risks find c^L at least as desirable as c^H.

For $J = H$, L and contract $c = (c_1, c_2)$, let $V_J(c)$ denote the expected utility of c for a consumer of class J. That is,

$$V_J(c) = p_J u(w_2 + c_2) + (1 - p_J) u(w_1 - c_1).$$

Let $\pi_J(c)$ be the expected profit from the sale of contract c to a consumer of risk class J. Then

$$\pi_J(c) = -p_J c_2 + (1 - p_J) c_1.$$

Suppose that firm 1 offers the contract pair $a_1 = (c^H(1), c^L(1))$ and firm 2, $a_2 = (c^H(2), c^L(2))$. Firm 1's expected profit from risk class J customers is therefore

$$\pi_1^J(a_1, a_2) = \begin{cases} n_J \pi_J(c^J(1)), & \text{if } V_J(c^J(1)) > V_J(c^J(2)) \\ (1/2) n_J \pi_J(c^J(1)), & \text{if } V_J(c^J(1)) = V_J(c^J(2)) \\ 0 & \text{otherwise.} \end{cases}$$

Naturally, π_2^J is defined symmetrically. Firm i's total expected profit is therefore,

$$U_i(a_1, a_2) = \pi_i^H(a_1, a_2) + \pi_i^L(a_1, a_2).$$

Consider the contract pair $a^* = (c^{*H}, c^{*L})$ satisfying $w_2 + c_2^{*H} = w_1 - c_1^{*H}$ (i.e. high risks are perfectly insured); $\pi_H(c^{*H}) = \pi_L(c^{*L}) = 0$ (i.e. each of the two risk classes generates zero expected profit); and $V_H(c^{*H}) = V_H(c^{*L})$, (i.e. high risk customers are indifferent between the low risk contract and their own). (See Figure 2.) We shall call a^* the "Rothschild-Stiglitz-Wilson" (R-S-W) contract pair. It is easy to see that, if a pure strategy equilibrium exists, both firms must offer a^*. First, no contract can earn positive profit because if, say, firm 1 offered such a contract, firm 2 could offer a contract with slightly better terms for high or low risks (whichever is the profitable class), thereby getting nearly all firm 1's profit for itself. (This is the "Bertrand"-like feature of the R-S-W model.) Second, among high risk contracts that earn zero profit, the most desirable from the customers' viewpoint is c^{*H}. Therefore c^{*H} is the only high risk contract not subject to the "undercutting" argument that ruled out positive profits. Finally, the most favourable zero-profit contract for low risks that has the property that high risks do not prefer it to their own is c^{*L}.

However, a^* may not be an equilibrium contract because if there is a sufficiently high proportion of low risks in the population, there exists a "pooling" contract c^{**} that earns positive profit overall (i.e. $\pi_H(c^{**}) + \pi_L(c^{**}) > 0$), and which both high and low risks prefer to a^*. (See Figure 2.)[11] We shall show below that nonetheless, a mixed strategy equilibrium always exists.

To place the insurance problem within the framework of our general existence theorem, we shall identify a contract c with the utility pair $(V_H(c), V_L(c))$. This identification is legitimate, since the mapping $c \to (V_H(c), V_L(c))$ is one-to-one. A strategy thus consists of offering a quadruple (V_H, V_L, V'_H, V'_L), where, by the convention we have adopted, $V_H \geqq V'_H$ and $V'_L \geqq V_L$. That is, (V_H, V_L) corresponds to the high risk contract and (V'_H, V'_L) to the low risk contract. We can therefore confine attention to the compact set

$$V = \{(V_H, V_L) | \exists c \text{ with } (V_H(c), V_L(c)) = (V_H, V_L),\ \pi_L(c) \geqq 0,\ \max\{V_L, V_H\} \geqq 0\},$$

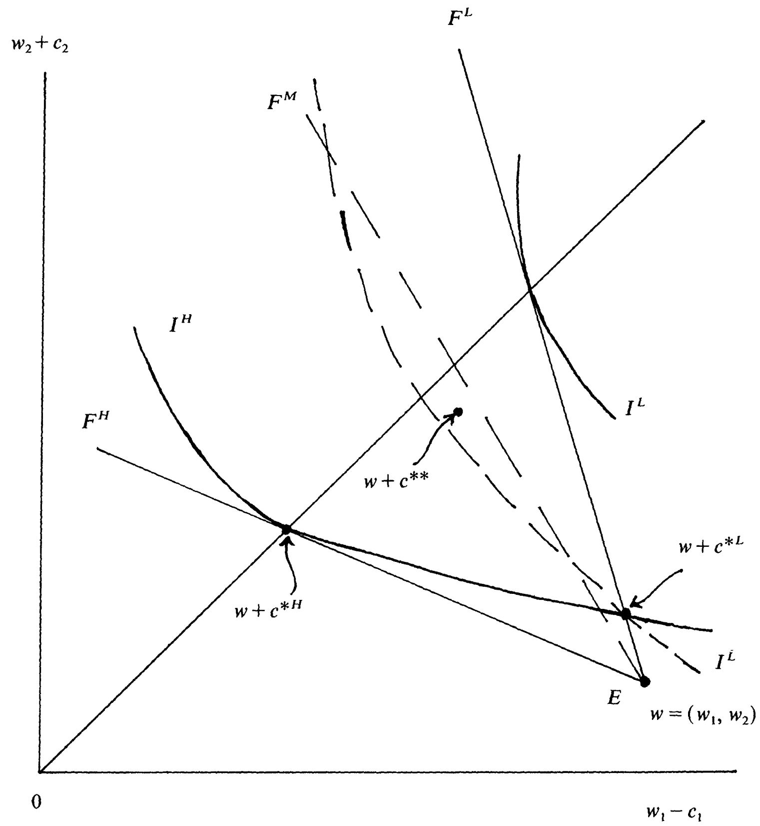

FIGURE 2

I^H is a high-risk indifference curve and I^L is a low-risk indifference curve. EF^H denotes the "fair-odds" line for high risks, (the locus $\pi_H = 0$), and EF^L denotes the "fair-odds" line for low-risks. EF^M is the "zero-profit" pooling line. (c^{*H}, c^{*L}) is the R-S-W contract pair. c^{**} is a pooling contract. The figure depicts a case where a pure strategy equilibrium does not exist

since any other contract either earns negative profit or is never purchased. Thus we can take firm i's strategy space to be

$$A_i = \{(V_H, V_L, V'_H, V'_L) | \{(V_H, V_L), (V'_H, V'_L)\} \subseteq V, V'_L \geqq V_L \text{ and } V_H \geqq V'_H\}.$$

We can now state,

Theorem 5. *The insurance market game has a symmetric equilibrium* (μ^*, μ^*). *Furthermore, for all* $(c^H, c^L) \neq (c^{*H}, c^{*L})$, with $\pi_H(c^H) = \pi_L(c^L) = 0$, $\mu^*(c^H, c^L) = 0$.

Proof. We must verify that the hypotheses of Theorem 6* in D-M (1986) are satisfied. However, as we have defined the payoff functions, the sum $U_1 + U_2$ is not continuous, nor even upper semicontinuous. This is because the discontinuities in firms' payoffs entail a shift of clientele from one firm's contracts to the other's, and so, if the contracts are not equally profitable, total profit changes discontinuously.[12] We, therefore, slightly modify the payoff functions to restore upper semicontinuity. After establishing the existence of equilibrium with these modified payoff functions, we show that such an equilibrium applies to the original payoffs.

Observe first that discontinuities in firm i's payoff function are confined to the set

$$A^*(i) = \{(a_1, a_2) \in A_1 \times A_2 | V_H(1) = V_H(2) \text{ or } V'_L(1) = V'_H(2) \text{ where } a_i = (V_H(i), V_L(i), V'_H(i), V'_L(i))\}.$$

For $(a_1, a_2) \in A^*(1)$ and $i = 1, 2$, define

$$\bar{U}_i(a_1, a_2) = \lim_{\varepsilon \to 0} \sup_{\substack{\|a'_1 - a_1\| < \varepsilon \\ \|a'_2 - a_2\| < \varepsilon}} U_1(a'_1, a'_2).$$

Take

$$U_i^*(a_1, a_2) = \begin{cases} \bar{U}_i(a_1, a_2), & \text{if } (a_1, a_2) \in A^*(1) \text{ and } a_1 \neq a_2 \\ U_i(a_1, a_2), & \text{otherwise.} \end{cases}$$

Clearly U_i^* is bounded. To see that $U_1^* + U_2^*$ is upper semicontinuous, consider a sequence $\{(a_1^n, a_2^n)\}$ converging to (a_1, a_2). If $(a_1, a_2) \notin A^*(1)$ then U_1^* and U_2^* are continuous at (a_1, a_2). If $a_1 = a_2$ then $U_1^* + U_2^*$ is continuous at (a_1, a_2) because the discontinuities in U_1 and U_2 simply entail shifting profit from one firm to another. If $(a_1, a_2) \in A^*(1)$ and $a_1 \neq a_2$, then U_i^* is upper semicontinuous at (a_1, a_2) by construction. Therefore $U_1^* + U_2^*$ is upper semicontinuous. Observe that the discontinuities in firm 1's payoff function, U_1^*, are confined to $A^*(1)$, which meets the requirements of Theorem 2 on the form of the discontinuity set. Consider a point $(a_1, a_2) \in A^*(1)$. Let $a_1 = (c^H(1), c^L(1))$. Suppose that $V_H(1) = V_H(2)$. (The argument is similar if $V'_L(1) = V'_L(2)$). If $\pi_H(c^H(1)) > 0$, then U_1 and hence U_1^* is right lower semicontinuous in the $V_H(1)$ component at (a_1, a_2). If $\pi_H(c^H(1)) \leqq 0$, then U_1^* is left lower semicontinuous in that component.

Now

$$A_1^{**}(1) = \{(c^H, c^L) \in A_1^*(1) | \pi_H(c^H) \neq 0 \text{ or } \pi_L(c^L) \neq 0\}.$$

Consider $(c^H, c^L) \in A_1^{**}(1)$. If, say, $\pi^H(c^H) < 0$, then for every $(\hat{c}^H, \hat{c}^L)$ such that $(c^H, c^L, \hat{c}^H, \hat{c}^L) \in A_1^{**}(1)$ U_1^* is left lower semicontinuous in V_H. Moreover, when $(\hat{c}^H, \hat{c}^L) = (c^H, c^L)$, U_1, and hence U_1^*, fails to be left upper semicontinuous in V_H. Similar conclusions can be reached when $\pi_H(c^H) > 0$ or $\pi_L(c^L) \neq 0$. Therefore property (α^*) of Theorem 6* in D–M (1986) holds. From this theorem, we conclude that there exists a symmetric equilibrium (μ^*, μ^*) for the payoff functions U_1^* and U_2^*, where μ^* is atomless on $A_i^{**}(i)$. Because μ^* is atomless and U_i and U_i^* differ only on A_i^{**},

$$\int U_1(\cdot, a_2) d\mu^*(a_2) = \int U_1^*(\cdot, a_2) d\mu^*(a_2).$$

Therefore (μ^*, μ^*) is an equilibrium for U_1 and U_2 as well.

To see that $\mu^*(\bar{c}^H, \bar{c}^L) = 0$, if $\pi_H(\bar{c}^H) = \pi_L(\bar{c}^L) = 0$ but $(\bar{c}^H, \bar{c}^L) \neq (c^{*H}, c^{*L})$, observe that if, say, firm 2 placed positive probability on $(\bar{c}^H, \bar{c}^L)$, where $\bar{c}^H \neq c^{*H}$, firm 1 could obtain a positive expected profit by choosing $(\hat{c}^H, \bar{c}^L)$ where $\pi_H(\hat{c}^H) > 0$ but $V_H(\hat{c}^H) > V_H(\bar{c}^L)$, contradicting the supposition that a zero-profit contract pair is part of an equilibrium strategy. ||

Appealing to Dasgupta and Maskin (1977), Appendix, we can say more about the nature of mixed strategy equilibrium.[13] First, as in pure strategy equilibrium, firms earn zero expected profit. Second, any high risk contract offered provides full insurance and either loses money or breaks even. That is, if (μ_1^*, μ_2^*) is an equilibrium, $\mu_i^*\{(c^H, c^L) | \pi_H(c^H) \leqq 0 \text{ and } w_2 + c_2^H = w_1 - c_1^H\} = 1$, for $i = 1, 2$. Third, any low risk contract offered provides less than full insurance and either makes money or breaks even.

That is, $\mu_i^*\{(c^H, c^L)|\pi_L(c^L)\geqq 0$ and $w_2+c_2^L< w_1-c_1^L\}=1$, $i=1,2$. Finally, high risk customers are indifferent between any high risk-low risk pair offered: $\mu_i^*\{(c^H, c^L)|V_H(c^H)=V_H(c^L)\}=1$, $i=1,2$.[14]

5. CLASSIFICATION

Because they take prices as given, agents in the Arrow-Debreu theory of resource allocation have payoff functions that are continuous (see Arrow and Debreu (1954)). Therefore it is no accident that the discontinuities that we have reviewed arise in models of imperfect competition.

One conclusion we can draw from our work, however, is that these discontinuities are *inessential*—not only in the sense that they do not prevent the existence of mixed-strategy equilibrium—but, more importantly, that the discontinuous games can be approximated arbitrarily closely by games to which the classical existence theorems (Theorem 3 of D-M (1986)) apply. In the Introduction to D-M (1986) we mentioned one form of approximation, namely the selection of a finite set of strategies (to which Nash's (1951) theorem then applies). Indeed, our method of proving Theorems 5* and 6* in D-M (1986) is through successively finer finite approximations. Alternatively, we could have retained the infinite strategy spaces, but introduced *exogenous* uncertainty in such a way that agents' *expected* payoffs are continuous. (For an application of this device to patent race games, see Dasgupta and Stiglitz (1980)). If the uncertainty is "small" the game is an approximation to the original discontinuous game. By analogy with our "finite" arguments one can show that as the uncertainty diminishes a subsequence of mixed-strategy equilibria in these smoothed games converges to a mixed-strategy equilibrium in the discontinuous game. Therefore in this alternative sense the equilibria of the discontinuous games are "robust".

The economic games we have studied can be classified according to the extent that there are "winners" and "losers". One should note that in all the examples, except for Hotelling's location model, a firm can always increase its market share by moving in a single consistent direction. Thus, under price competition firms increase their market share by cutting prices. In extreme cases, such as the Rothschild-Stiglitz-Wilson game, a firm can capture the entire market by undercutting its rivals infinitesimally. Such models are pure "winner-loser" games, or *contests*, in the sense that the winner takes all by outdoing its rivals. Other examples of such games are wars of attrition (Riley (1980)), patent races (Dasgupta and Stiglitz (1980)) and auctions where the higher bid wins (e.g. Dasgupta (1982)). In the symmetric versions of pure winner-loser games symmetric equilibria invariably involve a zero expected payoff for each player (where zero is the utility level of a non-participating player).[15] Some models that we have examined, namely the Bertrand-Edgeworth and Hotelling price competition models, share with contests the property that undercutting increases market share; however these are not pure winner-loser games because either the firm cannot physically accommodate all customers (as in Bertrand-Edgeworth) or else has inherent monopoly power over some portion of the market (as in Hotelling). Even the symmetric equilibria of such games need not entail zero expected payoffs.

By contrast, Hotelling's spatial location model is not a contest at all. It has no winners and losers and the direction of increasing market share depends on the actions of other firms. If, say, in the case of three firms on a line the firm in the middle moves in either direction it encroaches on the market area of the firm it is moving towards, but at the same time loses clientele to the third firm, and these effects counter-balance.

The fact that, in contests, and partial contests (like Bertrand–Edgeworth), there is a favoured direction of movement—and hence a distinguished point at the extremity of this direction—helps us understand the nature of equilibrium to such games. In particular, it helps explain why symmetric equilibrium may involve atoms at such distinguished points. For example, symmetric equilibrium in the insurance game may have an atom at the Rothschild–Stiglitz–Wilson contract. Similarly the pure Bertrand game has all firms setting the competitive price.[16] Viewed in the light of Theorem 6* in D–M (1986) these atoms are possible because at the distinguished point firms' payoff functions may not be discontinuous, since "undercutting" is no longer possible.

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NOTES

1. The Bertrand–Edgeworth model has been much studied. See e.g. Chamberlin (1956), Shubik (1955), Beckmann (1965) and d'Aspremont and Gabszewicz (1980). The original references are Bertrand (1883) and Edgeworth (1925).

2. When $a_1 = a_2$, firm 1 serves the minimum of S_1 and $Q(a_1)S_1/(S_1+S_2)$ customers as long as firm 2 is not capacity constrained (i.e. as long as $S_2 \geqq Q(a_1)S_2/(S_1+S_2)$). If firm 2 *is* capacity constrained, then firm 1 supplies the remainder of the market. When $a_1 > a_2$, if $Q(a_2) < S_2$ then clearly $U_1 = 0$. If, however, $Q(a_2) > S_2$, then the fraction of consumers that are served by firm 2 is $\alpha \equiv S_2/Q(a_2)$. Thus $(1-\alpha)$ is the proportion of consumers not served by 2. Total sales by firm 1 amount to $(1-\alpha)Q(a_1)$.

3. There cannot be an equilibrium with $a_1 < a_2$, since, if $Q(a_1) > S_1$, firm 1 will wish to raise its price and, if $Q(a_1) \leqq S_1$, firm 2 will wish to lower its price. Similarly, no equilibrium is possible with $a_2 > a_1$. There cannot be an equilibrium with $a_1 = a_2 > \bar{a}$ (the competitive price), since both firms have the incentive to lower their prices slightly. Consider now the case where $\bar{a} > 0$. If $Q(a)$ is inelastic at $\bar{a}$ then $a_1 = a_2 = \bar{a}$ cannot be an equilibrium because either firm can increase its profit by raising its price slightly above $\bar{a}$ if the other firm were to charge $\bar{a}$. In this case an equilibrium does not exist. One may argue that non-existence of an equilibrium in pure strategies is due to the fact that a firm's payoff function is not quasi-concave in its own price. To see this consider a slight modification of the above model. Suppose that there is a positive number η, such that if the firms choose prices *within* a distance of η of each other both receive zero profits; but if their price difference exceeds or is equal to η their profit functions are as in (1) and (2). It is easy to check that if η is small enough the model does not have an equilibrium in pure strategies if market demand is inelastic at $\bar{a}$. However, the payoff functions of this modified game are both upper-semi-continuous and graph continuous (Definitions (2), and (3) in D–M (1986)). From Theorem 3 of D–M (1986) we may conclude that the failure of the modified game to possess an equilibrium in pure strategies is due to the fact that a firm's profit function is not quasi-concave in its own price.

4. This theorem also helps explain why the technique, used by Beckmann (1965), of expressing the equilibrium conditions as a differential equation actually works.

5. We are indebted to Martin Hellwig for drawing our attention to an error in an earlier version of the proof of this Corollary.

6. We assume a uniform distribution for expository ease. Any non-atomic distribution can be assumed.

7. See d'Aspremont *et al.* (1979). To see this, one argues first that if an equilibrium exists, say (a_1^*, a_2^*), then it must be the case that $|a_1^* - a_2^*| \leqq c(x_2 - x_1)$, and so both serve the market. (If $|a_1^* - a_2^*| > c(x_2 - x_1)$, the firm charging the higher price makes zero profit, and can gain by lowering its price sufficiently.) But if, say, firm 1 sets $a_1 = a_2^* - c(x_2 - x_1) - \varepsilon$ (where ε is positive and "small") it captures the entire market. If the firms are located near each other then firm 1 does better by charging this than a_1^*.

8. This is easy to check. If all firms coincide at a point, any one can increase its profit by moving slightly away in some direction. But if they do not coincide, the outer flank firms can gain by moving closer to the firm in the middle. Hence there is no equilibrium configuration of locations. Eaton and Lipsey demonstrate, however, that for any value of N other than three, an equilibrium in pure strategies exists in the one-dimensional case.

9. We could as easily handle more than two.

10. To see this, suppose C is the set of contracts that firm i $(i = 1, 2)$ offers. C can be subdivided into C_H, C_L, C_K, the contracts which, given the contracts offered by the other firm, are optimal for high risks, low risks, and neither class, respectively. But since u is strictly concave, there are unique (expected) profit maximizing contracts c^H and c^L within C_H and C_L, respectively. Hence, offering (c^H, c^L) is at least as good as offering C and strictly better if C_H or C_L contain contracts other than c^H and c^L.

11. One can argue, as we did when commenting on the Bertrand-Edgeworth problem in footnote 3, that the reason for the non-existence of an equilibrium in pure strategies in the R-S-W model can be traced to the fact that a firm's payoff is not quasi-concave in its own strategy. To see this modify the R-S-W model slightly by introducing "minimal sensibility" on the part of customers. Suppose that there exist positive numbers η_H and η_L such that if contracts C_1 and C_2 are offered and C_1 is purchased by high risk customers then C_2 is also purchased by them if and only if

$$-\eta_H < V_H(C_2) - V_H(C_1) \leqq \eta_H,$$

and, likewise, if C_1 is purchased by low risk customers then C_2 is also purchased by them if and only if

$$-\eta_L \leqq V_L(C_2) - V_L(C_1) < \eta_L.$$

One can now verify that with this modification the expected profit function of insurance firms are upper semi-continuous. (See Dasgupta and Maskin (1977) for details.) They are of course graph-continuous even in the original R-S-W model. But if η_H and η_L are small enough an equilibrium in pure strategies does not exist. We may therefore conclude from Theorem 2 in D-M (1986) that the reason for the non-existence of an equilibrium in pure strategies in this modified model is the fact that the expected profit function of a firm is not quasi-concave in its own strategy.

12. We are grateful to David Salant, who reminded us of this point, and therefore helped correct an error in a previous version.

13. Lemma 8 in the Appendix of Dasgupta and Maskin (1977) is false. The remaining Lemmas are correct and it is these that we are summarizing below in the text.

14. Rosenthal and Weiss (1983) have explicitly computed a symmetric equilibrium in mixed strategies for a particular specification of preferences in the R-S-W model of insurance.

15. Many winner-loser games have asymmetric equilibria that do not have the zero-expected-payoff property. See e.g. Riley (1980) and Maskin and Riley (1982).

16. By contrast, in the spatial location model with three or more firms no point is distinguished, and so *symmetric* equilibrium entails no atoms.

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