ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM. A CORRECTION

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In a paper entitled "On computable numbers, with an application to the Entscheidungsproblem"* the author gave a proof of the insolubility of the Entscheidungsproblem of the "engere Funktionenkalkül". This proof contained some formal errors† which will be corrected here: there are also some other statements in the same paper which should be modified, although they are not actually false as they stand.

The expression for Inst $\{q_i S_j S_k L q_l\}$ on p. 260 of the paper quoted should read

$$(x, y, x', y') \Big\{ \Big(R_{S_j}(x, y) \,\&\, I(x, y) \,\&\, K_{q_i}(x) \,\&\, F(x, x') \,\&\, F(y', y) \Big)$$

$$\rightarrow \Big(I(x', y') \,\&\, R_{S_k}(x', y) \,\&\, K_{q_l}(x') \,\&\, F(y', z) \vee \Big[\Big(R_{S_0}(x, z) \rightarrow R_{S_0}(x', z) \Big)$$

$$\& \Big(R_{S_1}(x, z) \rightarrow R_{S_1}(x', z) \Big) \,\&\, \ldots \,\&\, \Big(R_{S_M}(x, z) \rightarrow R_{S_M}(x', z) \Big) \Big] \Big) \Big\},$$

$S_0, S_1, \ldots, S_M$ being the symbols which $\mathcal{M}$ can print. The statement on p. 261, line 33, viz.

$$\text{"Inst}\, \{q_a S_b S_d L q_c\} \,\&\, F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

is provable" is false (even with the new expression for Inst $\{q_a S_b S_d L q_c\}$): we are unable for example to deduce $F^{(n+1)} \rightarrow \big(-F(u, u'')\big)$ and therefore can never use the term

$$F(y', z) \vee \Big[\Big(R_{S_0}(x, z) \rightarrow R_{S_0}(x', z) \Big) \,\&\, \ldots \,\&\, \Big(R_{S_M}(x, z) \rightarrow R_{S_M}(x', z) \Big) \Big]$$

* *Proc. London Math. Soc.* (2), 42 (1936–7), 230–265.

† The author is indebted to P. Bernays for pointing out these errors.

in $\text{Inst}\{q_a S_b S_d L q_c\}$. To correct this we introduce a new functional variable G [$G(x, y)$ to have the interpretation "x precedes y"]. Then, if Q is an abbreviation for

$$(x)(\exists w)(y, z)\Big\{F(x, w) \,\&\, \big(F(x, y) \to G(x, y)\big) \,\&\, \big(F(x, z) \,\&\, G(z, y) \to G(x, y)\big)$$
$$\&\Big[G(z, x) \vee \big(G(x, y) \,\&\, F(y, z)\big) \vee \big(F(x, y) \,\&\, F(z, y)\big) \to \big(-F(x, z)\big)\Big]\Big\}$$

the corrected formula $\text{Un}(\mathfrak{M})$ is to be

$$(\exists u) A(\mathfrak{M}) \to (\exists s)(\exists t) R_{S_1}(s, t),$$

where $A(\mathfrak{M})$ is an abbreviation for

$$Q \,\&\, (y) R_{S_0}(u, y) \,\&\, I(u, u) \,\&\, K_{q_1}(u) \,\&\, \text{Des}(\mathfrak{M}).$$

The statement on page 261 (line 33) must then read

$$\text{Inst}\{q_a S_b S_d L q_c\} \,\&\, Q \,\&\, F^{(n+1)} \to (CC_n \to CC_{n+1}),$$

and line 29 should read

$$r\big(n, i(n)\big) = b, \quad r\big(n+1, i(n)\big) = d, \quad k(n) = a, \quad k(n+1) = c.$$

For the words "logical sum" on p. 260, line 15, read "conjunction". With these modifications the proof is correct. $\text{Un}(\mathfrak{M})$ may be put in the form (I) (p. 263) with $n = 4$.

Some difficulty arises from the particular manner in which "computable number" was defined (p. 233). If the computable numbers are to satisfy intuitive requirements we should have:

If we can give a rule which associates with each positive integer n two rationals a_n, b_n satisfying $a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n$, $b_n - a_n < 2^{-n}$, then there is a computable number α for which $a_n \leqslant \alpha \leqslant b_n$ each n. (A)

A proof of this may be given, valid by ordinary mathematical standards, but involving an application of the principle of excluded middle. On the other hand the following is false:

There is a rule whereby, given the rule of formation of the sequences a_n, b_n in (A) we can obtain a D.N. for a machine to compute α. (B)

That (B) is false, at least if we adopt the convention that the decimals of numbers of the form $m/2^n$ shall always terminate with zeros, can be seen in this way. Let $\mathfrak{N}$ be some machine, and define c_n as follows: $c_n = \frac{1}{2}$ if $\mathfrak{N}$ has not printed a figure 0 by the time the n-th complete configuration is reached $c_n = \frac{1}{2} - 2^{-m-3}$ if 0 had first been printed at the m-th complete configuration ($m \leqslant n$). Put $a_n = c_n - 2^{-n-2}$, $b_n = c_n + 2^{-n-2}$. Then the inequalities of (A) are satisfied, and the first figure of α is 0 if $\mathcal{N}$ ever prints 0 and is 1 otherwise. If (B) were true we should have a means of finding the first figure of α given the D.N. of $\mathcal{N}$: *i.e.* we should be able to determine whether $\mathcal{N}$ ever prints 0, contrary to the results of § 8 of the paper quoted. Thus although (A) shows that there must be machines which compute the Euler constant (for example) we cannot at present describe any such machine, for we do not yet know whether the Euler constant is of the form $m/2^n$.

This disagreeable situation can be avoided by modifying the manner in which computable numbers are associated with computable sequences, the totality of computable numbers being left unaltered. It may be done in many ways* of which this is an example. Suppose that the first figure of a computable sequence γ is i and that this is followed by 1 repeated n times, then by 0 and finally by the sequence whose r-th figure is c_r; then the sequence γ is to correspond to the real number

$$(2i-1)n + \sum_{r=1}^{\infty} (2c_r - 1)(\tfrac{2}{3})^r.$$

If the machine which computes γ is regarded as computing also this real number then (B) holds. The uniqueness of representation of real numbers by sequences of figures is now lost, but this is of little theoretical importance, since the D.N.'s are not unique in any case.

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* This use of overlapping intervals for the definition of real numbers is due originally to Brouwer.