The Network Equilibrium Problem in Integers

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ABSTRACT

In the usual approach to network equilibrium models, the flow variables are modeled as continuous. When the problem under study involves discrete decision makers each controlling an indivisible unit of flow, another approach is called for. We treat the problem as an n-person noncooperative game with pure strategies corresponding to feasible paths through the network. It is shown that pure-strategy Nash equilibria exist and that any solution to an integer-variable analogue of the usual network equilibrium model is such a Nash equilibrium. It is also shown that when individuals can control more than a single unit of flow and want to minimize the sum of their costs, pure-strategy Nash equilibria do not necessarily exist.

1. INTRODUCTION

A directed network of m arcs is given. Each of n individuals must select a directed path from his origin to his destination. A directed path from an individual's origin to his destination is termed feasible for that individual. If x_k individuals choose feasible paths containing arc k in the network, the cost to each of these individuals is $c_k(x_k)$ for that part of the journey. c_k is assumed to be nonnegative and nondecreasing as a function of x_k for $k = 1, \ldots, m$. If all individuals have chosen their routes, the total cost to an individual traversing path S is $\sum_{k \in S} c_k(x_k)$. An equilibrium for the system is a set of feasible paths, one for each individual,

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such that no individual can decrease his total cost by switching unilaterally to some other feasible path. We shall assume in all that follows that at least one feasible path exists for each individual.

The standard application of this type of model is to interpret cost on an arc as the time needed to travel a road. The individual seeks to minimize his total travel time assuming that all other individuals' paths are fixed. The system is in equilibrium when no individual can improve his position by changing to another route.

There are at least two fruitful ways to approach this problem. Firstly, one might model the various flows as continuous, rather than discrete, variables. This approach, reminiscent of models of electrical networks, has been frequently taken. (See, for example, Beckmann et.al. [1956]; Charnes and Cooper [1958]; Charnes and Cooper [1961]; Dafermos and Sparrow [1969].) Secondly, one might view the problem as an n-person, noncooperative game in which the pure strategies correspond to paths in the network. Nash equilibria for such games are sought.

In section 2 we describe the usual continuous-flow approach to this problem and object to its use whenever the flows of the system must be integer-valued. In section 3 the model is described as an n-person noncooperative game for which Nash equilibria are sought. It is shown that pure-strategy Nash equilibria always exist for games of this type and that any solution to a certain integer-variable analogue of the continuous-flow problem of section 2 is such an equilibrium. In section 4, an example is presented which illustrates that pure-strategy Nash equilibria need not exist if individuals are allowed to control more than one unit of flow and want to minimize the sum of their costs.

2. THE CONTINUOUS-VARIABLES MODEL

Let x_{ik} denote the fraction of individual i's flow which passes through arc $k(i=1,\ldots,n;\ k=1,\ldots,m)$ where the $c_k(\cdot)$ are now assumed to be defined over entire intervals. Consider the problem:

$$\text{minimize} \sum_{k=1}^{m} \int_{0}^{x_k} c_k(t)\,dt$$

subject to $x_k = \sum_{i=1}^{n} x_{ik}$ for $k = 1, \ldots, m$; the equations characterizing conservation of flow for each individual at each node; and $0 \leq x_{ik} \leq 1$ for each individual and each arc. If the x_{ik} values which solve this problem are all integer-valued, they represent an equilibrium for the system. If not, and the individual flows are in actuality not divisible, one may have difficulty interpreting the solution. Firstly, if it is used as an approximation to a true equilibrium in a large problem, in what sense is it a good approximation? Secondly, how should one interpret the costs (or travel times) for fractional flows?

In the next section, we shall present an example which indicates at least one sense in which the approximation is not good. In the following example we discuss the interpretation of costs for fractional flows. Consider a single individual with two arcs leading from his origin to his destination; $c_1(x_1) = \sqrt{x_1}$ and $c_2(x_2) = \sqrt{x_2}$. Restricted to integer flows, the single unit of flow may be sent down either path with cost of one. Both are equilibria. The continuous-variables solution is to send 1/2 unit of flow through each arc. If the costs are viewed as travel times, then each half unit arrives in $\frac{\sqrt{2}}{2}$ time units. Thus, the total flow arrives in $\frac{\sqrt{2}}{2} < 1$ time units. We seem to have minimized the maximum travel time for any part of the flow. Alternatively, we have minimized the weighted average of the travel times; i.e., $x_1 \sqrt{x_1} + x_2 \sqrt{x_2}$. Both of these interpretations are valid in general. As in this example, however, it is not generally true that the sum of the travel times is minimized.

3. THE GAME-THEORETIC MODEL

The individuals are assumed to be playing a game in which the pure strategies for each are the individuals' feasible paths. The payoffs (to be minimized) are the sums of the costs of the arcs used. Nash equilibria are sought. In this case these correspond to equilibria for the system. For general n-person games, however, one is not guaranteed that any Nash equilibria must exist; unless the individual strategy sets are extended to include all possible randomizations over the sets of pure strategies. (See Nash [1951].) (The cost of playing a randomized strategy is taken to be the expected cost over the relevant pure strategies.) These randomizations do not correspond to fractional solutions to the continuous-variables model. For this class of games, however, it is not necessary to introduce randomizations, since pure-strategy Nash equilibria always exist.

Theorem: In games derived from network equilibrium models, pure-strategy Nash equilibria always exist. Furthermore, any solution to the following problem is a pure-strategy Nash equilibrium:

$$
* \begin{cases}
\textit{Minimize} \quad \sum_{k=1}^{m} \left(\sum_{t=0}^{x_k} c_k(t) \right) \\
\textit{subject to } x_k = \sum_{i=1}^{n} x_{ik} \textit{ for } k = 1, \ldots, m; \textit{ the} \\
\textit{equations characterizing conservation of flow at} \\
\textit{each node for each individual; and } x_{ik} = 0 \textit{ or } 1 \\
\textit{for } i = 1, \ldots, n;\ k = 1, \ldots, m.
\end{cases}
$$

Proof: Since solutions to (*) always exist (whenever the constraints are consistent) it suffices to establish that each is an equilibrium. Let (x'_{ik}) solve (*). Assume it does not give rise to an equilibrium. Then some individual j taking some path S under (x'_{ik}) can reduce his cost by switching to some path T; i.e. $\sum_{k \varepsilon T \setminus S} c_k(x'_k + 1) < \sum_{k \varepsilon S \setminus T} c_k(x'_k)$. (Note that it suffices to show that no pure strategy is better). Consider the new values

$$
x^o_{ik} = \begin{Bmatrix} x'_{ik} + 1 \text{ if } i = j,\ k \varepsilon T \setminus S \\ x'_{ik} - 1 \text{ if } i = j,\ k \varepsilon S \setminus T \\ x'_{ik} \quad \text{otherwise} \end{Bmatrix}
$$

(x^o_{ik}) is clearly feasible for (*). The objective function evaluated at (x^o_{ik}) is:

$$
\sum_{k \varepsilon T \setminus S} \sum_{t=0}^{x'_k+1} c_k(t) + \sum_{k \varepsilon S \setminus T} \sum_{t=0}^{x'_k-1} c_k(t) + \sum_{k \varepsilon (S \cap T) \cup (S^c \cap T^c)} \sum_{t=0}^{x'_k} c_k(t)
$$

$$
= \sum_{k=1}^{m} \sum_{t=0}^{x'_k} c_k(t) + \sum_{k \varepsilon T \setminus S} c_k(x'_k + 1) - \sum_{k \varepsilon S \setminus T} c_k(x'_k)
$$

$$
< \sum_{k=1}^{m} \left(\sum_{t=0}^{x'_k} c_k(t) \right). \quad \text{A contradiction.} \,||
$$

The problem (*) is, of course, closely related to the minimization problem associated with the continuous-variables model. In particular, if the $c_k(\cdot)$ in the continuous-variables problem happen to be appropriate step functions, the two minimands are the same. (Note that the proof does not make use of the assumption that the $c_k(\cdot)$ are nonnegative or nondecreasing.)

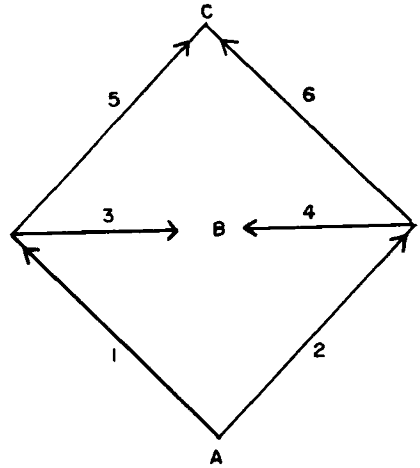

Fig. 1

Not every pure-strategy equilibrium solves (*). In Figure 1, individual 1 travels from A to B; individual 2 travels from A to C. The arc costs are:

$$c_1(x_1) = x_1^2, \; c_2(x_2) = x_2^2, \; c_3(x_3) = 0,$$

$$c_4(x_4) = 0, \; c_5(x_5) = 0, \; c_6(x_6) = 1.$$

If individual 1 takes arcs 1 and 3 and individual 2 takes arcs 2 and 6, this clearly results in an equilibrium. Also an equilibrium results if individual 1 takes arcs 2 and 4 and individual 2 takes arcs 1 and 5. The second equilibrium solves (*) and the continuous-variables problem. The first solves neither.

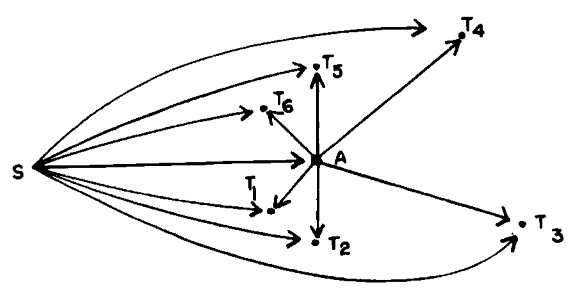

Fig. 2

The continuous-variables model is sometimes used as an approximation to the discrete model for large problems. While this approach may be good enough for many practical purposes, one should keep in mind the potential pitfalls of rounding to nearby integer solutions. In particular, the amount of flow on a particular arc may be greatly distorted. Consider the network in Figure 2. Six individuals originate at node S, each traveling to a different T_i (i=1,...,6). Each has two routes: a direct route which costs 40 times the square of the flow; and an indirect route through node A. The cost on the arc from S to A is seven plus the flow. The arcs from A to the various T_i all possess zero costs. At the continuous-variables solution each individual sends half of his flow on each of his feasible paths. At the unique equilibrium all of the flow passes through node A. Thus the traffic on the arc from S to A is 3 at the "approximate" solution and 6 at the equilibrium.

4. DISPATCHING

In this section, we extend the model to allow for the possibility that an individual may control more than one unit of flow. An example from traffic flow might be central dispatching of taxicabs. For the extended model we shall show by example that pure-strategy equilibria need not exist when a dispatcher wants to minimize the sum of his costs. This may help to point up the somewhat surprising nature of the theorem in the previous section.

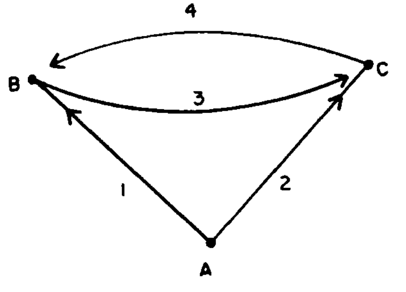

Fig. 3

In Figure 3, one vehicle travels from A to B under the control of player 1. Two vehicles travel from A to C. The two vehicles travelling from A to C are under the control of the same individual, player 2. The costs are:

$$c_1(x_1) = 2x_1 + 3, \quad c_2(x_2) = \begin{cases} 0 \text{ if } x_2 = 1 \\ 6 \text{ if } x_2 = 2, \\ 7 \text{ if } x_2 = 3 \end{cases} \quad c_3(x_3) = 4x_3, \quad c_4(x_4) = 0.$$

Denote a path to either destination which commences on arc 1 as L and on arc 2 as R. Represent the first player's pure strategies as rows, the second player's as columns. The payoffs in Table 1 will be the total costs to players 1 and 2, respectively, of the relevant strategy combinations. The game in Table 1 has no pure-strategy Nash equilibria when both players are minimizers.

	2L,OR	1L,1R	OL,2R
L	9,34	7,11	5,12
R	0,30	6,15	7,14

Table 1

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REFERENCES

1. Beckmann, M., C. McGuire and C. Winsten, *Studies in the Economics of Transportation*, Yale University Press, New Haven, 1956, Chapter 3.

2. Charnes, A. and W. Cooper, "Extremal Principles for Simulating Traffic Flow in a Network," *Proc. of National Academy of Sciences*, Vol. 44, 1958, pp. 201-204.

3. Charnes, A. and W. Cooper, *Management Models and Industrial Applications of Linear Programming*, Vol. II, 1961, John Wiley & Sons, New York, pp. 785-797.

4. Dafermos, S. and F. Sparrow, "The Traffic Assignment Problem for a General Network," *Journal of Research of the National Bureau of Standards*, Vol. 73B, 1969, pp. 91-118.

5. Nash, J., "Noncooperative Games," *Annals of Mathematics*, Vol. 54, 1951, pp. 286-295.

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