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Positional Strategies for Mean Payoff Games

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Abstract: We study some games of perfect information in which two players move alternately along the edges of a finite directed graph with weights attached to its edges. One of them wants to maximize and the other to minimize some means of the encountered weights.

Introduction

We study two games of perfect information, one infinite, Γ, and another finite, G. They are two person games but the first is not 0-sum. The players take turns moving a pawn from a designated starting position over the edges of a finite directed graph with real numbers attached to its edges. The payoffs are certain means of the encountered numbers. Γ originally motivated this study, but G seems equally interesting. The results assert the existence of optimal positional strategies, i.e., strategies securing the optimal payoff, if used against a perfect opponent, and such that the choices depend only on the position of the pawn and do not depend on the previous choices. An amusing feature of our proofs is that we have to use both games to establish our claims about any one of them.

Some other facts about the existence of optimal positional strategies are known, see *Mycielski*[1966, section 0].

Results

Let A be a finite set, $a_0 \in A$, $P_0 \subseteq A \times A$, $Q_0 \subseteq A \times A$, $P = P_0 \times \{0\}$ and $Q = Q_0 \times \{1\}$. We think of P and Q as of two directed graphs over the finite set of vertices A; if $x = ((a, b), c) \in P \cup Q$ then x is called an arrow, a is called the tail of x, b is called the head of x and c the color of x. We assume that a_0 is a tail of some arrow of P, the head of every arrow of P is the tail of some arrow of Q and the head of every arrow of Q is the tail of some arrow of P. Let $\varphi: P \cup Q \to \mathbf{R}$, where $\mathbf{R}$ is the real line.

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We define a two person game $\Gamma = \Gamma(a_0, P, Q, \varphi)$ with perfect information and with the following rules. Player I chooses any arrow $x_0 \in P$ with tail a_0, then player II chooses $x_1 \in Q$ with tail equal to the head of x_0 and again I chooses $x_2 \in P$ with tail equal to the head of x_1 etc. After the infinite sequence $x = (x_0, x_1, \ldots)$ is completed I looses the value

$$\nu_{\mathrm{I}}(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i)$$

and II wins the value

$$\nu_{\mathrm{II}}(x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i).$$

Let D_0 be the domain of P_0, i.e., $D_0 = \{a\colon \exists b \in A\ [(a, b) \in P_0]\}$, and D_1 be the domain of Q_0. A *positional strategy for player I* is a function $f\colon D_0 \to A$ such that $(a, f(a)) \in P_0$ for all $a \in D_0$ and a *positional strategy for player II* is a function g: $D_1 \to A$ such that $(a, g(a)) \in Q_0$ for all $a \in D_1$. If I decides to use f and II decides to use g then the resulting play is

$$((a_0, f(a_0)), 0), ((f(a_0), gf(a_0)), 1), ((gf(a_0), fgf(a_0)), 0), \ldots$$

Theorem 1: There exists a value $\nu(\Gamma)$ such that player I has a positional strategy which secures $\nu_{\mathrm{I}}(x) \leqslant \nu(\Gamma)$ and player II has a positional strategy which secures $\nu_{\mathrm{II}}(x) \geqslant \nu(\Gamma)$.

Now we define a finite game $G = G(a_0, P, Q, \varphi)$ which is also a game of perfect information and is played in the same way as Γ except that it ends as soon as one of the players chooses an arrow x_n whose head equals the tail of an arrow x_m chosen earlier by the other player. (Notice that $m + n$ is odd and the plays of G have no more than $2\,|A|$ moves.) Then player I pays to II the value

$$\frac{1}{n-m+1} \sum_{i=m}^{n} \varphi(x_i).$$

Since G is a finite 0-sum game of perfect information hence it has a value $\nu(G)$. Positional strategies for G are defined in the same way as for Γ.

Theorem 2: $\nu(G) = \nu(\Gamma)$ and there are positional strategies for each of the players which secure $\nu(G)$ both in G and in Γ.

This result about the existence of good positional strategies in G is somewhat surprising, as one may think that the knowledge of previous positions may be necessary for closing a loop securing $\nu(G)$. Our tempo conditions are essential for the validity of Theorem 2, i.e., if the definition of G was modified so that the game ends when the first loop is formed, disregarding the requirement that it must be the arrival of the same player to the same position, then Theorem 2 may fail. Figure 1 is such an example.

Here all the arrows are assumed to be bicolored and all the values $\varphi(x)$ are 0 except at the vertical arrow where it is 1. Clearly, in this game, player II has a strategy which se-

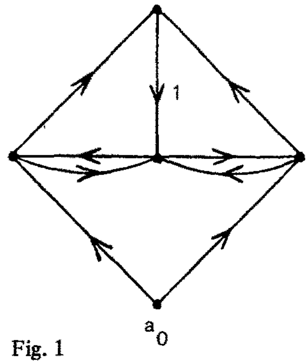

Fig. 1

cures an outcome $\geqslant 1/3$ but he has no positional strategy which secures an outcome > 0.

Our proofs are roundabout, we use the infinite game Γ to establish facts about the finite game G and vice versa. Perhaps more direct proofs would be desirable.

Our results were announced in *Ehrenfeucht/Mycielski* [1973].

Proofs

Let s be any strategy for player X in G. Then s defines a strategy $\tilde{s}$ for X in Γ. Namely

$$\tilde{s}(x_0, \ldots, x_{n-1}) = s(y_0, \ldots, y_{m-1}),$$

where $(y_0, \ldots, y_{m-1})$ is obtained from $(x_0, \ldots, x_{n-1})$ by the following process. Given any sequence $(z_0, \ldots, z_{r-1})$ of arrows suppose that there exist $p < q < r$ such that $p + q$ is odd and the head of z_q equals the tail of z_p. Suppose that q_0 is the least q for which there exists such a p and let p_0 be the (unique) corresponding p. Then we put

$$(z_0, \ldots, z_{r-1})' = (z_0, \ldots, z_{p_0-1}, z_{q_0+1}, \ldots, z_{r-1}).$$

{$q_0 = r - 1$ can happen and in this case $(z_0, \ldots, z_{r-1})' = (z_0, \ldots, z_{p_0-1})$, also $p_0 = 0$ can happen and then $(z_0, \ldots, z_{r-1})' = (z_{q_0+1}, \ldots, z_{r-1})$, and if r is even then both $p_0 = 0$ and $q_0 = r - 1$ can happen and then $(z_0, \ldots, z_{r-1})'$ is the empty sequence.} Now the sequence $(y_0, \ldots, y_{m-1})$ is obtained from $(x_0, \ldots, x_{n-1})$ by applying the operation $'$ as long as feasible.

This concludes the definition of $\tilde{s}$. Notice that $m < 2\,|A\,|$, since $'$ is applicable to any sequence of arrows of length $\geqslant 2\,|A\,|$. If s is equivalent to a positional strategy then so is $\tilde{s}$.

Lemma 1: If s secures v for player X in G then $\tilde{s}$ secures v for player X in Γ.

Proof: Case 1. $X = \mathrm{I}$. Let $x = (x_0, x_1, \ldots)$ be a play of Γ in which I has been using $\tilde{s}$. Since s secures v and by the definition of $\tilde{s}$ we infer the following: For every n the sequence $\varphi(x_0), \ldots, \varphi(x_{n-1})$ can be partitioned into subsequences

$$(\varphi_{11}, \ldots, \varphi_{1r(1)}), \ldots, (\varphi_{t1}, \ldots, \varphi_{tr(t)}),$$

each of length $\leqslant 2\,|A\,|$ and such that the mean of each of them, except at most one, is $\leqslant v$. Let $(\varphi_{11}, \ldots, \varphi_{1r(1)})$ be the one whose mean may be $> v$.
Then $r(1) + \ldots + r(t) = n$, $r(i) \leqslant 2\,|A\,|$ for $i = 1, \ldots, t$ and

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(x_i) = \frac{1}{n}\left(\sum_{i=1}^{t}\sum_{j=1}^{r(i)} \varphi_{ij}\right) \leqslant \frac{1}{n}\left(\sum_{i=1}^{t} vr(i) - vr(1) + \sum_{j=1}^{r(1)} \varphi_{1j}\right) \leqslant v + \frac{c}{n},$$

where c is a suitable constant independent of n. Hence $v_{\mathrm{I}}(x) \leqslant v$.

Case 2. $X = \mathrm{II}$. The proof is similar.

Corollary 1: Γ has a value $v(\Gamma)$ and $v(\Gamma) = v(G)$.

Proof: By Lemma 1.

For any $a \in D_0$ = domain (P_0) we define an auxiliary game G_a. G_a is the same as G until II chooses an arrow x_n whose head is a. Then the sequence $x_0, \ldots, x_n$ is "forgotten" and the game continues until one of the players chooses an arrow x_q whose head equals the tail of some arrow x_p with $p + q$ odd and $n < p < q$. Then I pays to II the mean

$$\frac{1}{q-p+1}\sum_{i=p}^{q} \varphi(x_i).$$

In other words, if no arrow chosen by II has the head a then G_a proceeds in the same way as G, but in the other case the game becomes $G(a, P, Q, \varphi)$.

Of course G_a is a finite 0-sum game with perfect information and hence it has a value $v(G_a)$.

Lemma 2: For every $a \in D_0$ we have $v(G_a) = v(G)$.

Proof: Let s be a strategy for player X which secures $v(G)$ in G. If no play in which X uses s has an arrow chosen by II whose head is a then of course s secures $v(G)$ in G_a. In the other case we consider the strategy $\tilde{s}$ and notice that there is a play of Γ consistent with $\tilde{s}$ in which an arrow x_n chosen by II has the head a. But since the payoffs in Γ are limits which do not depend on the choices prior to x_{n+1} hence $\Gamma(a, P, Q, \varphi)$ is not worse for X than Γ. Hence by Corollary 1 $G(a, P, Q, \varphi)$ is not worse for X than G. And $v(G_a) = v(G)$ follows.

Proofs of Theorems 1 and 2: By Lemma 1 and Corollary 1 it is enough to prove the existence of positional strategies which secure $v(G)$ in G.

Step 1: The existence of a positional strategy for I which secures $\nu(G)$ in G. The proof goes by induction with respect to $|P| - |D_0|$. If $|P| = |D_0|$ the assertion is obvious. Let $n > 0$ and assume that the positional strategy exists whenever $|P| - |D_0| < n$. Let now $|P| - |D_0| = n$. Then there is an $a \in D_0$ such that the set P_a of arrows in P whose tails equal a has more than one element. Consider the game G_a. By Lemma 2 there is a strategy s for I which secures $\nu(G)$ in G_a. We may assume that s uses at most one arrow in P_a, say $((a, b), 0)$ (since any return by II to a would end the game and the positions prior to a do not count). Hence we may remove from P all the arrows of P_a except $((a, b), 0)$ without destroying s. Let $P^{(0)} = P - P_a \cup$ $\cup \{((a, b), 0)\}$ and $G^{(0)} = G(a_0, P^{(0)}, Q, \varphi$ restricted to $P^{(0)} \cup Q)$. Thus s secures $\nu(G)$ in $G_a^{(0)}$. By Lemma 2 the value of $G_a^{(0)}$ equals the value of $G^{(0)}$. Hence, since $|P^{(0)}| - |D_0| < n$ and by the inductive assumption, there exists a positional strategy s^* for I which secures $\nu(G)$ in $G^{(0)}$. Clearly s^* secures also $\nu(G)$ in G.

Step 2: The existence of a positional strategy for II which secures $\nu(G)$ in G. The proof follows by symmetry, i.e., one can define a game $G' = G(a_0', P', Q', \varphi')$ such that the existence of a positional strategy for I which secures $\nu(G')$ in G' is equivalent to the existence of a positional strategy for II which secures $\nu(G)$ in G.

Remark

Let A and B be two disjoint compact metric spaces, $a_0 \in A$, $P = B \times A$, $Q = A \times B$ and $\varphi: P \cup Q \to \mathbf{R}$ be a continuous function. Consider the game $\Gamma(a_0, P, Q, \varphi)$ defined similarly as Γ. Then there is a value ν such that for every $\epsilon > 0$ there exists a positional strategy for I which secures $\nu + \epsilon$ and a positional strategy for II which secures $\nu - \epsilon$. The proof follows from Theorem 1 by an approximation of A and B by finite spaces. Can one get rid of ϵ? Is the above true for all Borel-measurable functions φ?

References

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