Chapter 12: Fuzzy Languages

George Rahonis

Department of Mathematics, Aristotle University of Thessaloniki,
54124 Thessaloniki, Greece
grahonis@math.auth.gr

1 Introduction

Classical logic, as founded by the Greek philosopher Aristotle, is based on the *principle of bivalence* which states that every proposition can be assigned exactly one of the logical values *true* or *false*. However, Aristotle himself observed that this principle cannot describe the status of *all* propositions especially the ones which refer to *future contingents*. In his treatise *On Interpretation 9*, the philosopher formulated the famous sentence "There will be a sea-battle tomorrow", which is actually neither true nor false. Clearly, (at least) a *third* logical value is required in order to describe such situations. Actually this third value spoils the principle of bivalence. Nevertheless, despite the efforts

M. Droste, W. Kuich, H. Vogler (eds.), *Handbook of Weighted Automata*, Monographs in Theoretical Computer Science. An EATCS Series,
DOI 10.1007/978-3-642-01492-5_12,

of philosophers and mathematicians in the Middle Ages, a *three-valued propositional logic* was successfully established by Łukasiewicz and Post only in 1920 (see [48, 59]). However, it came up that even that three-valued logic was not sufficient enough to describe the logical status of real world statements. Therefore, the three-valued logic has been extended to *multi-valued* (or *many-valued*) *logic* by considering (finitely or infinitely) many logical values. For textbooks on multi-valued logic, we refer the reader to [50, 64] (see also [66] for historical details for the multi-valued logic's progress and the contribution of Gr. Moisil and A. Salomaa to this field).

On the other hand, Zadeh [78] introduced in 1965 the concept of *fuzzy sets.* He was motivated by the real world where sentences like "the class of real numbers that are much greater than 1" or "the class of beautiful women" are naturally imprecise, and they do not determine sets in the usual mathematical sense. In 1973, Zadeh founded his *fuzzy logic* as a multi-valued logic over the interval $[0,1] \subseteq \mathbb{R}$, enriched with further fuzzy quantifiers like *most*, *few*, *many*, and *several.* In the meantime, Wee [75] introduced the *fuzzy automaton* as a model of learning systems. The fuzzy automaton model is the natural *fuzzification* of the classical finite automaton and it is actually a weighted automaton model (over the fuzzy semiring $\langle [0,1], \mathsf{max}, \mathsf{min}, 0, 1\rangle$) in the sense of [24]. Since then, fuzzy automata theory has been extended to more general structures like lattices, residuated lattices, and ℓ-monoids. However, in all these cases, the corresponding fuzzy automata act on semirings induced by the original structures. Therefore, all the well-known results for recognizable formal power series over semirings hold in particular for *fuzzy recognizable languages* accepted by fuzzy automata. More specific results can be obtained for fuzzy automata and their behaviors due to the special properties of their underlying semirings inherited by the original structures. For instance, the determinization problem is effectively solved for fuzzy automata and the equality is decidable for fuzzy recognizable languages over bounded distributive lattices.

Fuzzy structures and fuzzy logic contribute to a wide range of real world applications because they can effectively incorporate the impreciseness of practical problems. It is the purpose of this chapter, to present the theory of fuzzy recognizable languages as a paradigm of recognizable formal power series. Our fuzzy languages are defined over bounded distributive lattices. This is a more general case than the very first definition of fuzzy languages over the interval $[0,1]$, but still almost all the recognizability properties remain valid. In our development, we refer only briefly to those results which are inherited from the general theory of weighted automata and power series. Instead, we focus on results which do not hold for power series over arbitrary semirings. More precisely, our fuzzy recognizable languages are obtained as behaviors of multi-valued automata. We show that for every such multi-valued automaton we can effectively construct an equivalent trim deterministic one which moreover has a minimum counterpart. Furthermore, the equivalence problem for multi-valued automata is decidable and a pumping lemma holds for fuzzy recognizable languages. The equivalence problem turns out also to be decidable for multi-valued automata over infinite words. Our treatment of fuzzy recognizable languages is based on automata-theoretic techniques. It is worth noting that fuzzy recognizability over finite words, especially over the fuzzy semiring, has been also defined by means of finite monoid representations, syntactic congruences, syntactic monoids, and (left and right) derivatives (see [8, 9, 36, 47, 54, 57]). On the other hand, several authors have fuzzified notions like monoids [36], trees [22, 28], and algebras [46, 70–72, 74].

In the sequel, we briefly describe the contents of the chapter. First, we introduce basic notions like (bounded distributive) lattices and the more particular class of De Morgan algebras. We show that the collection of De Morgan algebras coincides with the family of semirings with complement function. We define the concept of fuzzy languages as formal power series over bounded distributive lattices. Then we deal with fuzzy recognizable languages over finite (resp. infinite) words obtained as behaviors of multi-valued (resp. multi-valued Büchi and Muller) automata. An MSO logic characterization of fuzzy recognizable languages is also provided. Next, we briefly investigate fuzzy languages over bounded ℓ-monoids and residuated lattices. These are the most general classes of fuzzy languages, but still they are special cases of power series. Finally, we refer to practical applications of fuzzy languages. The material concerning multi-valued automata over infinite words, De Morgan algebras, and the MSO logic is contained in [22].

For monographs presenting fuzzy logic, fuzzy languages, and fuzzy automata, we refer the reader to [32, 35, 54, 74]. Our list of references includes only these ones which are connected with the context of the chapter. In [1, 35, 54], there are extended lists of references until 2002. Also, the journal *Fuzzy Sets and Systems* publishes periodically an article entitled *Recent Literature*, and it presents the latest developments in fuzzy theory (see for instance volume 159 (2008), pages 857–865).

2 Lattices and Fuzzy Languages

A partially ordered set $(L, \leq)$ is called a *lattice* if the supremum (called also least upper bound or join) $a \vee b$ and the infimum (called also greatest lower bound or meet) $a \wedge b$ exist in L for every $a, b \in L$ (see [21]). A lattice $(L, \leq)$ (which is simply denoted by L if the order relation is understood) is *distributive* if it satisfies the equation $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (which in turn implies $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$) for every $a, b, c \in L$. The supremum (resp. the infimum) of every $A \subseteq L$ is denoted (if it exists in L) by $\vee A$ (resp. $\wedge A$). If $A = (a_i \mid i \in I)$, then we also use the notation $\bigvee_{i \in I} a_i$ (resp. $\bigwedge_{i \in I} a_i$). A lattice L is *bounded* if it contains two distinguished elements $0, 1 \in L$ such that $0 \leq a \leq 1$ for every $a \in L$. Furthermore, a lattice L is called *complete* if $\vee A$ and $\wedge A$ exist for every $A \subseteq L$. Observe that a complete lattice is also bounded with $0 = \vee\emptyset$ and $1 = \wedge\emptyset$. It is well known that if L is any distributive lattice and $A \subseteq L$ a finite subset, then the sub-lattice L_A of L generated by A is finite. In fact, if $A' = \{\wedge B \mid B \subseteq A\}$, then we have $L_A = \{\vee C \mid C \subseteq A'\}$ due to the distributivity law. Obviously, every finite lattice L is complete. A bounded distributive lattice L forms a semiring $\langle L, \vee, \wedge, 0, 1\rangle$ whose operations are both idempotent. An element $a \neq 0$ of a lattice L is called *join-irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$ for every $b, c \in L$. We denote by $J(L)$ the set of all join-irreducible elements of L. If the lattice L is finite, then

$$a = \vee\{b \in J(L) \mid b \leq a\}$$

for every $a \in L$. Moreover, if L is distributive, then every join-irreducible element $a \in L$ is *prime*, i.e., whenever $a \leq b \vee c$ with $b, c \in L$, then $a \leq b$ or $a \leq c$ (cf. [6, 16]).

Let $(L, \leq)$ be a bounded distributive lattice and $^{-} : L \to L$ be any function with $\overline{0} = 1$ and $\overline{1} = 0$. Then we call $^{-}$ a (*general*) *negation function* and $(L, \leq, ^{-})$ a *bounded distributive lattice with negation function*. Note that every bounded distributive lattice L can be equipped with a negation function $^{-}$ by letting for instance $\overline{0} = 1$ and $\overline{x} = 0$ for every $x \in L \setminus \{0\}$. De Morgan algebras, Heyting algebras, and variants of pseudo-complemented lattices are well-investigated classes of distributive lattices with negation function (see [3, 16]). Recently, De Morgan algebras have been investigated intensively for multi-valued model checking (see [13, 31, 39]). More precisely, a *De Morgan* (or *quasi-Boolean*) *algebra* is a distributive lattice $(L, \leq, ^{-})$ with a *complement mapping* $^{-}$ satisfying the involution $\overline{\overline{a}} = a$ and De Morgan laws, i.e., $\overline{a \vee b} = \overline{a} \wedge \overline{b}$ and $\overline{a \wedge b} = \overline{a} \vee \overline{b}$ for every $a, b \in L$. Then $a \leq b$ implies $\overline{b} \leq \overline{a}$ for every $a, b \in L$. Furthermore, if L is bounded, then $\overline{0} = 1$ and $\overline{1} = 0$, i.e., the function $^{-}$ is a negation function. Moreover, the mapping $^{-} : (L, \leq) \to (L, \geq)$ is an order-isomorphism. Hence, if $(a_i \mid i \in I) \subseteq L$ is a family of elements of L for which $\bigvee_{i\in I} a_i$ exists, then $\overline{\bigvee_{i\in I} a_i} = \bigwedge_{i\in I} \overline{a_i}$. For instance, the lattice $([0, 1], \leq, ^{-})$ with $\leq$ the usual order of real numbers, and $\overline{a} = 1 - a$ for every $a \in [0, 1]$ is a De Morgan algebra. The induced semiring $\langle [0, 1], \mathsf{max}, \mathsf{min}, 0, 1\rangle$ is referred to as the *fuzzy semiring*. In the sequel, without any further notation, for every De Morgan algebra $(L, \leq, ^{-})$, we require the lattice L to be bounded. On the other hand, every bounded distributive lattice can be endowed with a negation function, therefore, lattices with negation function constitute a much larger class than De Morgan algebras. In particular, any bounded distributive lattice which is not anti-isomorphic to itself, does not have a complement operation, and thus cannot be structured to a De Morgan algebra.

Next, we investigate the relationship between De Morgan algebras and semirings. Given a semiring $\langle S, +, \cdot, 0, 1\rangle$, a mapping $f : S \to S$ is called a *complement function*, if it satisfies the following statements:

(i) f is an *involution*, i.e., $f(f(a)) = a$ for every $a \in S$.
(ii) f is a monoid morphism from $\langle S, +, 0\rangle$ to $\langle S, \cdot, 1\rangle$, i.e., $f(0) = 1$ and $f(a + b) = f(a) \cdot f(b)$ for every $a, b \in S$.

It is easily seen that $f(1) = 0$ and $f(a \cdot b) = f(a) + f(b)$ for every $a, b \in S$, hence f is a monoid isomorphism from $\langle S, +, 0\rangle$ to $\langle S, \cdot, 1\rangle$ and from $\langle S, \cdot, 1\rangle$ to $\langle S, +, 0\rangle$. Every De Morgan algebra $(L, \leq, ^{-})$ induces a semiring $\langle L, \vee, \wedge, 0, 1\rangle$ with complement mapping $^{-}$, therefore, the following result concludes that De Morgan algebras and semirings with complement function coincide. This indicates the relation between the MSO logic over De Morgan algebras (see Sect. 3.3) and semirings (see [18, 23, 20]).

Proposition 2.1 ([22]). *Let $\langle S, +, \cdot, 0, 1\rangle$ be a semiring with complement function f. For every $a, b \in S$, we put $a \leq b$ iff $a + b = b$. Then $(S, \leq, f)$ is a De Morgan algebra.*

Proof. We have $f(0) = 1$, f is an involution, $f(a + b) = f(a) \cdot f(b)$ and $f(a \cdot b) = f(a) + f(b)$. Hence, $0 \cdot 0 = 0$ implies $1 + 1 = 1$, so $\langle S, +, 0\rangle$, and hence also $\langle S, \cdot, 1\rangle$ are idempotent. Thus, $\leq$ is a partial order on S (see Proposition 20.19 in [30]) and $a + b$ is the supremum of a and b in this partial order. Moreover, $0 \leq a$ for every $a \in S$, and $a \cdot 0 = 0$ implies $f(a) + 1 = 1$, so $f(a) \leq 1$, showing also $a \leq 1$ for every $a \in S$.

Next, observe that if $a \leq b$, then by distributivity we have $a \cdot c \leq b \cdot c$ for every $a, b, c \in S$. We show that $a \cdot b$ is the infimum of a and b in $(S, \leq)$ for every $a, b \in S$. Since $a \leq 1$, the previous remark implies $a \cdot b \leq b$ and similarly $a \cdot b \leq a$. Now if $c \in S$ with $c \leq a$ and $c \leq b$, then $c = c \cdot c \leq a \cdot c \leq a \cdot b$, proving that $a \cdot b = a \wedge b$. Hence, $(S, \leq)$ is a distributive lattice with $+$ being the operation supremum and $\cdot$ being the infimum. Moreover, $(S, \leq)$ is bounded, and f is a complement mapping satisfying De Morgan laws. Thus, the proof is completed. $\square$

The interested reader should find further characterizations of bounded distributive lattices by means of semirings in Example 1.5 of [30].

Given two lattices $(L, \leq)$ and $(L', \leq)$, a mapping $f : L \to L'$ is a *lattice morphism* if it preserves suprema and infima, i.e., for every $a, b \in L$

$$f(a \vee b) = f(a) \vee f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \wedge f(b).$$

Then $a \leq b$ implies $f(a) \leq f(b)$ for every $a, b \in L$. Furthermore, if $(L, \leq)$ and $(L', \leq)$ are bounded distributive lattices, then a lattice morphism $f : L \to L'$ satisfying $f(0) = 0$ and $f(1) = 1$ is a semiring morphism from $\langle L, \vee, \wedge, 0, 1\rangle$ to $\langle L', \vee, \wedge, 0, 1\rangle$.

Now we turn to fuzzy sets originally introduced by Zadeh in [78]. Given a non-empty set X, a *fuzzy set A in X* (or a *fuzzy subset A of X*) is defined by a membership function

$$f_A : X \to [0, 1].$$

A fuzzy subset of a free monoid is called a *fuzzy language* [40]. Thus, a fuzzy language is nothing else but a formal power series over the fuzzy semiring $\langle [0, 1], \mathsf{max}, \mathsf{min}, 0, 1\rangle$. So far, the term fuzzy language has been also used for power series over lattices, residuated lattices, and ℓ-monoids (see Sect. 4).

Here, we deal with fuzzy languages over bounded distributive lattices. More precisely, let S be a set and L be a bounded distributive lattice. A *formal power series* (*over* S *and* L) is a mapping $r : S \to L$. Such a series is called (*finitary*) *fuzzy language* (resp. *infinitary fuzzy language*) over some finite alphabet Σ if $S = \Sigma^*$ (resp. $S = \Sigma^\omega$, i.e., the set of all infinite words over Σ). Subsequently, we will only need the cases where $S = \Sigma^*$ or $S = \Sigma^\omega$. The support $\mathrm{supp}(r)$ of a series r over S and L is defined as usually by $\mathrm{supp}(r) = \{s \in S \mid (r, s) \neq 0\}$, and the *image* of r is the set $\{l \in L \mid \exists s \in S : (r, s) = l\}$. The collection $L\langle\langle S\rangle\rangle$ of all power series over S and L is itself a bounded distributive lattice $(L\langle\langle S\rangle\rangle, \leq)$; for $r, r' \in L\langle\langle S\rangle\rangle$ the partial order $\leq$ is determined by $r \leq r'$ iff $(r, s) \leq (r', s)$ for every $s \in S$. Then the supremum $r \vee r'$ and the infimum $r \wedge r'$ are defined elementwise by $(r \vee r', s) = (r, s) \vee (r', s)$ and $(r \wedge r', s) = (r, s) \wedge (r', s)$ for every $s \in S$. Furthermore, for every $k \in L$, the scalar infimum $k \wedge r$ is determined by $(k \wedge r, s) = k \wedge (r, s)$ for every $s \in S$. If $(L, \leq, ^-)$ is a bounded distributive lattice with negation function (resp. a De Morgan algebra), then $(L\langle\langle S\rangle\rangle, \leq, ^-)$ constitutes also a bounded distributive lattice with negation function (resp. a De Morgan algebra); for every $r \in L\langle\langle S\rangle\rangle$ its negation $\overline{r} \in L\langle\langle S\rangle\rangle$ is defined by $(\overline{r}, s) = \overline{(r, s)}$ for every $s \in S$.

Assume that $(L, \leq)$ and $(L', \leq)$ are two distributive lattices, and let $f : L \to L'$ be any mapping. Then f is extended to a mapping $f : L\langle\langle S\rangle\rangle \to L'\langle\langle S\rangle\rangle$ in the following way. For every $r \in L\langle\langle S\rangle\rangle$, the series $f(r) \in L'\langle\langle S\rangle\rangle$ is determined by $(f(r), s) = f((r, s))$ for every $s \in S$.

3 Fuzzy Recognizability over Bounded Distributive Lattices

We consider the concept of fuzzy recognizable languages obtained as behaviors of weighted automata over bounded distributive lattices. Such automata are called *multi-valued,* and they have recently contributed to multi-valued logics [22] and multi-valued model checking employing distributive lattices [10, 39]. Several other names occur in the literature for automata over lattices, like fuzzy automaton, max-min automaton, L-fuzzy automaton, and lattice automaton depending on the properties of the underlying lattice (see, for instance, [43, 54]).

First, we deal with fuzzy recognizable languages over finite words. For these languages a Kleene–Schützenberger theorem is obtained as a special case of the corresponding theorem for recognizable series over commutative semirings. Then we show that fuzzy recognizable languages have an elegant characterization, namely they are written as fuzzy recognizable step languages. This enables us to give short proofs for well-known results concerning multi-valued automata. More precisely, we show that:

(i) For every multi-valued automaton, we can effectively construct an equivalent minimum trim deterministic one.

(ii) The equivalence problem for multi-valued automata is decidable.
(iii) A pumping lemma holds for fuzzy recognizable languages.
(iv) A fuzzy language is recognizable iff it has finite image and each of its cut languages is recognizable.

It is worth noting that these results do not hold in general for weighted automata over arbitrary semirings. Next, we introduce Büchi and Muller multi-valued automata working on infinite words. As in the finitary case, we show that fuzzy Büchi recognizable languages can be written as fuzzy Büchi recognizable step languages. Using this simple characterization, we give elegant proofs for two important results. Namely, the classes of fuzzy Büchi and fuzzy Muller recognizable languages coincide, and a Kleene theorem holds for them. Moreover, we introduce a multi-valued MSO logic and we show the fundamental theorem of Büchi, i.e., fuzzy definable languages over infinite words coincide with fuzzy Büchi recognizable languages. For the rest of this section, Σ will denote an arbitrary finite alphabet and L an arbitrary bounded distributive lattice.

3.1 Fuzzy Recognizability over Finite Words

We start with the concept of multi-valued automata.

Definition 3.1. *A* multi-valued automaton *(MVA for short)* over Σ and L *is a quadruple* $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$, *where* Q *is the* finite state set, $\mathrm{in} : Q \to L$ *is the* initial distribution, $\mathrm{wt} : Q \times \Sigma \times Q \to L$ *is the mapping assigning* weights *to the transitions of the automaton, and* $\mathrm{out} : Q \to L$ *is the* final distribution.

Let $w = a_0 \dots a_{n-1} \in \Sigma^*$ where $a_0, \dots, a_{n-1} \in \Sigma$. A *path of* $\mathcal{A}$ *over* w is a sequence $P_w = (t_i)_{0 \leq i \leq n-1}$ of transitions, such that $t_i = (q_i, a_i, q_{i+1}) \in Q \times \Sigma \times Q$ for every $0 \leq i \leq n-1$. The *weight of* P_w is defined by

$$\mathrm{weight}(P_w) = \mathrm{in}(q_0) \wedge \bigwedge_{0 \leq i \leq n-1} \mathrm{wt}(t_i) \wedge \mathrm{out}(q_n).$$

We shall denote by $L_{\mathcal{A}}$ the finite sub-lattice of L generated by $\{0,1\} \cup \{\mathrm{in}(q) \mid q \in Q\} \cup \{\mathrm{out}(q) \mid q \in Q\} \cup \{\mathrm{wt}(t) \mid t \in Q \times \Sigma \times Q\}$. Clearly $\mathrm{weight}(P_w) \in L_{\mathcal{A}}$. The *behavior* of $\mathcal{A}$ is the fuzzy language

$$\|\mathcal{A}\| : \Sigma^* \to L$$

which is defined by

$$(\|\mathcal{A}\|, w) = \bigvee_{P_w} \mathrm{weight}(P_w)$$

for $w \in \Sigma^*$, where the supremum is taken over all paths P_w of $\mathcal{A}$ over w. It should be clear that $(\|\mathcal{A}\|, \varepsilon) = \bigvee_{q \in Q} \mathrm{in}(q) \wedge \mathrm{out}(q)$. Again, $(\|\mathcal{A}\|, w) \in L_{\mathcal{A}}$ for every $w \in \Sigma^*$.

Two multi-valued automata $\mathcal{A}$ and $\mathcal{A}'$ over Σ and L are called *equivalent* if they have the same behavior, i.e., $\|\mathcal{A}\| = \|\mathcal{A}'\|$.

A fuzzy language $r \in L\langle\langle \Sigma^* \rangle\rangle$ is said to be *fuzzy recognizable over Σ and L* if there is an MVA $\mathcal{A}$ such that $r = \|\mathcal{A}\|$. We denote the family of all fuzzy recognizable languages over Σ and L by $L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$. The reader should observe that an MVA over Σ and L is just a weighted automaton over Σ and the semiring $\langle L, \vee, \wedge, 0, 1\rangle$ in the sense of [24, 38, 67] (see also Theorems 2.2 and 3.6 in [27]). Thus, the class $L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$ coincides with the collection of all recognizable series over Σ and the semiring L. Therefore, as a consequence of the general Kleene–Schützenberger theorem for series over arbitrary semirings (see, for instance, [65]), we immediately obtain its reformulation for fuzzy languages as follows. Let us first reconsider the rational operations of formal power series in the setting of fuzzy languages. Let $r, r' \in L\langle\langle \Sigma^* \rangle\rangle$. The *Cauchy product rr' of r and r'* is a fuzzy language in $L\langle\langle \Sigma^* \rangle\rangle$ which is determined by $(rr', w) = \bigvee_{uu'=w} (r, u) \wedge (r', u')$ for every $w \in \Sigma^*$. If r is *proper*, i.e., $(r, \varepsilon) = 0$, then we define the *star $r^* \in L\langle\langle \Sigma^* \rangle\rangle$ of r* by $(r^*, w) = \vee\{(r, u_1) \wedge \cdots \wedge (r, u_n) \mid u_1 \ldots u_n = w, u_1, \ldots, u_n \in \Sigma^*\}$ for every $w \in \Sigma^*$. The *rational operations of* fuzzy languages in $L\langle\langle \Sigma^* \rangle\rangle$ are the supremum, the Cauchy product, and the star. We denote by $L^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$ the least class of fuzzy languages from $L\langle\langle \Sigma^* \rangle\rangle$ which contains the polynomials, i.e., the fuzzy languages with finite support, and is closed under the rational operations.

Theorem 3.2 (Kleene–Schützenberger). *Let Σ be an alphabet and L be a bounded distributive lattice. Then $L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle = L^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$.*

Let Σ, Δ be alphabets and $h : \Sigma^* \to \Delta^*$ be any morphism. Then we can define the mapping $h^{-1} : L\langle\langle \Delta^* \rangle\rangle \to L\langle\langle \Sigma^* \rangle\rangle$ (see [21]); if L is a complete lattice or h is non-deleting, then the mapping $h : L\langle\langle \Sigma^* \rangle\rangle \to L\langle\langle \Delta^* \rangle\rangle$ is also well defined.

Proposition 3.3 ([24]). *Let Σ, Δ be two alphabets and $h : \Sigma^* \to \Delta^*$ be any morphism. Then:*

(i) $h^{-1} : L\langle\langle \Delta^* \rangle\rangle \to L\langle\langle \Sigma^* \rangle\rangle$ preserves fuzzy recognizability.
(ii) If h is non-deleting, then $h : L\langle\langle \Sigma^* \rangle\rangle \to L\langle\langle \Delta^* \rangle\rangle$ preserves fuzzy recognizability.

Recall that for every language $R \subseteq \Sigma^*$, its characteristic series $1_R \in L\langle\langle \Sigma^* \rangle\rangle$ is defined by $(1_R, w) = 1$ if $w \in R$, and 0 otherwise, for every $w \in \Sigma^*$. Here, we call 1_R the *characteristic fuzzy language of R*. Every unweighted finite automaton with input alphabet Σ can be considered in the obvious way, as an MVA over Σ and L with weights only 0 and 1. Therefore, for every recognizable language R, its characteristic language 1_R is fuzzy recognizable. Assume now that $R_1, \ldots, R_n \subseteq \Sigma^*$ are recognizable languages and $k_1, \ldots, k_n \in L$. Clearly, the fuzzy language $k_i \wedge 1_{R_i}$ is recognizable for every $1 \leq i \leq n$. Then by Theorem 3.2, the fuzzy language

$$r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$$

is also recognizable. Such a language r is called a *fuzzy recognizable step language.* The class of recognizable languages is closed under the Boolean operations; hence, for every fuzzy recognizable step language $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$, we may assume that the family $(R_i \mid 1 \leq i \leq n)$ forms a partition of Σ^*. Next, we show that fuzzy recognizable languages and fuzzy recognizable step languages coincide. This important result has been firstly proved in [19] for power series over locally finite semirings. Therefore, it can be applied to the class of fuzzy languages over bounded distributive lattices (recall that for every bounded distributive lattice L, the semiring $\langle L, \vee, \wedge, 0, 1\rangle$ is locally finite). However, here we give an alternative proof based on lattices. The same proof has been also used in [22] for the corresponding result for infinitary fuzzy languages (see Sect. 3.2). We shall need the following lemma which is easily proved by a standard automata construction.

Lemma 3.4. *Let $(L, \leq)$ and $(L', \leq)$ be two bounded distributive lattices and $f : L \to L'$ be a lattice morphism. Then for every fuzzy recognizable language r in $L\langle\langle \Sigma^* \rangle\rangle$, the fuzzy language $f(r) \in L'\langle\langle \Sigma^* \rangle\rangle$ is again recognizable.*

Theorem 3.5. *Let Σ be an alphabet and L be a bounded distributive lattice. Then a fuzzy language $r \in L\langle\langle \Sigma^* \rangle\rangle$ is recognizable iff it is a fuzzy recognizable step language.*

Proof. Let r be fuzzy recognizable and $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ be an MVA over Σ and L such that $r = \|\mathcal{A}\|$ and $L_{\mathcal{A}} = \{k_1, \ldots, k_n\}$. We set $R_i = \{w \in \Sigma^* \mid (r, w) = k_i\}$ for every $1 \leq i \leq n$. Then

$$r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}.$$

We shall show that the languages R_i $(1 \leq i \leq n)$ are recognizable. Let $\mathbb{B} = (\{0, 1\}, \leq)$ be the two-valued Boolean lattice. For every join-irreducible element p of $L_{\mathcal{A}}$, we define a mapping $f_p : L_{\mathcal{A}} \to \{0, 1\}$ by putting

$$f_p(a) = \begin{cases} 1 & \text{if } p \leq a, \\ 0 & \text{otherwise} \end{cases}$$

for every $a \in L$.

We claim that f_p is a lattice morphism. Indeed, $p \neq 0$; hence, $f_p(0) = 0$ and $f_p(1) = 1$. Next, note that if $a, a' \in L_{\mathcal{A}}$ and $f_p(a \vee a') = 1$, then $p \leq a \vee a'$ which implies $p \leq a$ or $p \leq a'$ since p is prime, proving $f_p(a \vee a') = f_p(a) \vee f_p(a')$. Clearly, $f_p(a \wedge a') = f_p(a) \wedge f_p(a')$. By Lemma 3.4, the fuzzy language $f_p(r)$ of $\mathbb{B}\langle\langle \Sigma^* \rangle\rangle$ is recognizable and, therefore, the language $\mathrm{supp}(f_p(r)) = \{w \in \Sigma^* \mid p \leq (r, w)\}$ is recognizable. Now let $1 \leq i \leq n$. Since the element k_i of $L_{\mathcal{A}}$ is the supremum of the join-irreducible elements of $L_{\mathcal{A}}$ below k_i, the language R_i is obtained as the intersection of the languages $\mathrm{supp}(f_p(r))$ ($p \leq k_i$ and join-irreducible) and of the complements of the languages $\mathrm{supp}(f_p(r))$ ($p \not\leq k_i$ and join-irreducible). The class of recognizable languages is closed under the Boolean operations. Therefore, we conclude that R_i is a recognizable language, as required.

The converse is also true as already noted. □

Observe that the proof of the above theorem is effective. Indeed, starting from the weights of the multi-valued automaton $\mathcal{A}$, we compute the sub-lattice $L_{\mathcal{A}}$ in finitely many steps. Then following our proof, we obtain finite automata for the languages R_i ($1 \leq i \leq n$).

Due to Theorem 3.5, in the sequel, we write every fuzzy recognizable language as a fuzzy recognizable step language. This has very interesting consequences. Firstly, generalizing Lemma 3.4, we show that fuzzy recognizability is preserved even by arbitrary mappings between lattices.

Proposition 3.6. *Let $(L, \leq)$ and $(L', \leq)$ be two bounded distributive lattices and $f : L \to L'$ be any mapping. Then for every fuzzy recognizable language $r \in L\langle\langle \Sigma^* \rangle\rangle$ the language $f(r) \in L'\langle\langle \Sigma^* \rangle\rangle$ is again fuzzy recognizable.*

Proof. Let $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$. Then $f(r) = \bigvee_{1 \leq i \leq n} f(k_i) \wedge 1_{R_i}$ and so $f(r)$ is fuzzy recognizable. □

Next, we get a classical result from fuzzy language theory. More precisely, given a fuzzy language $r \in L\langle\langle \Sigma^* \rangle\rangle$ and $l \in L$, the *l-cut of r* is the language $r_{\geq l} = \{w \in \Sigma^* \mid (r, w) \geq l\}$. Furthermore, for every $l \in L$, we let $r_{=l} = r^{-1}(l) = \{w \in \Sigma^* \mid (r, w) = l\}$.

Proposition 3.7 ([43]). *For every fuzzy language $r \in L\langle\langle \Sigma^* \rangle\rangle$, the following statements are equivalent:*

(i) r is fuzzy recognizable.
(ii) r has finite image, and for every $l \in L$, $r_{=l}$ is a recognizable language.
(iii) r has finite image, and for every $l \in L$, $r_{\geq l}$ is a recognizable language.

Proof. The equivalence of (i) and (ii) is immediate by Theorem 3.5. We show the implication (i) $\Rightarrow$ (iii). Let $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$ with pairwise disjoint recognizable languages R_i. Consider an $l \in L$. If there is no $i \in \{1, \ldots, n\}$ such that $k_i \geq l$, then $r_{\geq l} = \emptyset$. Otherwise, let $k_{i_1}, \ldots, k_{i_m}$ ($1 \leq i_1 < \cdots < i_m \leq n$) be all the values of r with $k_{i_1}, \ldots, k_{i_m} \geq l$. Then $r_{\geq l} = R_{i_1} \cup \cdots \cup R_{i_m}$, hence $r_{\geq l}$ is recognizable. Finally, assume that statement (iii) is true. For every $l \in L$, we have $r_{=l} = r_{\geq l} \setminus \bigcup_{l' \in L, l < l'} r_{\geq l'}$, and thus $r_{=l}$ is recognizable which concludes our proof. □

In the sequel, we deal with the determinization and minimization problems of multi-valued automata. These problems do not always have a solution for weighted automata over arbitrary semirings (see [12, 34, 52]) or even over residuated lattices and ℓ-monoids (see [33, 42]). However, due to the local finiteness property of distributive lattices, we show that for every multi-valued automaton we can effectively construct an equivalent trim deterministic one. The determinization problem for fuzzy automata over $([0,1], \leq)$ was first solved in [51]. Borchardt in [7] showed that weighted tree automata over locally finite semirings can be effectively determinized (see [34] for the word case). A reformulation of the same method is used in [42] for fuzzy automata over bounded ℓ-monoids, and in [5, 43] (resp. in [33]) for the special case of fuzzy automata over bounded distributive lattices (resp. over residuated lattices). In all the aforementioned papers, the authors followed the well-known subset construction (or even the accessible subset construction in [33]) in the weighted setting. Here, we use the result of Theorem 3.5 and we reduce the determinization of multi-valued automata to the determinization of classical finite automata (as indicated in [19]). Then we minimize the trim deterministic multi-valued automaton. For this minimization procedure, we use the classical *reduction algorithm* (see [24]). In [4, 68] (resp. in [58]), the size (number of states) of a non-deterministic fuzzy automaton (over the fuzzy semiring) is reduced by means of equivalences (resp. congruences) on the set of states.

A *deterministic multi-valued automaton* (DMVA for short) *over* Σ *and* L is an MVA $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ such that the following two conditions hold:

(i) There is exactly one $q_0 \in Q$ such that $\mathrm{in}(q_0) = 1$ and for every $p \in Q$ with $p \neq q_0$ we have $\mathrm{in}(p) = 0$.
(ii) For every $q \in Q$ and $\sigma \in \Sigma$, there is at most one state $q' \in Q$ such that $\mathrm{wt}(q, \sigma, q') = 1$ and for every $p \in Q$ with $p \neq q'$ we have $\mathrm{wt}(q, \sigma, p) = 0$.

Clearly, for a DMVA $\mathcal{A}$, the function wt can be equivalently expressed by a (partial) function $\delta : Q \times \Sigma \to Q$ in the obvious way. Therefore, we will denote in the sequel a DMVA by $(Q, q_0, \delta, \mathrm{out})$ with $q_0 \in Q$ and $\delta : Q \times \Sigma \to Q$ as a partial function. Thus, a DMVA $\mathcal{A}$ can be considered as a classical deterministic automaton with weights attached only to the final states. The DMVA $\mathcal{A} = (Q, q_0, \delta, \mathrm{out})$ is called *accessible* if for every state $q \in Q$ there exists a word $w \in \Sigma^*$ such that $\delta(q_0, w) = q$. Furthermore, $\mathcal{A}$ is *co-accessible* if for every $q \in Q$ there exists $w \in \Sigma^*$ such that $\mathrm{out}(\delta(q, w)) > 0$. A DMVA is called *trim* if it is accessible and co-accessible. Observe that in a DMVA $\mathcal{A}$, for every word $w = a_0 \ldots a_{n-1} \in \Sigma^*$ and for every path $P_w = (p_i, a_i, p_{i+1})_{0 \leq i \leq n-1}$ of $\mathcal{A}$ over w such that $\delta(p_i, a) = p_{i+1}$, we have

$$\mathrm{weight}(P_w) = \begin{cases} \mathrm{out}(p_n) & \text{if } p_0 = q_0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.8. *Let* Σ *be an alphabet and* L *be a bounded distributive lattice. For every MVA* $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ *over* Σ *and* L*, we can effectively construct a trim DMVA* $\mathcal{A}'$ *over* Σ *and* L *such that* $\|\mathcal{A}'\| = \|\mathcal{A}\|$.

Proof. Let $\|\mathcal{A}\| = \bigvee_{1\le i\le n} k_i \wedge 1_{R_i}$ with pairwise disjoint recognizable languages R_i. Clearly, we may assume that $k_i \neq 0$ for every $1 \le i \le n$. Let $\mathcal{A}_i = (Q_i, \Sigma, q_{0i}, \delta_i, F_i)$ $(1 \le i \le n)$ be a complete deterministic (i.e., δ_i is a total mapping) finite automaton accepting R_i. Now we perform a classical construction of an automaton accepting a union of languages. Consider the finite automaton $\widetilde{\mathcal{A}} = (\widetilde{Q}, \Sigma, q_0, \widetilde{\delta}, \widetilde{F})$ with $\widetilde{Q} = Q_1 \times \cdots \times Q_n$, $q_0 = (q_{01}, \ldots, q_{0n})$, and $\widetilde{F} = \bigcup_{1\le i\le n} Q_1 \times \cdots \times Q_{i-1} \times F_i \times Q_{i+1} \times \cdots \times Q_n$. The (total) mapping $\widetilde{\delta} : \widetilde{Q} \times \Sigma \to \widetilde{Q}$ is determined by $\widetilde{\delta}((q_1, \ldots, q_n), a) = (\delta_1(q_1, a), \ldots, \delta_n(q_n, a))$ for every $(q_1, \ldots, q_n) \in \widetilde{Q}, a \in \Sigma$. Obviously, $\widetilde{\mathcal{A}}$ is deterministic with behavior $R_1 \cup \cdots \cup R_n$. Now let $\overline{\mathcal{A}} = (Q, \Sigma, q_0, \delta, F)$ be the trim part of $\widetilde{\mathcal{A}}$ (see [24]). We consider the DMVA $\mathcal{A}' = (Q, q_0, \delta, \mathrm{out})$ over Σ and L with $\mathrm{out}((q_1, \ldots, q_n)) = \bigvee_{1\le i\le n} \mathrm{out}_i(q_i)$ where

$$\mathrm{out}_i(q_i) = \begin{cases} k_i & \text{if } q_i \in F_i, \\ 0 & \text{otherwise.} \end{cases}$$

The finite automaton $\overline{\mathcal{A}}$ is accessible, and thus the DMVA $\mathcal{A}'$ is also accessible. Moreover, $\mathcal{A}'$ is trim. Indeed, let $(q_1, \ldots, q_n) \in Q$. Since $\overline{\mathcal{A}}$ is co-accessible there is a $w \in \Sigma^*$ such that $\delta((q_1, \ldots, q_n), w) \in F$, i.e., there is an index $1 \le i \le n$ such that $\delta_i(q_i, w) \in F_i$ which in turn implies that $\mathrm{out}_i(\delta_i(q_i, w)) = k_i$. In fact, since the languages R_i are pairwise disjoint, there is exactly one index i with this property, and for every other $1 \le j \le n$ with $j \neq i$, we have $\delta_j(q_j, w) \in Q_j \setminus F_j$. Hence, $\mathrm{out}(\delta((q_1, \ldots, q_n), w)) = k_i > 0$. Now for every $w \in \Sigma^*$,

$$\begin{aligned} (\|\mathcal{A}'\|, w) &= \mathrm{out}\big(\delta(q_0, w)\big) = \mathrm{out}\big(\big(\delta_1(q_{01}, w), \ldots, \delta_n(q_{0n}, w)\big)\big) \\ &= \bigvee_{1\le i\le n} \mathrm{out}_i\big(\delta_i(q_{0i}, w)\big) = \bigvee_{1\le i\le n} (k_i \wedge 1_{R_i}, w) \end{aligned}$$

i.e., $\|\mathcal{A}'\| = \|\mathcal{A}\|$ as required. □

Let $\mathcal{A} = (Q, q_0, \delta, \mathrm{out})$ and $\mathcal{A}' = (Q', q_0', \delta', \mathrm{out}')$ be two DMVA over Σ and L, and let $\varphi : Q \to Q'$ be a mapping such that:

(i) $\varphi(q_0) = q_0'$.
(ii) If $\delta(q, a)$ exists, then $\delta'(\varphi(q), a)$ exists and $\varphi(\delta(q, a)) = \delta'(\varphi(q), a)$ for every $q \in Q, a \in \Sigma$.

Then φ is called a *homomorphism from* $\mathcal{A}$ *to* $\mathcal{A}'$ and is denoted by $\varphi : \mathcal{A} \to \mathcal{A}'$. If $\mathrm{out}'(\varphi(q)) = \mathrm{out}(q)$ for every $q \in Q$, then φ is termed a *strong homomorphism.* A bijective strong homomorphism φ is an *isomorphism.*

Lemma 3.9. *Let* $\mathcal{A} = (Q, q_0, \delta, \mathrm{out})$ *and* $\mathcal{A}' = (Q', q_0', \delta', \mathrm{out}')$ *be two equivalent trim DMVA. Then there is at most one homomorphism* $\varphi : \mathcal{A} \to \mathcal{A}'$*. Every such homomorphism is surjective and strong.*

Proof. Assume that there are two homomorphisms $\varphi : \mathcal{A} \to \mathcal{A}'$ and $\psi : \mathcal{A} \to \mathcal{A}'$. For every $q \in Q$, there exists a word $w \in \Sigma^*$ such that $\delta(q_0, w) = q$. Then $\varphi(q) = \varphi(\delta(q_0, w)) = \delta'(\varphi(q_0), w) = \delta'(q_0', w)$. Similarly, we show that $\psi(q) = \delta'(q_0', w)$, and thus $\varphi(q) = \psi(q)$, i.e., $\varphi = \psi$. Next, we show that φ, whenever it exists, is surjective and strong. Consider $q' \in Q'$. Since $\mathcal{A}'$ is accessible, there is $w \in \Sigma^*$ with $q' = \delta'(q_0', w)$. Moreover, there exists $w' \in \Sigma^*$ such that $0 < \mathrm{out}'(\delta'(q', w')) = (\|\mathcal{A}'\|, ww') = (\|\mathcal{A}\|, ww')$. So, there exists $q \in Q$ with $q = \delta(q_0, w)$. Therefore, $\varphi(q) = \varphi(\delta(q_0, w)) = \delta'(\varphi(q_0), w) = \delta'(q_0', w) = q'$, showing that φ is surjective. Keeping the same notations, we have $\mathrm{out}'(q') = (\|\mathcal{A}'\|, w) = (\|\mathcal{A}\|, w) = \mathrm{out}(q)$ yielding that φ is a strong homomorphism. □

For every $r \in L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$, let $\mathrm{TR}(r)$ be the collection of all trim DMVA accepting r. We define a pre-order $\leq$ in $\mathrm{TR}(r)$; for every $\mathcal{A}, \mathcal{A}' \in \mathrm{TR}(r)$, we set $\mathcal{A}' \leq \mathcal{A}$ iff there exists an homomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$. We show that if $\mathcal{A} \leq \mathcal{A}'$ and $\mathcal{A}' \leq \mathcal{A}$, then $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic. Indeed, $\mathcal{A} \leq \mathcal{A}'$ and $\mathcal{A}' \leq \mathcal{A}$ imply that there exist homomorphisms $\varphi' : \mathcal{A}' \to \mathcal{A}$, $\varphi : \mathcal{A} \to \mathcal{A}'$. Then $\varphi' \circ \varphi : \mathcal{A} \to \mathcal{A}$, $\varphi \circ \varphi' : \mathcal{A}' \to \mathcal{A}'$ are also homomorphisms and by Lemma 3.9, $\varphi' \circ \varphi = 1_{\mathcal{A}}$ and $\varphi \circ \varphi' = 1_{\mathcal{A}'}$ where $1_{\mathcal{A}}$ and $1_{\mathcal{A}'}$ are the identity isomorphisms of $\mathcal{A}$ and $\mathcal{A}'$, respectively, and φ is strong. So, φ is an isomorphism. We conclude that the collection of the isomorphism classes of all trim DMVA accepting r forms a partial order. Clearly, the question of the existence (up to an isomorphism) of a minimum trim DMVA accepting r arises. Here, minimum refers to a trim DMVA in $\mathrm{TR}(r)$ which has as few states as any other automaton in $\mathrm{TR}(r)$. In the following, we show that such a minimum trim DMVA accepting r, always can be constructed and is unique up to isomorphism.

Given a fuzzy language $r \in L\langle\langle \Sigma^* \rangle\rangle$, we define an equivalence relation $\equiv_r$ on Σ^* as follows. For every $w_1, w_2 \in \Sigma^*$, $w_1 \equiv_r w_2$ iff $(r, w_1 w) = (r, w_2 w)$ for every $w \in \Sigma^*$. It is clear that $\equiv_r$ is a right congruence.

Proposition 3.10. *The fuzzy language $r \in L\langle\langle \Sigma^* \rangle\rangle$ is recognizable iff the right congruence $\equiv_r$ has finite index.*

Proof. Assume first that r is accepted by a trim DMVA $\mathcal{A} = (Q, q_0, \delta, \mathrm{out})$. We define an equivalence relation $\equiv_{\mathcal{A}}$ on Σ^* as follows. For every $w_1, w_2 \in \Sigma^*$, $w_1 \equiv_{\mathcal{A}} w_2$ iff $\delta(q_0, w_1) = \delta(q_0, w_2)$. Obviously, $\equiv_{\mathcal{A}}$ is a right congruence, i.e., $w_1 \equiv_{\mathcal{A}} w_2$ implies $w_1 w \equiv_{\mathcal{A}} w_2 w$ for every $w \in \Sigma^*$, and thus $(r, w_1 w) = (r, w_2 w)$; therefore, $\equiv_{\mathcal{A}} \subseteq \equiv_r$. Since Q is finite, $\equiv_{\mathcal{A}}$ has finite index, hence $\equiv_r$ has also finite index.

Conversely, assume that $\equiv_r$ has finite index and let $[w]$ denote the equivalence class of $w \in \Sigma^*$. We construct the accessible DMVA $\mathcal{A}' = (Q', [\varepsilon], \delta_r, \mathrm{out}_r)$ with $Q' = \{[w] \mid w \in \Sigma^*\}$. The function δ_r is determined by $\delta_r([w], a) = [wa]$ for every $[w] \in Q', a \in \Sigma$, and $\mathrm{out}_r([w]) = (r, w)$ for every $[w] \in Q'$. Then $\|\mathcal{A}'\| = r$ and thus r is fuzzy recognizable. By letting $Q_r = \{[w] \in Q' \mid \exists u \in \Sigma^* : (r, wu) > 0\}$, we get an equivalent trim DMVA $\mathcal{A}_r = (Q_r, [\varepsilon], \delta_r, \mathrm{out}_r)$. □

Keeping the notations of the previous proof, assume now that $r \in L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$ and let $\mathcal{A} = (Q, q_0, \delta, \mathrm{out})$ be a trim DMVA accepting r. Then for every $q \in Q$ there exists $w_q \in \Sigma^*$ such that $q = \delta(q_0, w_q)$. If w'_q is another word such that $q = \delta(q_0, w'_q)$, then $[w_q] = [w'_q]$. We define a mapping $\varphi_r : Q \to Q_r$ by $\varphi(q) = [w_q]$. Then $\varphi_r : \mathcal{A} \to \mathcal{A}_r$ is a homomorphism. Indeed, $\delta(q_0, \varepsilon) = q_0$, and thus $\varphi_r(q_0) = [\varepsilon]$. Furthermore, let $q \in Q$ and $w \in \Sigma^*$ such that $\delta(q, w)$ exists. Then $\varphi_r(\delta(q, w)) = \varphi_r(\delta(\delta(q_0, w_q), w)) = \varphi_r(\delta(q_0, w_q w)) = [w_q w] = \delta'([\varepsilon], w_q w) = \delta'([w_q], w) = \delta'(\varphi_r(q), w)$ proving our claim. Hence, we have obtained the following result.

Theorem 3.11. *Let Σ be an alphabet and L be a bounded distributive lattice. For every fuzzy recognizable language $r \in L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$, there exists a minimum trim DMVA $\mathcal{A}_r$ with $\|\mathcal{A}_r\| = r$.*

Any trim DMVA $\mathcal{A}$ which is isomorphic to $\mathcal{A}_r$ will be also called a *minimum* automaton for r. Next, we show that for every fuzzy recognizable language $r \in L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$, we can effectively construct a minimum automaton accepting r. Let us assume that $\mathcal{A} = (Q, q_0, \delta, \mathrm{out})$ is a trim DMVA with behavior $\|\mathcal{A}\| = r$. We define an equivalence relation $\equiv$ on Q as follows: for every $q, q' \in Q$, $q \equiv q'$ iff $\mathrm{out}(\delta(q, w)) = \mathrm{out}(\delta(q', w))$ for every $w \in \Sigma^*$. Then $\mathcal{A}$ is called *reduced* if $q \equiv q'$ implies $q = q'$ for every $q, q' \in Q$. It is easy to see that $\mathcal{A}$ is reduced iff the strong homomorphism $\varphi_r : \mathcal{A} \to \mathcal{A}_r$ is injective. Since φ_r is also surjective, we conclude the following proposition.

Proposition 3.12. *A DMVA $\mathcal{A}$ accepting $r \in L^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$ is minimum iff it is trim and reduced.*

The previous proposition actually points out a way to construct a minimum DMVA accepting r: we start from a trim DMVA $\mathcal{A} = (Q, q_0, \delta, \mathrm{out})$ with $\|\mathcal{A}\| = r$ and we merge its equivalent states. Therefore, we prove that the equivalence $q \equiv q'$ is decidable for every pair of states $q, q' \in Q$, and we give an algorithm which uses at most $\mathrm{card}(Q)$ iterations. To this end, we introduce the equivalence relations $\equiv_n$ $(n \geq 0)$ on Q, given by $q \equiv_n q'$ iff $\mathrm{out}(\delta(q, w)) = \mathrm{out}(\delta(q', w))$ for every $w \in \bigcup_{0 \leq k \leq n} \Sigma^k$. Obviously, $\equiv_0 \supseteq \equiv_1 \supseteq \cdots \supseteq \equiv_n \supseteq \cdots$ hence $\equiv = \bigcap_{n \geq 0} \equiv_n$. We show that if there exists an $n \geq 0$ such that $\equiv_n = \equiv_{n+1}$, then $\equiv_{n+1} = \equiv_{n+l}$ for every $l \geq 2$. Indeed, assume that $\equiv_n = \equiv_{n+1}$. Then for every $q, q' \in Q$,

$$
\begin{aligned}
& q \equiv_{n+1} q' \\
& \quad \Longleftrightarrow \quad \mathrm{out}\big(\delta(q, w)\big) = \mathrm{out}\big(\delta(q', w)\big) \quad \text{for every } w \in \bigcup_{0 \leq k \leq n+1} \Sigma^k \\
& \quad \Longleftrightarrow \quad \mathrm{out}\big(\delta(q, au)\big) = \mathrm{out}\big(\delta(q', au)\big) \quad \text{for every } a \in \Sigma,\ u \in \bigcup_{0 \leq k \leq n} \Sigma^k
\end{aligned}
$$

$$\begin{aligned}
&\iff \operatorname{out}\big(\delta(\delta(q,a),u)\big) = \operatorname{out}\big(\delta\big(\delta(q',a),u\big)\big) \\
&\qquad \text{for every } a \in \Sigma,\ u \in \bigcup_{0\le k\le n} \Sigma^k \\
&\iff \delta(q,a) \equiv_n \delta(q',a) \quad \text{for every } a \in \Sigma \\
&\iff \delta(q,a) \equiv_{n+1} \delta(q',a) \quad \text{for every } a \in \Sigma \quad \text{(by hypothesis)} \\
&\iff \operatorname{out}\big(\delta\big(\delta(q,a),u\big)\big) = \operatorname{out}\big(\delta\big(\delta(q',a),u\big)\big) \\
&\qquad \text{for every } a \in \Sigma, u \in \bigcup_{0\le k\le n+1} \Sigma^k \\
&\iff q \equiv_{n+2} q'.
\end{aligned}$$

Therefore, by induction, we have $\equiv_{n+1} = \equiv_{n+l}$ for every $l \geq 2$. Now we let $e_0, e_1, \ldots$ denote the numbers of the equivalence classes of $\equiv_0, \equiv_1, \ldots,$ respectively. Then $e_0 \leq e_1 \leq \cdots \leq \operatorname{card}(Q)$. Thus, there exists an $n \leq \operatorname{card}(Q)$ such that $e_n = e_{n+1}$, hence $\equiv_n = \equiv_{n+1}$ and so $\equiv = \equiv_n$. We conclude that the equivalence $q \equiv q'$ is decidable in at most $\operatorname{card}(Q)$ iterations.

We complete this subsection with two further important consequences of Theorem 3.5. First, a *pumping lemma* is valid within the class $L^{\text{rec}}\langle\langle \Sigma^* \rangle\rangle$.

Proposition 3.13. *Let $r \in L^{\text{rec}}\langle\langle \Sigma^* \rangle\rangle$. There exists an integer $m > 0$ such that for every $w \in \Sigma^*$ with $|w| > m$, the word w can be written as $w = w_1 u w_2$ with $|u| > 0$ and $|w_1 w_2| < m$, and $(r, w_1 u^k w_2) = (r, w)$ for every $k \geq 0$.*

Proof. Let $r = \bigvee_{1\le i\le n} k_i \wedge 1_{R_i}$. Then the pumping lemma holds for every recognizable language R_i $(1 \leq i \leq n)$, and let m_i be the corresponding integer for R_i. We conclude our proof by letting $m = \mathsf{max}\{m_1, \ldots, m_n\}$. □

A pumping lemma for fuzzy recognizable languages over the interval $[0, 1]$, has been proved in [8] by means of fuzzy monoid recognizability.

Now we show that the *equivalence problem* is decidable for multi-valued automata over Σ and L. In fact, we prove the following stronger result.

Theorem 3.14. *Let Σ be an alphabet and L be a bounded distributive lattice. For every two fuzzy recognizable languages $r, r' \in L^{\text{rec}}\langle\langle \Sigma^* \rangle\rangle$, the relations $r \leq r'$ and $r = r'$ are decidable.*

Proof. Let $r = \bigvee_{1\le i\le n} k_i \wedge 1_{R_i}$ with pairwise disjoint recognizable languages R_i and $r' = \bigvee_{1\le j\le m} k'_j \wedge 1_{R'_j}$ with pairwise disjoint recognizable languages R'_j. Clearly, our decidability problems reduce to well-known decidability problems for recognizable languages. For instance in case of equality, we check that whenever $R_i \cap R'_j \neq \emptyset$ then $k_i = k'_j$. □

3.2 Fuzzy Recognizability over Infinite Words

In this subsection, we introduce Büchi and Muller multi-valued automata consuming infinite words. We show that both models accept the same class of infinitary fuzzy languages, and a Kleene-type theorem holds for this class. The material of the present and the next subsection is based on [22].

Definition 3.15.

(a) A multi-valued Muller automaton *(MVMA for short)* over Σ and L *is a quadruple* $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathcal{F})$, *where* Q *is the* finite state set, $\mathrm{in} : Q \to L$ *is the* initial distribution, $\mathrm{wt} : Q \times \Sigma \times Q \to L$ *is the mapping assigning* weights *to the transitions of the automaton, and* $\mathcal{F} \subseteq \mathcal{P}(Q)$ *is the* family of final state sets.

(b) An MVMA $\mathcal{A}$ *is a* multi-valued Büchi automaton *(MVBA for short) if there is a set* $F \subseteq Q$ *such that* $\mathcal{F} = \{S \subseteq Q \mid S \cap F \neq \emptyset\}$.

Let $w = a_0 a_1 \ldots \in \Sigma^\omega$. A *path of* $\mathcal{A}$ *over* w is an infinite sequence of transitions $P_w = (t_i)_{i \geq 0}$, so that $t_i = (q_i, a_i, q_{i+1}) \in Q \times \Sigma \times Q$ for every $i \geq 0$. The *weight of* P_w is defined by

$$\mathrm{weight}(P_w) = \mathrm{in}(q_0) \wedge \bigwedge_{i \geq 0} \mathrm{wt}(t_i).$$

Observe that $\mathrm{weight}(P_w)$ is well-defined since $\mathrm{weight}(P_w) \in L_{\mathcal{A}}$, where $L_{\mathcal{A}}$ is the finite sub-lattice of L generated by $\{0,1\} \cup \{\mathrm{in}(q) \mid q \in Q\} \cup \{\mathrm{wt}(t) \mid t \in Q \times \Sigma \times Q\}$. The path P_w is called *successful* if the set of states that appear infinitely often along P_w constitutes a final state set. The *behavior* of $\mathcal{A}$ is the infinitary fuzzy language

$$\|\mathcal{A}\| : \Sigma^\omega \to L$$

which is defined by

$$(\|\mathcal{A}\|, w) = \bigvee_{P_w} \mathrm{weight}(P_w)$$

for $w \in \Sigma^\omega$, where the supremum is taken over all successful paths P_w of $\mathcal{A}$ over w. Since $L_{\mathcal{A}}$ is finite, $(\|\mathcal{A}\|, w)$ exists and $(\|\mathcal{A}\|, w) \in L_{\mathcal{A}}$ for every $w \in \Sigma^\omega$.

An infinitary fuzzy language $r \in L\langle\langle \Sigma^\omega \rangle\rangle$ is said to be *fuzzy Muller recognizable* (resp. *fuzzy Büchi recognizable* or *fuzzy ω-recognizable*) if there is an MVMA (resp. an MVBA) $\mathcal{A}$ so that $r = \|\mathcal{A}\|$. We denote the family of all fuzzy Muller recognizable (resp. fuzzy ω-recognizable) languages over Σ and L by $L^{\text{M-rec}}\langle\langle \Sigma^\omega \rangle\rangle$ (resp. $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$). It should be clear that the class $L^{\text{M-rec}}\langle\langle \Sigma^\omega \rangle\rangle$ (resp. $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$) coincides with the class of Muller recognizable (resp. ω-recognizable) series over Σ and the semiring $\langle L, \vee, \wedge, 0, 1 \rangle$ (see [22, 23]). Clearly $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle \subseteq L^{\text{M-rec}}\langle\langle \Sigma^\omega \rangle\rangle$. Later on, we shall prove that in fact the two classes coincide.

Two multi-valued Muller (resp. Büchi) automata $\mathcal{A}$ and $\mathcal{A}'$ over Σ and L are called *equivalent* if $\|\mathcal{A}\| = \|\mathcal{A}'\|$.

Given a language $R \subseteq \Sigma^\omega$, its characteristic infinitary fuzzy language $1_R \in L\langle\langle \Sigma^\omega \rangle\rangle$ is defined in a similar way as for finitary languages. Obviously, every unweighted Büchi automaton with input alphabet Σ can be considered as an

MVBA over Σ and L with weights only 0 and 1. Therefore, we immediately obtain the next proposition.

Proposition 3.16 ([23]). *Let $R \subseteq \Sigma^\omega$ be an ω-recognizable language. Then the characteristic infinitary fuzzy language $1_R \in L\langle\langle \Sigma^\omega \rangle\rangle$ is ω-recognizable.*

Assume now that $R_1, \ldots, R_n \subseteq \Sigma^\omega$ are ω-recognizable languages, $k_1, \ldots, k_n \in L$, and let

$$r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}.$$

Such a language r is called *fuzzy ω-recognizable step language* [23]. Actually a fuzzy ω-recognizable step language is fuzzy ω-recognizable. Indeed, let us assume that for every $1 \leq i \leq n$ we are given a Büchi automaton $\mathcal{A}_i = (Q_i, I_i, \Delta_i, \mathcal{F}_i)$ accepting R_i (see [56]). We fix an $1 \leq i \leq n$. Then as already noted above, $\mathcal{A}_i$ can be considered as an MVBA $(Q_i, \mathrm{in}_i, \mathrm{wt}_i, \mathcal{F}_i)$ over Σ and L with behavior 1_{R_i}. We consider the MVBA $\overline{\mathcal{A}_i} = (Q_i, k_i \wedge \mathrm{in}_i, \mathrm{wt}_i, \mathcal{F}_i)$. Obviously $\|\overline{\mathcal{A}_i}\| = k_i \wedge 1_{R_i}$. Now let $\mathcal{A}$ be the MVBA obtained as the disjoint union of all $\overline{\mathcal{A}_i}$ $(1 \leq i \leq n)$. Clearly, $\|\mathcal{A}\| = r$ proving our claim.

Theorem 3.17 ([22]). *Let Σ be an alphabet and L be a bounded distributive lattice. Then the following statements are equivalent for every infinitary fuzzy language $r \in L\langle\langle \Sigma^\omega \rangle\rangle$:*

(i) r is fuzzy Muller recognizable.
(ii) r is fuzzy ω-recognizable.
(iii) r is a fuzzy ω-recognizable step language.

Proof. We show that (i) implies (iii). Let $r \in L^{\text{M-rec}}\langle\langle \Sigma^\omega \rangle\rangle$ and $\mathcal{A}$ be an MVMA accepting r. Then $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$ where $L_{\mathcal{A}} = \{k_1, \ldots, k_n\}$ and $R_i = \{w \in \Sigma^\omega \mid (r, w) = k_i\}$ for every $1 \leq i \leq n$. Following the proof of Theorem 3.5, we can show that the languages R_i $(1 \leq i \leq n)$ are Muller recognizable, and thus ω-recognizable, which in turn implies that r is a fuzzy ω-recognizable step language.

The implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are also true as already shown. □

Observe that our proof above is effective (recall the discussion after Theorem 3.5). In the sequel without any further notation, we write every fuzzy ω-recognizable language r over Σ and L as $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$.

Theorem 3.17 has very interesting consequences. Firstly, we can easily obtain closure properties of fuzzy ω-recognizable languages.

Proposition 3.18. *The class $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$ of fuzzy ω-recognizable languages is closed under supremum, infimum, and scalar infimum.*

Proof. Closure under supremum is immediate and closure under scalar infimum is obtained by distributivity of L. Furthermore, for the closure under infimum one has to recall that the class of ω-recognizable languages is closed under intersection. □

Consider now two alphabets Σ, Δ and a non-deleting homomorphism $h : \Sigma^* \to \Delta^*$. Then h can be extended to a mapping $h : \Sigma^\omega \to \Delta^\omega$ by setting $h(a_0 a_1 \ldots) = h(a_0)h(a_1)\ldots$ for every infinite word $a_0 a_1 \ldots \in \Sigma^\omega$. Let $r \in L\langle\langle\Sigma^\omega\rangle\rangle$ be an infinitary fuzzy language having finite image, and $R \subseteq \Sigma^\omega$. We define the infinitary fuzzy language $h_R(r) \in L\langle\langle\Delta^\omega\rangle\rangle$ by

$$\big(h_R(r), u\big) = \bigvee_{w \in h^{-1}(u) \cap R} (r, w)$$

for every $u \in \Delta^\omega$. We denote the mapping h_{Σ^ω} simply by h. Furthermore, if $s \in L\langle\langle\Delta^\omega\rangle\rangle$, then the fuzzy language $h^{-1}(s) \in L\langle\langle\Sigma^\omega\rangle\rangle$ is specified by

$$\big(h^{-1}(s), w\big) = \big(s, h(w)\big)$$

for every $w \in \Sigma^\omega$.

Proposition 3.19.

(i) Let $(L, \leq)$ and $(L', \leq)$ be two bounded distributive lattices and $f : L \to L'$ be any mapping. Then for every fuzzy ω-recognizable language r in $L\langle\langle\Sigma^\omega\rangle\rangle$ the fuzzy language $f(r) \in L'\langle\langle\Sigma^\omega\rangle\rangle$ is again ω-recognizable.

(ii) Let $h : \Sigma^\omega \to \Delta^\omega$ be a non-deleting homomorphism and $R \subseteq \Sigma^\omega$ be an ω-recognizable language. Then $h_R : L\langle\langle\Sigma^\omega\rangle\rangle \to L\langle\langle\Delta^\omega\rangle\rangle$ and $h^{-1} : L\langle\langle\Delta^\omega\rangle\rangle \to L\langle\langle\Sigma^\omega\rangle\rangle$ preserve the ω-recognizability property of fuzzy languages.

Proof. Statement (i) can be shown as Proposition 3.6, using Theorem 3.17. Now let $r \in L^{\omega\text{-rec}}\langle\langle\Sigma^\omega\rangle\rangle$ with $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$. For every $u \in \Delta^\omega$, we have

$$\big(h_R(r), u\big) = \bigvee_{w \in h^{-1}(u) \cap R} (r, w) = \bigvee_{1 \leq i \leq n} \Bigg(k_i \wedge \bigvee_{w \in h^{-1}(u) \cap R} (1_{R_i}, w)\Bigg)$$

which is equal to $\bigvee_{1 \leq i \leq n} k_i \wedge (1_{h(R_i \cap R)}, u)$. Hence,

$$h_R(r) = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{h(R_i \cap R)}.$$

Since the class of ω-recognizable languages is closed under non-deleting homomorphisms [56], we obtain that the fuzzy language $h_R(r)$ is ω-recognizable.

Finally, assume that $s = \bigvee_{1 \leq j \leq m} k'_j \wedge 1_{R'_j}$. Then

$$h^{-1}(s) = \bigvee_{1 \leq j \leq m} k'_j \wedge 1_{h^{-1}(R'_j)}.$$

The class of ω-recognizable languages is closed under inverse non-deleting homomorphisms [56], therefore, $h^{-1}(s)$ is fuzzy ω-recognizable and our proof is completed. $\square$

As an immediate consequence of Proposition 3.19(i), we obtain the closure of fuzzy ω-recognizable languages under negation functions.

Corollary 3.20. *Let $(L, \leq, ^{-})$ be a bounded distributive lattice with negation function, and $r \in L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$. Then also $\overline{r} \in L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$.*

By Theorem 3.17, we get the statements of Proposition 3.7 in the setting of infinitary fuzzy languages. More precisely, for every $r \in L\langle\langle \Sigma^\omega \rangle\rangle$ and $l \in L$, we consider the infinitary languages $r_{\geq l} = \{w \in \Sigma^\omega \mid (r, w) \geq l\}$ and $r_{=l} = r^{-1}(l) = \{w \in \Sigma^\omega \mid (r, w) = l\}$.

Proposition 3.21. *For every fuzzy language $r \in L\langle\langle \Sigma^\omega \rangle\rangle$, the following statements are equivalent:*

(i) r is fuzzy ω-recognizable.
(ii) r has finite image, and for every $l \in L$, $r_{=l}$ is an ω-recognizable language.
(iii) r has finite image, and for every $l \in L$, $r_{\geq l}$ is an ω-recognizable language.

As a further consequence of Theorem 3.17, we prove that the *equivalence problem* is decidable for multi-valued Muller (resp. Büchi) automata over Σ and L. In fact, we get the subsequent stronger decidability result.

Theorem 3.22. *Let Σ be an alphabet and L be a bounded distributive lattice. For every two fuzzy ω-recognizable languages $r, r' \in L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$, the relations $r \leq r'$ and $r = r'$ are decidable.*

Proof. See the proof of Theorem 3.14. □

Finally, we show that a Kleene theorem holds for fuzzy ω-recognizable languages. We firstly recall the ω-rational operations of fuzzy languages (see [37, 63, 27]). Let $r \in L\langle\langle \Sigma^* \rangle\rangle$ and $r' \in L\langle\langle \Sigma^\omega \rangle\rangle$. Then the *Cauchy product $rr' \in L\langle\langle \Sigma^\omega \rangle\rangle$ of r and r'* is defined by $(rr', w) = \vee\{(r, u) \wedge (r', u') \mid w = uu', u \in \Sigma^*, u' \in \Sigma^\omega\}$ for every $w \in \Sigma^\omega$. Furthermore, whenever r is proper, i.e., $(r, \varepsilon) = 0$, we define the *ω-star $r^\omega \in L\langle\langle \Sigma^\omega \rangle\rangle$ of r* as follows: $(r^\omega, w) = \vee\{\wedge\{(r, w_1), (r, w_2), \ldots\} \mid w = w_1 w_2 \ldots$ with $w_1, w_2, \ldots \in \Sigma^*\}$ for every $w \in \Sigma^\omega$. Now the class of *fuzzy ω-rational languages over Σ and L*, denoted by $L^{\omega\text{-rat}}\langle\langle \Sigma^\omega \rangle\rangle$, is the least class of infinitary fuzzy languages generated by the finitary fuzzy languages (over Σ and L) with finite support, applying finitely many times the operations of supremum, Cauchy product, star, and ω-star. Every fuzzy ω-recognizable language $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$ over Σ and L is ω-rational. Indeed, for every $1 \leq i \leq n$ the language R_i is ω-rational and thus 1_{R_i} is a fuzzy ω-rational language with values 0 and 1. Then $k_i \wedge 1_{R_i}$ is just the Cauchy product of the series $k_i\varepsilon$ and 1_{R_i}, where the series $k_i\varepsilon$ is defined by $(k_i\varepsilon, w) = 1$ if $w = \varepsilon$ and $(k_i\varepsilon, w) = 0$ otherwise, for every $w \in \Sigma^*$. Conversely, we claim that $L^{\omega\text{-rat}}\langle\langle \Sigma^\omega \rangle\rangle \subseteq L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$. For this, it suffices to show that for every $r \in L^{\text{rec}}\langle\langle \Sigma^* \rangle\rangle, r' \in L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$ the fuzzy language $rr' \in L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$, and for every proper fuzzy language $r \in L^{\text{rec}}\langle\langle \Sigma^* \rangle\rangle$, the ω-star $r^\omega \in L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$. Once again by using Theorems 3.5 and 3.17, this is reduced to the well-known closure properties of recognizable languages under the ω-rational operations (see, for instance, [56]). Therefore, we get the subsequent Kleene theorem for infinitary fuzzy languages.

Theorem 3.23. *Let Σ be an alphabet and L be a bounded distributive lattice. Then $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle = L^{\omega\text{-rat}}\langle\langle \Sigma^\omega \rangle\rangle$.*

For a Kleene theorem for infinitary formal power series over a larger class of semirings than bounded distributive lattices, we refer the reader to [25–27].

3.3 Multi-valued MSO Logic

Following [22], we introduce a multi-valued monadic second-order logic (multi-valued MSO logic, for short) over infinite words, and we state a multi-valued version of Büchi's theorem [11] for fuzzy languages over bounded distributive lattices with negation function. A corresponding theory for finite words has been obtained as an application of weighted logics over locally finite semirings (see [18, 20]). Throughout this subsection, we assume that $(L, \leq, ^{-})$ is a bounded distributive lattice with negation function.

Every word $w = a_0 a_1 \ldots \in \Sigma^\omega$, with $a_0, a_1, \ldots \in \Sigma$, is also written as $w = w(0)w(1)\ldots$ with $w(i) = a_i$ for $i \geq 0$. Then every $w \in \Sigma^\omega$ is represented by the structure $(\omega, \leq, (R_a)_{a \in \Sigma})$ where $R_a = \{i \mid w(i) = a\}$ for $a \in \Sigma$. Given a finite set $\mathcal{V}$ of first- and second-order variables, a $(w, \mathcal{V})$-*assignment* σ is a mapping assigning elements of ω to first-order variables from $\mathcal{V}$, and subsets of ω to second-order variables from $\mathcal{V}$. If x is a first-order variable and $i \in \omega$, then $\sigma[x \to i]$ denotes the $(w, \mathcal{V} \cup \{x\})$-assignment which assigns i to x and acts as σ on $\mathcal{V} \setminus \{x\}$. For a second-order variable X and $I \subseteq \omega$, the notation $\sigma[X \to I]$ has a similar meaning.

By using the extended alphabet $\Sigma_{\mathcal{V}} = \Sigma \times \{0,1\}^{\mathcal{V}}$, we encode pairs (w, σ) for every $w \in \Sigma^\omega$ and every $(w, \mathcal{V})$-assignment σ. Every word in $\Sigma_{\mathcal{V}}^\omega$ is considered as a pair (w, σ) where w is the projection over Σ, and σ is the projection over $\{0,1\}^{\mathcal{V}}$. Then σ is a *valid* $(w, \mathcal{V})$-assignment if for every first-order variable $x \in \mathcal{V}$ the x-row contains exactly one 1. In this case, we identify σ with the $(w, \mathcal{V})$-assignment so that for every first-order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 on the x-row, and for every second-order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions labeled with 1 along the X-row. By standard automata constructions, it can be shown that the language

$$N_{\mathcal{V}} = \left\{ (w, \sigma) \in \Sigma_{\mathcal{V}}^\omega \;\middle|\; \sigma \text{ is a valid } (w, \mathcal{V})\text{-assignment} \right\}$$

is ω-recognizable.

Definition 3.24. *The set of all* MSO(L, Σ)*-formulas of the multi-valued MSO logic over Σ and L is defined to be the smallest set F such that:*

- *F contains all* atomic formulas k, $P_a(x)$, $x \leq y$, $x \in X$.
- *If $\varphi, \psi \in F$, then also $\neg\varphi, \varphi \vee \psi, \varphi \wedge \psi, \exists x \,.\, \varphi, \exists X \,.\, \varphi, \forall x \,.\, \varphi, \forall X \,.\, \varphi \in F$*

where $k \in L$, $a \in \Sigma$, x, y are first-order variables and X is a second-order variable.

We represent the semantics of the formulas in $\mathrm{MSO}(L, \Sigma)$ as infinitary fuzzy languages over the extended alphabet $\Sigma_{\mathcal{V}}$ and the lattice L. Here, our definition of semantics is more general that the one used in [18, 23, 20]. There, the authors assigned to every atomic formula $P_a(x), x \leq y$, or $x \in X$, respectively, the characteristic series of its associated MSO-language. These series take on only $0, 1$. Here, we assume that there is a function f assigning to every atomic formula φ of the form $P_a(x), x \leq y$, or $x \in X$, respectively, an infinitary fuzzy language $f(\varphi)$ in $L\langle\langle \Sigma_\varphi^\omega \rangle\rangle$ (where Σ_φ stands for $\Sigma_{\mathrm{Free}(\varphi)}$). This generalization has been already used in other logics. For instance, in many-valued predicate logic, every object variable is being assigned a value from an L-structure M, where L is a BL-algebra (see Sect. 5 in [32]). In [39], the atomic propositions of the multi-valued LTL take values from a subset of the underlying De Morgan algebra. Our assignment f here is called *ω-recognizable* if the fuzzy language $f(\varphi)$ is ω-recognizable for every atomic formula φ. Later on, we always require that f is an ω-recognizable assignment. Thus, the language $f(\varphi)$ will be taking on only finitely many values, for every atomic formula φ. Therefore, we will call f a *multi-valued atomic assignment over Σ*, if $f(\varphi)$ takes on only finitely many values, for every atomic formula φ.

Definition 3.25. *Let $\varphi \in \mathrm{MSO}(L, \Sigma)$, $\mathcal{V}$ be a finite set of variables containing* $\mathrm{Free}(\varphi)$, *and f be a multi-valued atomic assignment over Σ. We define the* f-semantics *of φ to be an infinitary fuzzy language $\|\varphi\|_{\mathcal{V}}^f \in L\langle\langle \Sigma_{\mathcal{V}}^\omega \rangle\rangle$ in the following way. Let $(w, \sigma) \in \Sigma_{\mathcal{V}}^\omega$. If σ is not a valid $(w, \mathcal{V})$-assignment, then we put $(\|\varphi\|_{\mathcal{V}}^f, (w, \sigma)) = 0$. Otherwise, we inductively define $(\|\varphi\|_{\mathcal{V}}^f, (w, \sigma)) \in L$ as follows:*

- $(\|k\|_{\mathcal{V}}^f, (w, \sigma)) = k$
- $(\|\varphi\|_{\mathcal{V}}^f, (w, \sigma)) = (f(\varphi), (w, \sigma|_{\mathrm{Free}(\varphi)}))$ *if φ is an atomic formula of the form $P_a(x), x \leq y$, or $x \in X$*
- $(\|\neg\varphi\|_{\mathcal{V}}^f, (w, \sigma)) = \overline{(\|\varphi\|_{\mathcal{V}}^f, (w, \sigma))}$
- $(\|\varphi \vee \psi\|_{\mathcal{V}}^f, (w, \sigma)) = (\|\varphi\|_{\mathcal{V}}^f, (w, \sigma)) \vee (\|\psi\|_{\mathcal{V}}^f, (w, \sigma))$
- $(\|\varphi \wedge \psi\|_{\mathcal{V}}^f, (w, \sigma)) = (\|\varphi\|_{\mathcal{V}}^f, (w, \sigma)) \wedge (\|\psi\|_{\mathcal{V}}^f, (w, \sigma))$
- $(\|\exists x \,.\, \varphi\|_{\mathcal{V}}^f, (w, \sigma)) = \bigvee_{i \in \omega} (\|\varphi\|_{\mathcal{V} \cup \{x\}}^f, (w, \sigma[x \to i]))$
- $(\|\exists X \,.\, \varphi\|_{\mathcal{V}}^f, (w, \sigma)) = \bigvee_{I \subseteq \omega} (\|\varphi\|_{\mathcal{V} \cup \{X\}}^f, (w, \sigma[X \to I]))$
- $(\|\forall x \,.\, \varphi\|_{\mathcal{V}}^f, (w, \sigma)) = \bigwedge_{i \in \omega} (\|\varphi\|_{\mathcal{V} \cup \{x\}}^f, (w, \sigma[x \to i]))$
- $(\|\forall X \,.\, \varphi\|_{\mathcal{V}}^f, (w, \sigma)) = \bigwedge_{I \subseteq \omega} (\|\varphi\|_{\mathcal{V} \cup \{X\}}^f, (w, \sigma[X \to I]))$.

It should be clear that in Definition 3.25 all the occurring infinite suprema and infima exist in L (without any further completeness assumption). More precisely, one can show by induction on the structure of formulas φ that $\|\varphi\|^f_{\mathcal{V}}$ takes on only finitely many values. Indeed, for atomic formulas, this is clear by assumption, and the property is preserved by negation, disjunction, and conjunction. Since L is a lattice, the property is also preserved by infinite suprema and infima, proving our claim.

If the multi-valued atomic assignment is well-known, then we omit the superscript f from $\|\varphi\|^f_{\mathcal{V}}$. Furthermore, we simply write $\|\varphi\|$ for $\|\varphi\|_{\mathrm{Free}(\varphi)}$. If φ has no free variables, i.e., if it is a sentence, then $\|\varphi\| \in L\langle\langle \Sigma^\omega \rangle\rangle$.

An infinitary fuzzy language $r \in L\langle\langle \Sigma^\omega \rangle\rangle$ is called *MSO-f-definable* if there is a sentence $\varphi \in \mathrm{MSO}(L, \Sigma)$ such that $r = \|\varphi\|^f$. We let $L^{f\text{-mso}}\langle\langle \Sigma^\omega \rangle\rangle$ comprise all fuzzy languages from $L\langle\langle \Sigma^\omega \rangle\rangle$ which are f-definable by some sentence in $\mathrm{MSO}(L, \Sigma)$. In the sequel, we show that the classes $L^{f\text{-mso}}\langle\langle \Sigma^\omega \rangle\rangle$ and $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$ coincide.

Let us first give an example of possible interpretations of multi-valued MSO-formulas. The reader can find more examples in [18, 22, 23, 20].

Example 3.26 ([22]). We consider the bounded distributive lattice $(\mathbb{N} \cup \{\infty\}, \leq, ^-)$ (where $\mathbb{N}$ is the set of natural numbers and $^-$ is an arbitrary negation function). Let $\Sigma = \{a, b, c\}$ and f be the multi-valued atomic assignment over Σ, determined in the following way. For every $w \in \Sigma^\omega$ and every valid $(w, \{x\})$-assignment σ, we set:

- $(f(P_a(x)), (w, \sigma)) = 0$
- $(f(P_b(x)), (w, \sigma)) = \begin{cases} 1 & \text{if } w(\sigma(x)) = b, \\ 0 & \text{otherwise} \end{cases}$
- $(f(P_c(x)), (w, \sigma)) = \begin{cases} 2 & \text{if } w(\sigma(x)) = c, \\ 0 & \text{otherwise.} \end{cases}$

For every other atomic formula φ, $f(\varphi)$ is the fuzzy language with image $\{0\}$. Let $\varphi = \forall x \,.\, (P_a(x) \vee P_b(x) \vee P_c(x))$. In fact, φ is a sentence, and for every word $w \in \Sigma^\omega$ the semantics $\|\varphi\|^f$ returns the value 0 if the letter a occurs at least once in w, the value 1 if no a appears in w but b occurs at least once, and it returns the value 2 if $w = c^\omega$.

The reader should observe that the above definition of semantics is valid for every formula $\varphi \in \mathrm{MSO}(L, \Sigma)$ and every finite set $\mathcal{V}$ of variables containing $\mathrm{Free}(\varphi)$. The following proposition states that the f-semantics $\|\varphi\|^f_{\mathcal{V}}$ is in fact independent of the set $\mathcal{V}$; it depends only on $\mathrm{Free}(\varphi)$. For a proof, we refer the reader to [18, 20].

Proposition 3.27. *For every $\varphi \in \mathrm{MSO}(L, \Sigma)$, every finite set $\mathcal{V}$ of variables with $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$, and every multi-valued atomic assignment f over Σ, it holds that*

$$\left(\|\varphi\|_{\mathcal{V}}^{f},(w,\sigma)\right)=\left(\|\varphi\|^{f},(w,\sigma|_{\mathrm{Free}(\varphi)})\right)$$

for every $(w,\sigma)\in\Sigma_{\mathcal{V}}^{\omega}$*, where* σ *is a valid* $(w,\mathcal{V})$*-assignment. Furthermore, the fuzzy language* $\|\varphi\|^{f}$ *is* ω*-recognizable iff the fuzzy language* $\|\varphi\|_{\mathcal{V}}^{f}$ *is* ω*-recognizable.*

The next lemma states a further closure property of the class of fuzzy ω-recognizable languages.

Lemma 3.28. *Let* $h:\Sigma^{\omega}\to\Delta^{\omega}$ *be a non-deleting homomorphism,* $R\subseteq\Sigma^{\omega}$ *be an* ω*-recognizable language, and* $r\in L^{\omega\text{-rec}}\langle\!\langle\Sigma^{\omega}\rangle\!\rangle$ *be a fuzzy* ω*-recognizable language. Then the language* $\bigwedge_{h,R}(r)\in L\langle\!\langle\Delta^{\omega}\rangle\!\rangle$ *defined by* $(\bigwedge_{h,R}(r),u)=\bigwedge_{w\in h^{-1}(u)\cap R}(r,w)$ *is fuzzy* ω*-recognizable.*

Proof. Let $(L^{d},\leq^{d})=(L,\geq)$ be the dual lattice of L, which is obtained by interchanging suprema and infima. Since r takes on only finitely many values and each value on an ω-recognizable language, r is also fuzzy ω-recognizable over L^{d}. Consider the transformation $h_{R}^{d}:L^{d}\langle\!\langle\Sigma^{\omega}\rangle\!\rangle\to L^{d}\langle\!\langle\Delta^{\omega}\rangle\!\rangle$. By Proposition 3.19(ii), we obtain $h_{R}^{d}(r)\in(L^{d})^{\omega\text{-rec}}\langle\!\langle\Delta^{\omega}\rangle\!\rangle$ which in turn means that $h_{R}^{d}(r)\in L^{\omega\text{-rec}}\langle\!\langle\Delta^{\omega}\rangle\!\rangle$. Since suprema in L^{d} equal infima in L, we have $h_{R}^{d}(r)=\bigwedge_{h,R}(r)$ and our proof is completed. □

Proposition 3.29. *Let* $\varphi,\psi\in\mathrm{MSO}(L,\Sigma)$ *such that* $\|\varphi\|_{\mathcal{V}}^{f},\|\psi\|_{\mathcal{V}}^{f}$ *are fuzzy* ω*-recognizable languages where* f *is a multi-valued atomic assignment, and* $\mathcal{V}$ *is a finite set of variables with* $\mathrm{Free}(\varphi)\cup\mathrm{Free}(\psi)\subseteq\mathcal{V}$*. Then the languages* $\|\neg\varphi\|_{\mathcal{V}}^{f}$, $\|\varphi\vee\psi\|_{\mathcal{V}}^{f}$, $\|\varphi\wedge\psi\|_{\mathcal{V}}^{f}$, $\|\exists x\,.\,\varphi\|_{\mathcal{V}}^{f}$, $\|\exists X\,.\,\varphi\|_{\mathcal{V}}^{f}$, $\|\forall x\,.\,\varphi\|_{\mathcal{V}}^{f}$*, and* $\|\forall X\,.\,\varphi\|_{\mathcal{V}}^{f}$ *are fuzzy* ω*-recognizable.*

Proof. The f-semantics of the negation of φ is fuzzy ω-recognizable by Corollary 3.20. The f-semantics of disjunction and conjunction of φ and ψ are fuzzy ω-recognizable by Proposition 3.18. Next, we deal with existential and universal quantifiers. By assumption, $\|\varphi\|_{\mathcal{V}}^{f}$ is fuzzy ω-recognizable. Let $\|\varphi\|_{\mathcal{V}}^{f}=\bigvee_{1\leq i\leq n}k_i\wedge 1_{R_i}$, and

$$h:\Sigma_{\mathcal{V}\cup\{x\}}^{\omega}\to\Sigma_{\mathcal{V}}^{\omega}\quad\text{and}\quad h':\Sigma_{\mathcal{V}\cup\{X\}}^{\omega}\to\Sigma_{\mathcal{V}}^{\omega}$$

be the non-deleting homomorphisms erasing the x-row and the X-row, respectively. Clearly,

$$\begin{aligned}&\|\exists x\,.\,\varphi\|_{\mathcal{V}}^{f}=h\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}^{f}\right) && \|\exists X\,.\,\varphi\|_{\mathcal{V}}^{f}=h'\left(\|\varphi\|_{\mathcal{V}\cup\{X\}}^{f}\right)\\ &\|\forall x\,.\,\varphi\|_{\mathcal{V}}^{f}=\bigwedge\nolimits_{h,N_{\mathcal{V}\cup\{x\}}}\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}^{f}\right) && \|\forall X\,.\,\varphi\|_{\mathcal{V}}^{f}=\bigwedge\nolimits_{h',\Sigma_{\mathcal{V}\cup\{X\}}^{\omega}}\left(\|\varphi\|_{\mathcal{V}\cup\{X\}}^{f}\right).\end{aligned}$$

We conclude our proof by applying Proposition 3.19(ii) and Lemma 3.28. □

Proposition 3.30. *Let* f *be any* ω*-recognizable multi-valued atomic assignment. Then* $L^{f\text{-mso}}\langle\!\langle\Sigma^{\omega}\rangle\!\rangle\subseteq L^{\omega\text{-rec}}\langle\!\langle\Sigma^{\omega}\rangle\!\rangle$.

Proof. We apply induction on the structure of MSO(L, Σ)-formulas using Proposition 3.29. □

Next, we define the *crisp atomic assignment* cf for atomic formulas. More precisely, let φ be an atomic formula of the form $P_a(x), x \leq y$, or $x \in X$. Then for every $(w, \sigma) \in \Sigma^\omega_\varphi$ with σ a valid assignment, we set:

- $(cf(P_a(x)), (w, \sigma)) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a, \\ 0 & \text{otherwise} \end{cases}$
- $(cf(x \leq y), (w, \sigma)) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y), \\ 0 & \text{otherwise} \end{cases}$
- $(cf(x \in X), (w, \sigma)) = \begin{cases} 1 & \text{if } \sigma(x) \in \sigma(X), \\ 0 & \text{otherwise.} \end{cases}$

Note that if φ is an atomic formula of this form, then $(\|\neg\varphi\|^{cf}, (w, \sigma)) = \overline{(cf(\varphi), (w, \sigma))}$ for every $(w, \sigma) \in N_\varphi$, and by the property of $^-$ that $\overline{1} = 0$ and $\overline{0} = 1$, the semantics of $\neg\varphi$ coincides with the one given in [23, 20]. Furthermore, the crisp atomic assignment is ω-recognizable [23]. We denote the class $L^{cf\text{-mso}}\langle\langle \Sigma^\omega \rangle\rangle$ simply by $L^{\text{mso}}\langle\langle \Sigma^\omega \rangle\rangle$.

Now we can state our Büchi-type characterization of the class $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$.

Theorem 3.31. *Let Σ be an alphabet and L be a bounded distributive lattice with any negation function. Then*

$$L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle = \bigcup_f L^{f\text{-mso}}\langle\langle \Sigma^\omega \rangle\rangle = L^{\text{mso}}\langle\langle \Sigma^\omega \rangle\rangle$$

where the union is taken over all ω-recognizable multi-valued atomic assignments.

Proof. Let $r \in L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle$ with $r = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$. We fix an $1 \leq i \leq n$. By Büchi's theorem [11], R_i is definable by a classical MSO-sentence φ_i. Clearly, φ_i can be considered as a multi-valued sentence over Σ and L. Then $\|\varphi_i\| = 1_{R_i}$ which in turn implies that $\|\bigvee_{1 \leq i \leq n} k_i \wedge \varphi_i\| = r$. Thus, $L^{\omega\text{-rec}}\langle\langle \Sigma^\omega \rangle\rangle \subseteq L^{\text{mso}}\langle\langle \Sigma^\omega \rangle\rangle$. Now Proposition 3.30 completes our proof. □

This result shows that for every formula $\varphi \in$ MSO(L, Σ), whose semantics is defined with any ω-recognizable multi-valued atomic assignment, we can construct an equivalent MSO(L, Σ)-formula with the crisp atomic assignment. In case of De Morgan algebras, an alternative simpler syntax of formulas of multi-valued MSO logic can be given by the grammar

$$\varphi ::= k \mid P_a(x) \mid x \leq y \mid x \in X \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x \,.\, \varphi \mid \exists X \,.\, \varphi.$$

We define the semantics $\|\varphi\|$ of formulas φ of this syntax exactly as in Definition 3.25. Given a multi-valued atomic assignment f, let $L^{\text{dm-}f\text{-mso}}\langle\langle \Sigma^\omega \rangle\rangle$ be the collection of all infinitary fuzzy languages definable in this logic. Then conjunction and universal quantifiers are determined by:

- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- $\forall x \mathbin{.} \varphi = \neg(\exists x \mathbin{.} \neg\varphi)$
- $\forall X \mathbin{.} \varphi = \neg(\exists X \mathbin{.} \neg\varphi)$

for every $\varphi, \psi \in \mathrm{MSO}(L, \Sigma)$. By using the De Morgan laws, we have the following equalities for every $(w, \sigma) \in \Sigma_{\mathcal{V}}^{\omega}$ where σ is a valid assignment:

- $(\|\varphi \wedge \psi\|_{\mathcal{V}}^{f}, (w,\sigma)) = (\|\varphi\|_{\mathcal{V}}^{f}, (w,\sigma)) \wedge (\|\psi\|_{\mathcal{V}}^{f}, (w,\sigma))$
- $(\|\forall x \mathbin{.} \varphi\|_{\mathcal{V}}^{f}, (w,\sigma)) = \bigwedge_{i\in\omega}(\|\varphi\|_{\mathcal{V}\cup\{x\}}^{f}, (w, \sigma[x \to i]))$
- $(\|\forall X \mathbin{.} \varphi\|_{\mathcal{V}}^{f}, (w,\sigma)) = \bigwedge_{I\subseteq\omega}(\|\varphi\|_{\mathcal{V}\cup\{X\}}^{f}, (w, \sigma[X \to I]))$.

The crisp atomic assignment cf is also defined as before, and we denote again the class $L^{\text{dm-}cf\text{-mso}}\langle\langle \Sigma^{\omega} \rangle\rangle$ simply by $L^{\text{dm-mso}}\langle\langle \Sigma^{\omega} \rangle\rangle$. Then the next result is an immediate consequence of Theorem 3.31 and the above equalities.

Corollary 3.32. *Let Σ be an alphabet and $(L, \leq, ^{-})$ be a De Morgan algebra. Then*
$$L^{\omega\text{-rec}}\langle\langle \Sigma^{\omega} \rangle\rangle = \bigcup_{f} L^{\text{dm-}f\text{-mso}}\langle\langle \Sigma^{\omega} \rangle\rangle = L^{\text{dm-mso}}\langle\langle \Sigma^{\omega} \rangle\rangle$$
where the union is taken over all ω-recognizable multi-valued atomic assignments.

4 Fuzzy Languages: An Overview

In the previous section, we have focused on fuzzy languages over bounded distributive lattices. Several other concepts of fuzzy languages occur in the literature, and they mainly differ in their underlying structure. The most general cases are covered by fuzzy languages over bounded ℓ-monoids and residuated lattices. Since every residuated lattice is a bounded ℓ-monoid, fuzzy languages over residuated lattices constitute a subclass of fuzzy languages over bounded ℓ-monoids. Fuzzy automata over these two concepts have been investigated recently. Actually, they are weighted automata over the corresponding induced semirings. Therefore, the properties of their behaviors mostly follow from the general theory of recognizable formal power series. Further properties of fuzzy recognizable languages over residuated lattices and bounded ℓ-monoids require specific restrictions for their underlying structures. In this section, we only highlight the most interesting results (without proofs) for fuzzy recognizable languages over bounded ℓ-monoids and residuated lattices. We also succinctly refer to fuzzy automata with outputs, to families of fuzzy languages, and to fuzzy tree languages. For the rest of this section, Σ will denote an arbitrary finite alphabet.

4.1 Fuzzy Languages over ℓ-Monoids

A *lattice-ordered monoid* (or *ℓ-monoid* for short) is a lattice $(L,\leq)$ equipped with an operation $\cdot$ and a distinguished element $e \in L$ such that the following conditions hold:

(i) $\langle L, \cdot, e\rangle$ is a monoid
(ii) $a \cdot (b \vee c) = a \cdot b \vee a \cdot c$ and $(a \vee b) \cdot c = a \cdot c \vee b \cdot c$

for every $a, b, c \in L$ (see [6]). Note that then $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$, for every $a, b, c \in L$.

The ℓ-monoid defined above is denoted by $(L, \vee, \cdot)$, and is called *bounded* if the lattice $(L,\leq)$ is bounded and

(iii) $a \cdot 0 = 0 \cdot a = 0$

for every $a \in L$. Furthermore, if $(L,\leq)$ is a complete lattice satisfying

(iv) $a \cdot (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \cdot b_i)$ and $(\bigvee_{i\in I} b_i) \cdot a = \bigvee_{i\in I}(b_i \cdot a)$

for every $a \in L$ and every countable family $(b_i \mid i \in I) \subseteq L$ of elements of L, then $(L, \vee, \cdot)$ is called *countably distributive*. Every bounded distributive lattice is a bounded ℓ-monoid with $\cdot = \wedge$ and $e = 1$. A further example of a bounded ℓ-monoid with $e = 1$ is given by any residuated lattice $\mathcal{L}$ (see Sect. 4.2 below). Given a bounded ℓ-monoid $(L, \vee, \cdot)$, the structure $\langle L, \vee, \cdot, 0, e\rangle$ is a semiring. An *L-valued language r over Σ and* $(L, \vee, \cdot)$ is a formal power series $r \in L\langle\langle \Sigma^*\rangle\rangle$. Automata over bounded ℓ-monoids, called L-fuzzy automata, were introduced in [42]. More precisely, an *L-fuzzy automaton over Σ and* $(L, \vee, \cdot)$ is just a weighted automaton over Σ and the semiring $\langle L, \vee, \cdot, 0, e\rangle$. A Kleene theorem for L-valued recognizable languages is stated in [42] under the assumption that the ℓ-monoid $(L, \vee, \cdot)$ is countably distributive or $e = 1$. In fact, this is actually an application of the Kleene–Schützenberger theorem for recognizable series (see for instance [27, 65]). A *deterministic L-fuzzy automaton over Σ and* $(L, \vee, \cdot)$ is defined in the same way as the deterministic multi-valued automaton (see Sect. 3.1). The subsequent theorem indicates the requirements for the determinization of L-fuzzy automata over bounded l-monoids.

Theorem 4.1 ([42]). *Let $(L, \vee, \cdot)$ be a bounded ℓ-monoid. Then for every L-fuzzy automaton over $(L, \vee, \cdot)$, there exists an equivalent deterministic L-fuzzy automaton over $(L, \vee, \cdot)$ iff the semiring $\langle L, \vee, \cdot, 0, e\rangle$ is locally finite.*

For the "if" part, we refer the reader also to [7, 19] where corresponding statements for arbitrary locally finite semirings are shown. For the "only if" part, we consider an arbitrary finite subset A of L, and we construct an L-fuzzy automaton $\mathcal{A}$ having A as its set of weights and the submonoid L_A of $\langle L, \cdot, e\rangle$ generated by A as the image of its behavior (see [42]). Since every deterministic L-fuzzy automaton takes on only finitely many values and $\mathcal{A}$

can be determinized, L_A is finite. Since the monoid $\langle L, \vee, 0\rangle$ is clearly locally finite, it follows that the semiring $\langle L, \vee, \cdot, 0, e\rangle$ is locally finite.

If the semiring $\langle L, \vee, \cdot, 0, e\rangle$ is locally finite, then by [19] we get that an L-valued language $r \in L\langle\langle \Sigma^* \rangle\rangle$ is recognizable iff it is an L-valued recognizable step language (in the sense of Sect. 3 and [19]). Then following our constructions in Sect. 3.1, we get similar results as for fuzzy recognizable languages over bounded distributive lattices. We collect these results in the subsequent theorem.

Theorem 4.2. *Let Σ be an alphabet and $(L, \vee, \cdot)$ be a bounded ℓ-monoid such that the semiring $\langle L, \vee, \cdot, 0, e\rangle$ is locally finite. Then:*

- *For every L-valued recognizable language $r \in L\langle\langle \Sigma^* \rangle\rangle$ we can effectively construct a minimum trim deterministic L-fuzzy automaton with behavior r.*
- *An L-valued language $r \in L\langle\langle \Sigma^* \rangle\rangle$ is recognizable iff r has finite image and each of its cut languages is recognizable.*[1]
- *A pumping lemma holds for L-valued recognizable languages in $L\langle\langle \Sigma^* \rangle\rangle$.*
- *The equivalence problem is decidable for L-fuzzy automata over Σ and $(L, \vee, \cdot)$.*

4.2 Fuzzy Languages over Residuated Lattices

Now we turn to residuated lattices. A *residuated lattice* is an algebra $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$ where $(L, \leq)$ is a bounded lattice equipped with two operations $\otimes, \rightarrow$ such that the following conditions hold:

(i) $\langle L, \otimes, 1\rangle$ is a commutative monoid
(ii) $\otimes$ and $\rightarrow$ form an *adjoint pair*, i.e., $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$

for every $a, b, c \in L$ (see [73]).

If the lattice L is complete, then $\mathcal{L}$ is called a *complete residuated lattice*. Examples of (complete) residuated lattices are provided by the fuzzy semiring $\langle [0, 1], \mathsf{max}, \mathsf{min}, 0, 1\rangle$ equipped with operations $\otimes$ and $\rightarrow$ defined, respectively, as follows. For every $a, b \in [0, 1]$, let:

- $a \otimes b = \mathsf{max}(a + b - 1, 0)$, and $a \rightarrow b = \mathsf{min}(1 - a + b, 1)$ (the *Łukasiewicz structure*)
- $a \otimes b = a \cdot b$, and $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b/a$ (the usual quotient of real numbers) otherwise (the *product structure*)
- $a \otimes b = \mathsf{min}(a, b)$, and $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b$ otherwise (the *Gödel structure*).

Next, we claim that if $(a_i \mid i \in I) \subseteq L$ is a family of elements of L such that $\bigvee_{i \in I} a_i$ exists, then for every $a \in L$ we have

[1] This statement has been also derived in [42].

$$\Big(\bigvee_{i\in I} a_i\Big)\otimes a=\bigvee_{i\in I}(a_i\otimes a).$$

Indeed, for every $j\in I$, we have

$$a_j\le\bigvee_{i\in I} a_i\le\Big(a\to\Big(\Big(\bigvee_{i\in I} a_i\Big)\otimes a\Big)\Big),$$

so $a_j\otimes a\le(\bigvee_{i\in I} a_i)\otimes a$. Now let $c\in L$ such that $a_i\otimes a\le c$ for every $i\in I$, hence $a_i\le a\to c$. Then we get $\bigvee_{i\in I} a_i\le a\to c$, and thus $(\bigvee_{i\in I} a_i)\otimes a\le c$ proving our claim.

By (ii), we obtain also $0\otimes a=0$ for every $a\in L$.

Clearly, for every residuated lattice $\mathcal{L}=\langle L,\vee,\wedge,\otimes,\to,0,1\rangle$, the triple $(L,\vee,\otimes)$ is a bounded ℓ-monoid (with $e=1$). Moreover, $\langle L,\vee,\otimes,0,1\rangle$ is a commutative semiring which is called the *semiring reduct of* $\mathcal{L}$ and is denoted by $\mathcal{L}^*$. Obviously, the semiring reducts induced by Łukasiewicz and Gödel structures are locally finite, whereas this is not the case for the semiring reduct induced by the product structure. A *fuzzy language* r *over* Σ *and* $\mathcal{L}$ is a formal power series $r\in L\langle\langle\Sigma^*\rangle\rangle$. Then a *fuzzy automaton over* Σ *and* $\mathcal{L}$ is a weighted automaton over Σ and the semiring reduct $\mathcal{L}^*$ (see [60, 61]);[2] it is also an L-fuzzy automaton over Σ and $(L,\vee,\otimes)$. Fuzzy automata over the *product structure* occur in practical applications (see Sect. 5). As immediate consequences of Theorems 4.1 and 4.2, we obtain the following corollaries.

Corollary 4.3 ([33]). *Let $\mathcal{L}=\langle L,\vee,\wedge,\otimes,\to,0,1\rangle$ be a residuated lattice. Then for every fuzzy automaton over $\mathcal{L}$ there exists an equivalent deterministic fuzzy automaton over $\mathcal{L}$ iff the semiring reduct $\mathcal{L}^*$ is locally finite.*

Corollary 4.4. *Let Σ be an alphabet and $\mathcal{L}=\langle L,\vee,\wedge,\otimes,\to,0,1\rangle$ be a residuated lattice with locally finite semiring reduct $\mathcal{L}^*$. Then:*

- *For every fuzzy recognizable language $r\in L\langle\langle\Sigma^*\rangle\rangle$, we can effectively construct a minimum trim deterministic fuzzy automaton with behavior r.*
- *A fuzzy language $r\in L\langle\langle\Sigma^*\rangle\rangle$ is recognizable iff r has finite image and each of its cut languages is recognizable.*
- *A pumping lemma holds for fuzzy recognizable languages in $L\langle\langle\Sigma^*\rangle\rangle$.*
- *The equivalence problem is decidable for fuzzy automata over Σ and $\mathcal{L}$.*

Recently in [15], it has been proved that for every (non-deterministic) fuzzy automaton over a residuated lattice $\mathcal{L}$, a size (number of states) reduction algorithm exists, provided that $\mathcal{L}$ is a locally finite residuated lattice. The authors constructed a fuzzy automaton over the product structure (which is

[2] In [60, 61], the author considers also fuzzy automata over the semiring $\langle L,\vee,\wedge,0,1\rangle$ and for this requires $\mathcal{L}$ to be complete. For such automata, a type of pumping lemma is shown in [62]. The completeness axiom of $\mathcal{L}$ required in [15, 33, 60–62] is actually superfluous for fuzzy automata over the semiring reduct $\mathcal{L}^*$.

not locally finite) for which their reduction algorithm cannot be applied. The minimization problem for (either deterministic or non-deterministic) fuzzy automata over arbitrary residuated lattices remains open.

4.3 Fuzzy Automata with Outputs

Fuzzy automata with outputs have been mainly defined over the bounded distributive lattice $([0,1],\leq)$ (see [54, 17]). They are special cases of weighted transducers over the fuzzy semiring (for definitions on weighted transducers see [53]).

In [14, 49, 58, 69], size reduction algorithms have been developed for non-deterministic versions of fuzzy automata with outputs.

In [41], it is shown that a size reduction algorithm exists for complete L-fuzzy automata with outputs over a finite ℓ-monoid $(L,\vee,\cdot)$.

4.4 Fuzzy Abstract Families of Languages

In [2], a theory for full abstract families of fuzzy languages (full AFFLs) is presented. The underlying structure is a bounded ℓ-monoid $(L,\vee,\cdot)$ with $e=1$ and its operation $\cdot$ being commutative. Furthermore, the lattice $(L,\leq)$ is complete satisfying

$$a\wedge\left(\bigvee_{i\in I} b_i\right)=\bigvee_{i\in I}(a\wedge b_i)$$

for every $a\in L$ and every family $(b_i\mid i\in I)\subseteq L$ of elements of L.

Rational operations between fuzzy languages over $(L,\vee,\cdot)$ are defined as the rational operations of formal power series over the semiring $(L,\vee,\cdot,0,1)$ (see [27]). For every two alphabets Σ and Δ, every homomorphism $h:\Sigma^*\to\Delta^*$ induces a *fuzzy homomorphism* $\overline{h}:\Sigma^*\to L\langle\!\langle\Delta^*\rangle\!\rangle$ mapping every word $w\in\Sigma^*$ to a fuzzy language with support $\{h(w)\}$. Then a family of fuzzy languages $\mathcal{R}$ is called a *full abstract family of fuzzy languages (full AFFL)* if it is closed under the rational operations, fuzzy and inverse fuzzy homomorphisms, and infimum with fuzzy recognizable languages. It is proved that the class of fuzzy recognizable languages over the ℓ-monoid $(L,\vee,\cdot)$ is a full AFFL. Furthermore, the concept of a fuzzy substitution is introduced, and the closure property of fuzzy recognizable languages under fuzzy substitutions is investigated.

4.5 Fuzzy Tree Languages

Recently, several authors have dealt with fuzzy tree languages. These are tree series over (complete or bounded) distributive lattices (for definitions on tree series, see [29]).

In [28], the authors study fuzzy tree languages over *completely distributive lattices*, i.e., complete lattices in which arbitrary suprema distribute over arbitrary infima and vice versa. They show that fuzzy recognizable and fuzzy rational tree languages coincide, i.e., a Kleene theorem, and moreover fuzzy recognizable tree languages have an equational characterization.

In [22], a multi-valued MSO logic over infinite trees is introduced and a Rabin-type theorem is proved for infinitary fuzzy tree languages over bounded distributive lattices.

5 Applications

In this section, we present two applications of fuzzy languages, with an effect to real world problems. First, we refer to an alternative method of syntactic pattern recognition using fuzzy languages. Then we define fuzzy discrete event systems which have successfully contributed to medicine.

A popular method for pattern recognition is the *syntactic pattern recognition*, where a pattern is classified by checking its syntax. Usually, patterns are represented by finite words over a finite alphabet. The letters of the alphabet, which are called *primitives*, are aimed to describe the features of the patterns. A pattern class is a language of patterns. The method of syntactic pattern recognition is the following. First, we construct finitely many regular grammars (with their terminal alphabet to be the set of primitives) taking into account the syntactic features according to which we wish to classify any pattern. The languages generated by these grammars (pattern classes) should be pairwise disjoint. Then for every constructed grammar, we consider the corresponding equivalent finite automaton. Now given a pattern to be classified, we check from which automaton it is accepted, and we classify the pattern in the pattern class represented by this automaton.

However, in many practical applications, the structural information of the patterns is inherently vague, i.e., the patterns are distorted or imperfectly formed. For instance, consider the case of recognizing handwritten characters, or determining the type of a geometrical pattern which is not perfectly sketched. In such situations, we consider the pattern classes to be fuzzy languages. Therefore, we define the pattern classes by using *fuzzy grammars*. These are weighted right-linear grammars over the fuzzy semiring (see [37, 27]). The corresponding equivalent weighted automata are actually multi-valued automata over the fuzzy semiring. It should be noted that now the supports of the fuzzy languages (pattern classes) are not required to be pairwise disjoint. Given a pattern to be classified, we compute its membership value to every pattern class, by using the constructed multi-valued automata. Then we look for the greatest value, and we classify the pattern into the corresponding class. Fuzzy syntactic pattern recognition has been applied in the identification of the skeletal maturity of children by using X-ray images of radius [55] (see also [35]). More precisely, the shapes of the radius of children were considered as patterns. Nine pattern classes were generated featuring the maturity of the radius. Then the skeletal maturity of a child was being identified by classifying an X-ray image of its radius.

Now we turn to fuzzy discrete event systems [44]. A *fuzzy discrete event system* (*FDES* for short) is a system $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ $(n > 0)$ of weighted automata over the semiring $\langle [0, 1], \mathsf{max}, \cdot, 0, 1 \rangle$ where $\cdot$ is the multiplication of the real numbers. Such automata are usually called fuzzy automata and they are actually $[0, 1]$-fuzzy automata over the ℓ-monoid $([0, 1], \mathsf{max}, \cdot)$. They can also be considered as fuzzy automata over the product structure (see Sect. 4). The input alphabet is not required to be the same for all the automata, thus we consider every automaton $\mathcal{A}_i$ $(1 \leq i \leq n)$ over an individual alphabet Σ_i. For the purposes of FDES theory, fuzzy automata lack their final distribution, i.e., they are of the form $\mathcal{A}_i = (Q_i, \mathrm{in}_i, \mathrm{wt}_i)$ for every $1 \leq i \leq n$. The elements of $\Sigma_1 \cup \cdots \cup \Sigma_n$ are called *events*. The FDES can be considered as a composite fuzzy automaton $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt})$ over $\Sigma_1 \cup \cdots \cup \Sigma_n$ and $[0, 1]^n$, where:

- $Q = Q_1 \times \cdots \times Q_n$
- $\mathrm{in}((q_1, \ldots, q_n)) = (\mathrm{in}_1(q_1), \ldots, \mathrm{in}_n(q_n))$
- $\mathrm{wt}((q_1, \ldots, q_n), a, (q'_1, \ldots, q'_n)) = (r_1, \ldots, r_n)$ where $r_i = \mathrm{wt}_i(q_i, a, q'_i)$ for every $1 \leq i \leq n$ with $a \in \Sigma_i$, and $r_i = 1$ for every $1 \leq i \leq n$ with $a \notin \Sigma_i$ provided that $q'_i = q_i$; in any other case, we let $\mathrm{wt}((q_1, \ldots, q_n), a, (q'_1, \ldots, q'_n)) = 0$

for every $(q_1, \ldots, q_n), (q'_1, \ldots, q'_n) \in Q, a \in \Sigma$.

In the following, we describe an important application of FDES to the implementation of a self-learning system for the selection of the suitable regimen for the HIV/AIDS (see [45, 76, 77]). The HIV/AIDS is among the most complex diseases to treat. One of the reasons for this complexity is that there is no cure for it. A treatment can only suppress the HIV virus and boost the immune system. Currently, there are only four classes of available anti-retroviral drugs and a regimen consists of a combination of two or more classes. Unfortunately, the HIV virus can easily develop resistance to the drugs. Thus, a decision for the suitable drug regimen for every particular patient turns to be a difficult task and can be successfully done only by experts. A wrong decision should be devastating since the patient may run out of options on available drugs. According to the experts, the following parameters must be considered for the choice of a suitable regimen:

- *Potency of the regimen*: Unlike other diseases, it is not reasonable to use the most potent regimen in the first stage of HIV/AIDS. In fact, initiating anti-retroviral therapy when the immune system is still intact does not prolong survival. The term "intact immune system" is already vague and this makes the HIV/AIDS treatment more complex than other diseases.
- *Adherence of the patient to the regimen*: This factor is very crucial unlike other diseases. The probability that a patient will benefit from the anti-retroviral therapy reduces dramatically if the patient skips even 5%

of doses. Moreover, this increases the risk that the virus will easier develop resistance to concrete drugs or even to a whole class of drugs. Unfortunately, statistics for HIV/AIDS and other chronic diseases show that the patients take only 50–70% of the required doses of long-term medications.

- *Adverse events*: These consist of side effects which may be mild to severe, and toxicity. Side effects like abdominal discomfort, loss of appetite, etc. are common especially in the first stages of HIV/AIDS treatment. On the other hand, toxicity usually causes liver problems, pancreatitis, etc. Unfortunately, in some cases, these problems turn out to be fatal.
- *Future drug options*: The HIV frequently develops resistance to the drugs. Thus, it is critical for a doctor, before concluding to any regimen, to consider the future drugs options after a potential occurrence of the resistance.

Clearly, the HIV/AIDS disease can be treated only by expert doctors. Moreover, the number of infected people increases all over the world. Actually, the number of experts is too small, especially in poor countries. Therefore, a computer program for the HIV/AIDS treatment regimen selection is desirable. Such a program has been built by using an FDES [45, 76, 77]. Here, we will briefly describe the contribution of the FDES to the program. The FDES is composed by four fuzzy automata $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ (over the semiring $\langle [0,1], \mathsf{max}, \cdot, 0, 1\rangle$), every one corresponding to one of the four aforementioned factors, respectively. The input alphabet is the same for all the automata. Every letter corresponds to a possible regimen which is a combination of two or more classes of drugs. The fuzzy automaton $\mathcal{A}_1$ has three states "initial", "medium", and "high" simulating the three instances of potency of a regimen. The states of the fuzzy automaton $\mathcal{A}_2$ are "initial", "challenging", "moderate", and "easy" modeling the possible values of the adherence. The states of $\mathcal{A}_3$ are "initial", "medium", "low", and "very low" simulating the level of the adverse events. Finally, the fuzzy automaton $\mathcal{A}_4$ has the states "initial", "medium", and "high" modeling the several options of future regimens. The initial distribution of every automaton assigns the value 1 to the "initial" state of every automaton, and the value 0 to every other state. The values of the weight assigning mappings for all the automata are determined by the expert doctors according to statistics and clinical experiments. Assume now that we have a particular patient and we ask the program to choose the optimal regimen. Initially, the system using a set of generic algorithms determines four vectors w_1, w_2, w_3, and w_4 with dimensions 3, 4, 4, and 3, respectively (these are the numbers of states of the automata $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$, respectively). Then every one of the automata $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ takes as input a letter σ (i.e., a possible regimen) and produces a vector assigning a value to every one of its states. Let us denote these vectors by $q_{1\sigma}$, $q_{2\sigma}$, $q_{3\sigma}$, and $q_{4\sigma}$, respectively. Then the system computes the *performance index*

$$J(\sigma) = w_1^{\mathsf{T}} q_{1\sigma} + w_2^{\mathsf{T}} q_{2\sigma} + w_3^{\mathsf{T}} q_{3\sigma} + w_4^{\mathsf{T}} q_{4\sigma}$$

where $w_i^{\intercal}$ denotes the transpose of the vector w_i $(1 \leq i \leq 4)$. The optimal treatment corresponds to the maximum $J(\sigma)$ for all regimens σ. For a second round treatment, the procedure is repeated. The vector states of the automata are now $q_{1\sigma}$, $q_{2\sigma}$, $q_{3\sigma}$, and $q_{4\sigma}$, but the system will compute new vectors w taking into account the current situation of the patient's health. In clinical experiments, this system matches the experts selection of regimen for 80% of the patients (see [45]).

Acknowledgments. I am deeply grateful to the editors for inviting me to write this chapter, and especially to Manfred Droste and Heiko Vogler for fruitful discussions and valuable comments on several versions of the chapter. I should also like to thank Symeon Bozapalidis for fruitful discussions. Peter Asveld, Andreas Maletti, Eleni Mandrali, Witold Pedrycz, and Arto Salomaa read previous versions of the chapter and made valuable remarks and suggestions. Andreas Maletti suggested the treatment of fuzzy recognizable languages with the "recognizable step" characterization. I am deeply grateful to all of them. I thank Arto Salomaa for providing me with reference [66] and indicating the seminal papers [48, 59]. I also thank Miroslav Ćirić for providing me with references [15, 33], Daowen Qiu for references [60, 61], and Feng Lin for clarifications on the papers [76, 77].

References

1. P.R.J. Asveld. A bibliography on fuzzy automata, grammars and languages. *Bulletin of the European Association for Theoretical Computer Science*, 58:187–196, 1996.
2. P.R.J. Asveld. Algebraic aspects of families of fuzzy languages. *Theoretical Computer Science*, 293:417–445, 2003.
3. R. Balbes and P. Dwinger. *Distributive Lattices*. University of Missouri Press, Columbia, 1974.
4. N.C. Basak and A. Gupta. On quotient machines of a fuzzy automaton and the minimal machine. *Fuzzy Sets and Systems*, 125:223–229, 2002.
5. R. Bělohlávek. Determinism and fuzzy automata. *Information Sciences*, 143:205–209, 2002.
6. G. Birkhoff. *Lattice Theory*, revised edition, volume XXV of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, 1961.
7. B. Borchardt. A pumping lemma and decidability problems for recognizable tree series. *Acta Cybernetica*, 16:509–544, 2004.
8. S. Bozapalidis and O. Louscou-Bozapalidou. On the recognizability of fuzzy languages I. *Fuzzy Sets and Systems*, 157:2394–2402, 2006.
9. S. Bozapalidis and O. Louscou-Bozapalidou. On the recognizability of fuzzy languages II. *Fuzzy Sets and Systems*, 159:107–113, 2008.
10. G. Bruns and P. Godefroid. Model checking with multi-valued logics. In *Proceedings of ICALP 2004*, volume 3142 of *Lecture Notes in Computer Science*, pages 281–293. Springer, Berlin, 2004.

11. J.R. Büchi. On a decision method in restricted second order arithmetic. In *Proceedings of International Congress for Logic, Methodology and Philosophy of Science*, pages 1–11, 1962
12. A.L. Buchsbaum, R. Giancarlo, and J.R. Westbrook. On the determinization of weighted finite automata. *SIAM Journal on Computing*, 5:1502–1531, 2002.
13. M. Chechik, B. Devereux, and A. Gurfinkel. Model-checking infinite state-space systems with fine-grained abstractions using SPIN. In *Proceedings of SPIN 2001*, volume 2057 of *Lecture Notes in Computer Science*, pages 16–36. Springer, Berlin, 2001.
14. W. Cheng and Z. Mo. Minimization algorithm of fuzzy finite automata. *Fuzzy Sets and Systems*, 141:439–448, 2004.
15. M. Ćirić, A. Stamenković, J. Ignjatović, and T. Petković. Factorization of fuzzy automata. In *Proceedings of FCT 2007*, volume 4639 of *Lecture Notes in Computer Science*, pages 213–225. Springer, Berlin, 2007.
16. A.B. Davey and H.A. Priestley. *Introduction to Lattices and Order*, 2nd edition. Cambridge University Press, Cambridge, 2002.
17. M. Doostfatemeh and S.C. Kremer. New directions in fuzzy automata. *International Journal of Approximate Reasoning*, 38:175–214, 2005.
18. M. Droste and P. Gastin. Weighted automata and weighted logics. *Theoretical Computer Science*, 380:69–86, 2007. Extended abstract in: *Proceedings of ICALP 2005*, volume 3580 of *Lecture Notes in Computer Science*, pages 513–525. Springer, Berlin, 2005.
19. M. Droste and P. Gastin. On aperiodic and star-free formal power series in partially commuting variables. *Theory of Computing Systems*, 42:608–631, 2008. Extended abstract in: D. Krob, A.A. Milchalev, and A.V. Michalev, editors, *Proceedings of the 12th International Conference on Formal Power Series and Algebraic Combinatorics, Moscow*, pages 158–169. Springer, Berlin, 2000.
20. M. Droste and P. Gastin. Weighted automata and weighted logics. In this *Handbook*. Chapter 5. Springer, Berlin, 2009.
21. M. Droste and W. Kuich. Semirings and formal power series. In this *Handbook*. Chapter 1. Springer, Berlin, 2009.
22. M. Droste, W. Kuich, and G. Rahonis. Multi-valued MSO logics over words and trees. *Fundamenta Informaticae*, 84:305–327, 2008.
23. M. Droste and G. Rahonis. Weighted automata and weighted logics on infinite words. *Russian Mathematics (Iz. VUZ)*, 2009, in press. Extended abstract in: *Proceedings of DLT'06*, volume 4036 of *Lecture Notes in Computer Science*, pages 49–58. Springer, Berlin, 2006.
24. S. Eilenberg. *Automata, Languages and Machines, volume A*. Academic Press, San Diego, 1974.
25. Z. Ésik and W. Kuich. A semiring-semimodule generalization of ω-regular languages I. *Journal of Automata, Languages and Combinatorics*, 10:203–242, 2005.

26. Z. Ésik and W. Kuich. A semiring-semimodule generalization of ω-regular languages II. *Journal of Automata, Languages and Combinatorics*, 10:243–264, 2005.
27. Z. Ésik and W. Kuich. Finite automata. In this *Handbook*. Chapter 3. Springer, Berlin, 2009.
28. Z. Ésik and G. Liu. Fuzzy tree automata. *Fuzzy Sets and Systems*, 158:1450–1460, 2007.
29. Z. Fülöp and H. Vogler. Weighted tree automata and tree transducers. In this *Handbook*. Chapter 9. Springer, Berlin, 2009.
30. S. Golan. *Semirings and Their Applications*. Kluwer Academic, Dordrecht, 1999.
31. A. Gurfinkel and M. Chechik. Multi-valued model checking via classical model checking. In *Proceedings of CONCUR 2003*, volume 2761 of *Lecture Notes in Computer Science*, pages 266–280. Springer, Berlin, 2003.
32. P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic, Dordrecht, 1998.
33. J. Ignjatović, M. Ćirić, and S. Bogdanović. Determinization of fuzzy automata with membership values in complete residuated lattices. *Information Sciences*, 175:164–180, 2008.
34. D. Kirsten and I. Mäurer. On the determinization of weighted automata. *Journal of Automata, Languages and Combinatorics*, 10:287–312, 2005.
35. G. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic*. Prentice Hall, Englewood Cliffs, 1995.
36. S. Konstandinidis, N. Sântean, and S. Yu. Fuzzification of rational and recognizable sets. *Fundamenta Informaticae*, 76:413–447, 2007.
37. W. Kuich and G. Rahonis. Fuzzy regular languages over finite and infinite words. *Fuzzy Sets and Systems*, 157:1532–1549, 2006.
38. W. Kuich and A. Salomaa. *Semirings, Automata, Languages*, volume 5 of *Monographs in Theoretical Computer Science. An EATCS Series*. Springer, Berlin, 1986.
39. O. Kupferman and Y. Lustig. Lattice automata. In *Proceedings of VMCAI 2007*, volume 4349 of *Lecture Notes in Computer Science*, pages 199–213. Springer, Berlin, 2007.
40. L. Lee and L.A. Zadeh. Note on fuzzy languages. *Information Sciences*, 1:421–434, 1969.
41. H. Lei and Y. Li. Minimization of states in automata theory based on finite lattice-ordered monoids. *Information Sciences*, 177:1413–1421, 2007.
42. Y. Li and W. Pedrycz. Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids. *Fuzzy Sets and Systems*, 156:68–92, 2005.
43. Y. Li and W. Pedrycz. Minimization of lattice finite automata and its application to the decomposition of lattice languages. *Fuzzy Sets and Systems*, 158:1423–1436, 2007.

44. F. Lin and H. Ying. Modeling and control of fuzzy discrete event systems. *IEEE Transactions on Systems, Man and Cybernetics—Part B: Cybernetics*, 32:408–415, 2002.
45. F. Lin, H. Ying, R.D. MacArthur, J.A. Cohn, D.C. Barth-Jones, H. Ye, and L.R. Crane. Decision making in fuzzy discrete event systems. *Information Sciences*, 177:3749–3763, 2007.
46. O. Louskou-Bozapalidou. Fuzzy congruences on fuzzy algebras. *Applied Mathematical Sciences (Hikari)*, 1:815–819, 2007.
47. O. Louskou-Bozapalidou. Non-deterministic recognizability of fuzzy languages. *Applied Mathematical Sciences (Hikari)*, 1:821–826, 2007.
48. J. Łukasiewicz. O logice trojwartosciowej. *Ruch Filosoficzny*, 5:169–171, 1920.
49. D.S. Malik, J.N. Mordeson, and M.K. Sen. Minimization of fuzzy finite automata. *Information Sciences*, 113:323–330, 1999.
50. G. Malinowski. *Many-Valued Logics*. Oxford Science Publications, Oxford, 1993.
51. A. Mateescu, A. Salomaa, K. Salomaa, and S. Yu. Lexical analysis with a finite-fuzzy-automaton model. *Journal of Universal Computer Science*, 1:292–311, 1995.
52. M. Mohri. Finite-state transducers in language and speech processing. *Computational Linguistics*, 23:269–311, 1997.
53. M. Mohri. Weighted automata algorithms. In this *Handbook*. Chapter 6. Springer, Berlin, 2009.
54. J. Mordeson and D. Malik. *Fuzzy Automata and Languages: Theory and Applications*. Chapman & Hall/CRC, London, 2002.
55. A. Pathak and S.K. Pal. Fuzzy grammars in syntactic recognition of skeletal maturity from X-rays. *IEEE Transactions on Systems, Man and Cybernetics*, 16:657–667, 1986.
56. D. Perrin and E.J. Pin. *Infinite Words*. Elsevier, Amsterdam, 2004.
57. T. Petković. Varietes of fuzzy languages. In *Proceedings of the 1st International Conference on Algebraic Informatics*, pages 197–205. Aristotle University of Thessaloniki, Thessaloniki, 2005.
58. T. Petković. Congruences and homomorphisms of fuzzy automata. *Fuzzy Sets and Systems*, 157:444–458, 2006.
59. E.L. Post. Introduction to a general theory of elementary propositions. *American Journal of Mathematics*, 43:163–185, 1921.
60. D. Qiu. Automata theory based on complete residuated lattice-valued logic. *Science in China. Series F*, 44:419–429, 2001.
61. D. Qiu. Automata theory based on complete residuated lattice-valued logic (II). *Science in China. Series F*, 45:442–452, 2002.
62. D. Qiu. Pumping lemma in automata theory based on complete residuated lattice-valued logic: A note. *Fuzzy Sets and Systems*, 157:2128–2138, 2006.
63. G. Rahonis. Infinite fuzzy computations. *Fuzzy Sets and Systems*, 153:275–288, 2005.
64. J.B. Rosser and A.R. Turquette. *Many Valued Logics*. North-Holland, Amsterdam, 1952.

65. J. Sakarovitch. Rational and recognisable power series. In this *Handbook*. Chapter 4. Springer, Berlin, 2009.
66. A. Salomaa. Moisil and many-valued logic. In A. Iorgulescu, S. Marcus, S. Rudeanu, and D. Vaida, editors, *Grigore C. Moisil and his Followers in Theoretical Computer Science*, pages 29–39. Editura Academiei Romane, Bucharest, 2007.
67. A. Salomaa and M. Soittola. *Automata-Theoretic Aspects of Formal Power Series. Texts and Monographs in Computer Science*. Springer, Berlin, 1978.
68. E. Santos. Maximin automata. *Information and Control*, 13:363–377, 1968.
69. E. Santos. On reductions of maximin machines. *Journal of Mathematical Analysis and Applications*, 40:60–78, 1972.
70. B. Šešelja. Homomorphisms of poset-valued algebras. *Fuzzy Sets and Systems*, 121:333–340, 2001.
71. B. Šešelja and A. Tepavčević. On a generalization of fuzzy algebras and congruences. *Fuzzy Sets and Systems*, 65:85–94, 1994.
72. A. Tepavčević and G. Trajkovski. L-fuzzy lattices: An introduction. *Fuzzy Sets and Systems*, 223:209–216, 2001.
73. M. Ward and R.P. Dilworth. Residuated lattices. *Transactions of the American Mathematical Society*, 45:335–354, 1939.
74. W. Wechler. *The Concept of Fuzziness in Automata and Language Theory*. Akademie Verlag, Berlin, 1978.
75. W.G. Wee. On generalization of adaptive algorithm and applications of the fuzzy sets concept to pattern classification. PhD dissertation, Purdue University, Lafayette, 1967.
76. H. Ying, F. Lin, R.D. MacArthur, J.A. Cohn, D.C. Barth-Jones, H. Ye, and L.R. Crane. A fuzzy discrete event system approach determining optimal HIV/AIDS treatment regimens. *IEEE Transactions on Information Technology in Biomedicine*, 10:663–676, 2006.
77. H. Ying, F. Lin, R.D. MacArthur, J.A. Cohn, D.C. Barth-Jones, H. Ye, and L.R. Crane. A self-learning fuzzy discrete event system approach determining optimal HIV/AIDS treatment regimen selection. *IEEE Transactions on Systems, Man and Cybernetics—Part B: Cybernetics*, 37:966–979, 2007.
78. L.A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.