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Multi-Valued MSO Logics Over Words and Trees

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Abstract. We introduce multi-valued Büchi and Muller automata over distributive lattices and a multi-valued MSO logic for infinite words. For this logic, we prove the expressive equivalence of ω-recognizable and MSO-definable infinitary formal power series over distributive lattices with negation function. Then we consider multi-valued Muller tree automata and a multi-valued MSO logic for trees over distributive lattices. For this logic, we establish a version of Rabin's theorem for infinitary tree series.

Keywords: Distributive lattices, multi-valued automata over infinite words and trees, infinitary formal power series, multi-valued MSO logic.

1. Introduction

In 1962, Büchi [12], motivated by decision problems in logic, showed that the languages of infinite words accepted by finite automata coincide with the languages definable in monadic second order (MSO)

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logic. Büchi's fundamental theorem led to important practical applications for model checking of non-terminating processes (cf. [1, 38, 41, 45, 53, 54]). Recently, the concept of multi-valued logics over De Morgan or quasi-Boolean algebras has played a central role in the development of new tools for model checking techniques (cf. [10, 13, 32, 37]). In this approach, the values of the atomic formulas are given as elements of an underlying De Morgan algebra, thereby modelling uncertainty or partial information which often occurs when analyzing or specifying properties of systems. This has led to multi-valued practical tools [13, 32, 37]; for a survey, see [29].

It is the goal of this paper to establish Büchi's theorem for a multi-valued MSO logic. We will assume that the values of our logic are taken in an arbitrary bounded distributive lattice. The class of these lattices is much larger than the class of De Morgan algebras. We will also introduce multi-valued Büchi and Muller automata acting on infinite words, and we show their mutual equivalence in expressive power with our multi-valued logic.

Next we describe our approach in some more detail. The syntax of our multi-valued MSO logic is enriched with $\wedge$, $\forall x$, $\forall X$. In order to cope with negation of formulas, we assume that the underlying bounded lattice has an arbitrary negation function which is just supposed to interchange the largest and smallest elements of the lattice, respectively. Since any bounded distributive lattice carries such a function, this is no essential restriction of our class of lattices considered. In a first result, we show that the behaviors of multi-valued Büchi or Muller automata over bounded distributive lattices and infinite words admit a simple characterization. In our main result, we use this description to show the expressive equivalence of our logic with the automata models. Since all our proofs are constructive, we obtain decidability procedures for the equality and (multi-valued) implication problems for sentences of our logic.

We also indicate that our methods can be extended to cover multi-valued logics and automata on infinite trees. Here we provide an example using a lattice with negation function which is *not* a De Morgan algebra. Finally, we phrase our results for a logic based on De Morgan algebras, and we give a characterization of De Morgan algebras in terms of semirings with complement functions.

For the closely related strand of weighted automata see [2, 17, 23, 28, 33, 36, 49] for monographs and surveys. Recently, several authors have been interested in MSO logic equipped with weights from semirings. More precisely, Droste and Gastin [16] considered a weighted MSO logic and proved its expressive equivalence with weighted automata on finite words, thereby generalizing Büchi's and Elgot's theorem [11, 24]. Droste and Rahonis [19] considered the same logics and established the aforementioned fundamental result of Büchi for infinitary series. It is clear that in this case the underlying semirings have to satisfy special completeness properties permitting infinite sums and countably infinite products. More recently, in [20] the authors considered weighted automata and a weighted logics with discounting eliminating the completeness axioms of the underlying semirings. For further work on weighted logics and automata for trees, pictures, traces, texts, and distributive systems we refer the reader to [5, 21, 22, 30, 39, 40, 42, 48]. In our paper, we develop our theory for arbitrary bounded distributive lattices with negation function, without any further requirement of completeness axioms. Also, here negation can be applied to all formulas, whereas in [16, 19, 20] negation is restricted to atomic formulas.

2. Preliminaries

Let $(L, \leq)$ be a partially ordered set and $a, b \in L$ (resp. $S \subseteq L$). We denote *the least upper bound* (or *supremum*) of a and b (resp. of S), if it exists in L, by $a \vee b$ (resp. $\vee S$) and the *greatest lower bound* (or *infimum*), if it exists in L, by $a \wedge b$ (resp. $\wedge S$). If $S = (a_i \mid i \in I)$ then we also use the notations $\bigvee_{i \in I} a_i$ and $\bigwedge_{i \in I} a_i$, respectively. If the order relation $\leq$ is understood, then we simply denote a partially ordered set by L.

A partially ordered set L is called a *lattice*, if for all $a, b \in L$, $a \vee b$ and $a \wedge b$ exist. A lattice L is called *distributive* if it satisfies, for all $a, b, c \in L$, the equations

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and } (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c).$$

It is well-known that if L is any distributive lattice and $F \subseteq L$ a finite subset, then the sublattice L' of L generated by F is finite. In fact, if $F' = \{\wedge I \mid I \subseteq F\}$, due to the distributivity laws we have $L' = \{\vee J \mid J \subseteq F'\}$.

A partially ordered set $(L, \leq)$ is *bounded* if it contains two distinguished elements $0, 1 \in L$ such that $0 \leq a \leq 1$, for all $a \in L$. An element $a \neq 0$ of a lattice L is called *join-irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$ for all $b, c \in L$. We denote by $J(L)$ the set of all join-irreducible elements of L. If the lattice L is finite, then

$$a = \vee\{b \in J(L) \mid b \leq a\}$$

for any $a \in L$. Moreover, if L is distributive, any join-irreducible element $a \in L$ is *prime*, i.e. whenever $a \leq b \vee c$ with $b, c \in L$, then $a \leq b$ or $a \leq c$ (cf. [4, 14]).

Let $(L, \leq)$ be a bounded distributive lattice and $^- : L \to L$ be any function with $\overline{0} = 1$ and $\overline{1} = 0$. Then we call $^-$ a (*general*) *negation function* and $(L, \leq, ^-)$ a *bounded distributive lattice with negation function*. Note that any bounded distributive lattice L can be equipped with a negation function $^-$ by letting, e.g., $\overline{0} = 1$ and $\overline{x} = 0$ for each $x \in L \setminus \{0\}$. The more particular class of De Morgan algebras will be considered in Section 6 (see there for a more detailed discussion and comparison). Other well-investigated classes of distributive lattices with negation function include Heyting-algebras and variants of pseudocomplemented lattices.

Given two bounded lattices $(L, \leq)$, $(L', \leq)$, a mapping $f : L \to L'$ is called a *lattice morphism* if f preserves the greatest and smallest elements, respectively, and for any $a, b \in L$ we have

$$f(a \vee b) = f(a) \vee f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \wedge f(b).$$

Then $a \leq b$ implies $f(a) \leq f(b)$ for all $a, b \in L$.

Now we turn to semirings. By a *semiring* we mean a set A together with two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that

(i) $(A, +, 0)$ is a commutative monoid,

(ii) $(A, \cdot, 1)$ is a monoid,

(iii) the distributivity laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for all $a, b, c \in A$,

(iv) $0 \cdot a = a \cdot 0 = 0$ for all $a \in A$.

If the operations and the constant elements of A are understood, then we denote the semiring simply by A. Otherwise, we use the notation $(A, +, \cdot, 0, 1)$.

A semiring A is called *commutative* if $a \cdot b = b \cdot a$ for every $a, b \in A$. Clearly, any bounded distributive lattice with operations supremum as addition and infimum as multiplication constitutes an idempotent, commutative semiring. Particular examples are provided by the *fuzzy semiring* $([0,1], \vee, \wedge, 0, 1)$ and the *Boolean semiring* $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$.

In the following, we introduce infinitary formal power series over distributive lattices.

Let Σ be an alphabet. We denote the set of infinite words over Σ by Σ^ω. Let $w = x_0 x_1 \ldots \in \Sigma^\omega$, with $x_0, x_1, \ldots \in \Sigma$. We shall use the notation $w = w(0)w(1)\ldots$, with $w(i) = x_i$, for $i = 0, 1, \ldots$.

Furthermore, let $(L, \leq)$ be a bounded distributive lattice. A mapping $S : \Sigma^\omega \to L$ is called an *infinitary formal power series* (or *series* for short) *over* Σ *and* L. The values of S are denoted by (S, w), where $w \in \Sigma^\omega$, and are also referred to as the *coefficients* of the series. The series S can be written as a formal supremum

$$S = \bigvee_{w \in \Sigma^\omega} (S, w) w.$$

The *support* of a series $S : \Sigma^\omega \to L$ is the set $supp(S) = \{w \in \Sigma^\omega \mid (S, w) \neq 0\}$. The collection of all infinitary formal power series over Σ and L is denoted by $L\langle\langle \Sigma^\omega \rangle\rangle$. Then $(L\langle\langle \Sigma^\omega \rangle\rangle, \leq)$ is a bounded distributive lattice where for $S, T \in L\langle\langle \Sigma^\omega \rangle\rangle$ the partial order $S \leq T$ is defined by $S \leq T$ iff $(S, w) \leq (T, w)$ for all $w \in \Sigma^\omega$. Then, the supremum $S \vee T$, the infimum $S \wedge T$ and the scalar infimum $k \wedge S$ ($k \in L$) are defined elementwise

$$\begin{aligned}
(S \vee T, w) &= (S, w) \vee (T, w) \\
(S \wedge T, w) &= (S, w) \wedge (T, w) \\
(k \wedge S, w) &= k \wedge (S, w)
\end{aligned}$$

for any $w \in \Sigma^\omega$.

If $(L, \leq, ^-)$ is a bounded distributive lattice with negation function, then $(L\langle\langle \Sigma^\omega \rangle\rangle, \leq, ^-)$ constitutes also a bounded distributive lattice with negation function; for any series $S \in L\langle\langle \Sigma^\omega \rangle\rangle$ its negation $\overline{S} \in L\langle\langle \Sigma^\omega \rangle\rangle$ is defined by

$$(\overline{S}, w) = \overline{(S, w)}$$

for any $w \in \Sigma^\omega$.

Let $(L, \leq)$, $(L', \leq)$ be two bounded distributive lattices and $f : L \to L'$ be a mapping. Then f is extended to a mapping $f : L\langle\langle \Sigma^\omega \rangle\rangle \to L'\langle\langle \Sigma^\omega \rangle\rangle$ in the following way. For any series $S \in L\langle\langle \Sigma^\omega \rangle\rangle$ the series $f(S) \in L'\langle\langle \Sigma^\omega \rangle\rangle$ is determined by

$$(f(S), w)) = f((S, w))$$

for all $w \in \Sigma^\omega$.

In the sequel, we recall notions and results from classical monadic second order logic (MSO logic for short) over infinite words. Let Σ be an alphabet. The syntax of formulas of the MSO logic over Σ is given by:

$$\varphi := P_a(x) \mid x \leq y \mid x \in X \mid \neg\varphi \mid \varphi \vee \psi \mid \exists x \,.\, \varphi \mid \exists X \,.\, \varphi$$

where $a \in \Sigma$, x, y are first order variables and X is a second order (set) variable.

We shall denote the set of natural numbers $\mathbb{N}$ also by ω. An infinite word $w = w(0)w(1)\ldots \in \Sigma^\omega$ is represented by the structure $(\omega, \leq, (R_a)_{a \in \Sigma})$ where $R_a = \{i \mid w(i) = a\}$ for $a \in \Sigma$. Let $\mathcal{V}$ be a finite set of first and second order variables. A $(w, \mathcal{V})$-*assignment* σ is a mapping assigning elements of ω to first order variables from $\mathcal{V}$, and subsets of ω to second order variables from $\mathcal{V}$. If x is a first order variable and $i \in \omega$, then $\sigma[x \to i]$ denotes the $(w, \mathcal{V} \cup \{x\})$-assignment which assigns i to x and acts as σ on $\mathcal{V} \setminus \{x\}$. For a second order variable X and $I \subseteq \omega$, the notation $\sigma[X \to I]$ has a similar meaning.

In order to encode pairs (w, σ) for all $w \in \Sigma^\omega$ and any $(w, \mathcal{V})$-assignment σ, we use the extended alphabet $\Sigma_{\mathcal{V}} = \Sigma \times \{0,1\}^{\mathcal{V}}$. Each word in $\Sigma_{\mathcal{V}}^\omega$ can be considered as a pair (w, σ) where w is the projection over Σ and σ is the projection over $\{0,1\}^{\mathcal{V}}$. Then σ is a *valid* $(w, \mathcal{V})$-assignment if for each first order variable $x \in \mathcal{V}$ the x-row contains exactly one 1. In this case, we identify σ with the $(w, \mathcal{V})$-assignment so that for each first order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 on the x-row, and for each second order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions labelled with 1 along the X-row.

It is well-known that the set

$$N_{\mathcal{V}} = \{(w, \sigma) \in \Sigma_{\mathcal{V}}^\omega \mid \sigma \text{ is a valid } (w, \mathcal{V})\text{-assignment}\}$$

is ω-recognizable.

Let φ be an MSO-formula. We shall write Σ_φ for $\Sigma_{Free(\varphi)}$ and $N_\varphi = N_{Free(\varphi)}$. Furthermore, for $Free(\varphi) \subseteq \mathcal{V}$ we set

$$\mathcal{L}_{\mathcal{V}}(\varphi) = \{(w, \sigma) \in N_{\mathcal{V}} \mid (w, \sigma) \models \varphi\}$$

for the language defined by φ over $\Sigma_{\mathcal{V}}$. We simply write $\mathcal{L}(\varphi) = \mathcal{L}_{Free(\varphi)}(\varphi)$. Then, the fundamental theorem of Büchi [12] states that for each MSO-formula φ the language $\mathcal{L}_{\mathcal{V}}(\varphi)$ is ω-recognizable; conversely, each ω-recognizable language $R \subseteq \Sigma^\omega$ is definable by an MSO-sentence φ, i.e. $R = \mathcal{L}(\varphi)$.

3. Multi-valued automata

In this section, we introduce multi-valued Büchi and Muller automata over distributive lattices. Let Σ be an alphabet and L be a bounded distributive lattice.

Definition 3.1. (a) A *multi-valued Muller automaton* (MVMA for short) *over* Σ *and* L is a quadruple $\mathcal{A} = (Q, in, wt, \mathcal{F})$, where Q is the *finite state set*, $in : Q \to L$ is the *initial distribution*, $wt : Q \times \Sigma \times Q \to L$ is a mapping assigning *weights* to the transitions of the automaton, and $\mathcal{F} \subseteq \mathcal{P}(Q)$ is the *family of final state sets*.

(b) An MVMA $\mathcal{A}$ is a *multi-valued Büchi automaton* (MVBA for short) if there is a set $F \subseteq Q$ such that $\mathcal{F} = \{S \subseteq Q \mid S \cap F \neq \emptyset\}$.

Let $w = a_0 a_1 \ldots \in \Sigma^\omega$. A *path of* $\mathcal{A}$ *over* w is an infinite sequence of transitions $P_w := (t_i)_{i \geq 0}$, so that $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$. The *weight of* P_w is defined by

$$weight(P_w) := in(q_0) \wedge \bigwedge_{i \geq 0} wt(t_i).$$

Observe that the set $\{wt(t) \mid t \in Q \times \Sigma \times Q\}$ is finite and thus $weight(P_w)$ is well-defined. Furthermore, $weight(P_w) \in L'$ where L' is the (finite) sublattice of L generated by $\{0, 1, in(q), wt(t) \mid q \in Q, t \in Q \times \Sigma \times Q\}$. We denote by $In^Q(P_w)$ the set of states which appear infinitely many times in P_w, i.e.,

$$In^Q(P_w) = \{q \in Q \mid \exists^\omega i : t_i = (q, a_i, q_{i+1})\}.$$

The path P_w is called *successful* if the set of states that appear infinitely often along P_w constitute a final state set, i.e., $In^Q(P_w) \in \mathcal{F}$. The *behavior* of $\mathcal{A}$ is the infinitary formal power series

$$\|\mathcal{A}\| : \Sigma^\omega \to L$$

which is defined by

$$(\|\mathcal{A}\|, w) = \bigvee_{P_w} weight(P_w)$$

for $w \in \Sigma^\omega$, where the supremum is taken over all successful paths P_w of $\mathcal{A}$ over w. Since L' is finite, $(\|\mathcal{A}\|, w)$ exists and $(\|\mathcal{A}\|, w) \in L'$ for each $w \in \Sigma^\omega$.

An infinitary series $S : \Sigma^\omega \to L$ is said to be *Muller recognizable* (resp. *Büchi recognizable* or *ω-recognizable)* if there is an MVMA (resp. an MVBA) $\mathcal{A}$ so that $S = \|\mathcal{A}\|$. We shall denote the family of all Muller recognizable (resp. ω-recognizable) series over Σ by $L^{M-rec}\langle\langle\Sigma^\omega\rangle\rangle$ (resp. $L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle$). So, trivially we have $L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle \subseteq L^{M-rec}\langle\langle\Sigma^\omega\rangle\rangle$. In fact:

Theorem 3.1. [19] Let Σ be an alphabet and L be a bounded distributive lattice. Then

$$L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle = L^{M-rec}\langle\langle\Sigma^\omega\rangle\rangle .$$

Proof:
Choose any MVMA $\mathcal{A}$. Let L' be the sublattice of L generated by $0, 1$ and the weights of $\mathcal{A}$. Then L' is a finite bounded distributive lattice which in turn means that the semiring $(L', \vee, \wedge, 0, 1)$ satisfies all the technical assumptions of [19] (cf. [19], Example 1), i.e. it is totally commutative complete. By Theorem 25 in [19], we obtain $\|\mathcal{A}\| \in L'^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle \subseteq L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle$. □

Proposition 3.1. [19, 26, 27] The class of ω-recognizable power series $L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle$ is closed under supremum, infimum and scalar infimum.

Next we obtain the following Proposition by a standard automaton construction. Using this result, we will derive a generalization of it below in Proposition 3.4.

Proposition 3.2. Let $(L, \leq)$, $(L', \leq)$ be two bounded distributive lattices and $f : L \to L'$ be a lattice morphism. Then for any ω-recognizable series S in $L\langle\langle\Sigma^\omega\rangle\rangle$ the series $f(S) \in L'\langle\langle\Sigma^\omega\rangle\rangle$ is again ω-recognizable.

Proof:
Let $\mathcal{A}$ be an MVBA accepting S. Define the MVBA $\mathcal{A}'$ over L' by replacing in $\mathcal{A}$ all weights k by $f(k)$. Since f is a lattice morphism, it is easy to check that then $\|\mathcal{A}'\| = f(S)$. □

For any language $R \subseteq \Sigma^\omega$, the *characteristic series* $1_R \in L\langle\langle\Sigma^\omega\rangle\rangle$ *of* R, is defined by

$$(1_R, w) = \begin{cases} 1 & \text{if } w \in R \\ 0 & \text{otherwise} \end{cases}$$

for all $w \in \Sigma^\omega$.

Proposition 3.3. [19] Let $R \subseteq \Sigma^\omega$ be an ω-recognizable language. Then the characteristic series 1_R is ω-recognizable.

Assume now that $R_1, \ldots, R_n \subseteq \Sigma^\omega$ are ω-recognizable languages and $k_1, \ldots, k_n \in L$. Then by Propositions 3.1 and 3.3 the series

$$S = \bigvee_{1 \le i \le n} k_i \wedge 1_{R_i}$$

is ω-recognizable. Such a series S is called *ω-recognizable step function* [19]. Since the class of ω-recognizable languages over Σ^ω is closed under the Boolean operations, here we may assume the sets $(R_i)_{1 \le i \le n}$ to form a partition of Σ^ω. Hence S is an ω-recognizable step function iff the image of S is finite and for each $k \in L$, the language $S^{-1}(k) = \{w \in \Sigma^\omega \mid (S, w) = k\}$ is ω-recognizable.

Next, in our first main result, we show that ω-recognizable series are the same as ω-recognizable step functions in the context of distributive lattices.

Theorem 3.2. Let $(L, \le)$ be any bounded distributive lattice. Then, an infinitary power series $S \in L\langle\langle \Sigma^\omega \rangle\rangle$ is ω-recognizable iff it is an ω-recognizable step function.

Proof:
First, assume that S is ω-recognizable and let $\mathcal{A} = (Q, in, wt, \mathcal{F})$ be an MVMA over Σ such that $S = \|\mathcal{A}\|$. Let $L' = \{k_1, \ldots, k_n\}$ be the sublattice of L generated by 0, 1 and the weights of $\mathcal{A}$. For any $1 \le i \le n$ we set

$$R_i = \{w \in \Sigma^\omega \mid (S, w) = k_i\}.$$

Then

$$S = \bigvee_{1 \le i \le n} k_i \wedge 1_{R_i}.$$

We show that the languages R_i ($1 \le i \le n$) are ω-recognizable. Let $\mathbb{B} = (\{0, 1\}, \le)$ be the two-valued Boolean algebra. For each join-irreducible element p of L', we define a mapping $f_p : L' \to \{0, 1\}$ by putting

$$f_p(a) = \begin{cases} 1 & \text{if } p \le a \\ 0 & \text{otherwise} \end{cases}$$

for any $a \in L$.

We claim that f_p is a lattice morphism. Clearly, $f_p(0) = 0$ as $p \ne 0$, and $f_p(1) = 1$. Next, note that if $a, a' \in L'$ and $f_p(a \vee a') = 1$, then $p \le a \vee a'$, hence $p \le a$ or $p \le a'$ since p is prime, proving $f_p(a \vee a') = f_p(a) \vee f_p(a')$. Clearly, $f_p(a \wedge a') = f_p(a) \wedge f_p(a')$. By Proposition 3.2, the series $f_p(S)$ of $\mathbb{B}\langle\langle \Sigma^\omega \rangle\rangle$ is ω-recognizable and therefore the language $supp(f_p(S)) = \{w \in \Sigma^\omega \mid p \le (S, w)\}$ is ω-recognizable. Now let $1 \le i \le n$. Since the element k_i of L' is the supremum of the join-irreducible elements of L' below k_i, the infinitary language R_i is obtained as the intersection of the languages $supp(f_p(S))$ ($p \le k_i$ and join-irreducible) and of the complements of the languages $supp(f_p(S))$ ($p \not\le k_i$ and join-irreducible). Since the class of ω-recognizable languages is closed under Boolean operations, we conclude that R_i is an ω-recognizable language, as required.

The converse is immediate as noted before, by Propositions 3.1 and 3.3. $\square$

The reader should observe that the above proof is effective. Indeed, starting form the weights of the automaton $\mathcal{A}$, we compute the sublattice L' in finitely many steps. Then following our proof, we obtain Büchi automata for the languages R_i $(1 \leq i \leq n)$.

Consider two alphabets Σ, Δ and a non-deleting homomorphism $h : \Sigma^* \to \Delta^*$, i.e., $h(a) \neq \varepsilon$ for each $a \in \Sigma$. Then h can be extended to a mapping $h : \Sigma^\omega \to \Delta^\omega$ in the obvious way. For any infinitary series $S \in L\langle\langle \Sigma^\omega \rangle\rangle$ having finite image, and any infinitary language $R \subseteq \Sigma^\omega$, we can define the series $h_R(S) \in L\langle\langle \Delta^\omega \rangle\rangle$ by

$$(h_R(S), u) = \bigvee_{w \in h^{-1}(u) \cap R} (S, w)$$

for all $u \in \Delta^\omega$. We denote the series h_{Σ^ω} simply by h.

Furthermore, if $T \in L\langle\langle \Delta^\omega \rangle\rangle$ then the series $h^{-1}(T) \in L\langle\langle \Sigma^\omega \rangle\rangle$ is specified by

$$(h^{-1}(T), w) = (T, h(w))$$

for any $w \in \Sigma^\omega$.

Next, we show that given two distributive lattices L, L', any mapping $f : L \to L'$ preserves the ω-recognizability property of formal power series. This generalizes Proposition 3.2. Furthermore, we show that the ω-recognizability property of infinitary series is preserved by non-deleting and inverse non-deleting homomorphisms.

Proposition 3.4. (a) Let $(L, \leq)$, $(L', \leq)$ be two distributive lattices and $f : L \to L'$ be any mapping. Then for any ω-recognizable series S in $L\langle\langle \Sigma^\omega \rangle\rangle$ the series $f(S) \in L'\langle\langle \Sigma^\omega \rangle\rangle$ is again ω-recognizable.

(b) Let $h : \Sigma^\omega \to \Delta^\omega$ be a non-deleting homomorphism and $R \subseteq \Sigma^\omega$ be an ω-recognizable language. Then $h_R : L\langle\langle \Sigma^\omega \rangle\rangle \to L\langle\langle \Delta^\omega \rangle\rangle$ and $h^{-1} : L\langle\langle \Delta^\omega \rangle\rangle \to L\langle\langle \Sigma^\omega \rangle\rangle$ preserve the ω-recognizability property of formal power series.

Proof:
Due to Theorem 3.2, S is an ω-recognizable step function, i.e., $S = \bigvee_{1 \leq i \leq n} k_i \wedge 1_{R_i}$, where for $1 \leq i \leq n$, $k_i \in L$ and R_i is ω-recognizable. Then

$$f(S) = \bigvee_{1 \leq i \leq n} f(k_i) \wedge 1_{R_i}$$

and thus $f(S)$ is an ω-recognizable step function.

On the other hand, for any $u \in \Delta^\omega$ we have

$$\begin{aligned}
(h_R(S), u) &= \bigvee_{w \in h^{-1}(u) \cap R} (S, w) \\
&= \bigvee_{1 \leq i \leq n} \left(k_i \wedge \bigvee_{w \in h^{-1}(u) \cap R} (1_{R_i}, w) \right) \\
&= \bigvee_{1 \leq i \leq n} k_i \wedge \left(1_{h(R_i \cap R)}, u\right)
\end{aligned}$$

and hence

$$h_R(S) = \bigvee_{1 \le i \le n} k_i \wedge 1_{h(R_i \cap R)}.$$

Since the class of ω-recognizable languages is closed under non-deleting homomorphisms [45], we obtain that the series $h_R(S)$ is ω-recognizable.

Finally, assume that $T = \bigvee_{1 \le j \le m} k'_j \wedge 1_{R'_j}$ is an ω-recognizable series over Δ and L, where for $1 \le j \le m$, $k'_j \in L$ and $R'_j \subseteq \Delta^\omega$ is ω-recognizable. Then

$$h^{-1}(T) = \bigvee_{1 \le j \le m} k'_j \wedge 1_{h^{-1}(R'_j)}.$$

The class of ω-recognizable languages is closed under inverse non-deleting homomorphisms [45], therefore $h^{-1}(T)$ is ω-recognizable and our proof is completed. $\square$

As an immediate consequence of Proposition 3.4(a), we obtain:

Corollary 3.1. Let $(L, \le, ^-)$ be a bounded distributive lattice with negation function, and let $S \in L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle$. Then also $\overline{S} \in L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle$.

4. Multi-valued MSO logic

In this section, we introduce our multi-valued monadic second order logic over words, and we state our multi-valued Büchi theorem in the context of distributive lattices with negation function. Throughout this section, we assume that Σ is an alphabet and $(L, \le, ^-)$ is a bounded distributive lattice with negation function.

Definition 4.1. The syntax of formulas of the multi-valued MSO logic over Σ and L is given by:

$$\varphi := k \mid P_a(x) \mid x \le y \mid x \in X \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x \,.\, \varphi \mid \exists X \,.\, \varphi \mid \forall x \,.\, \varphi \mid \forall X \,.\, \varphi$$

where $k \in L, a \in \Sigma$. We shall denote by $MSO(L, \Sigma)$ the set of all such multi-valued MSO-formulas φ.

Next, we represent the semantics of the formulas in $MSO(L, \Sigma)$, as infinitary formal power series over the extended alphabet $\Sigma_{\mathcal{V}}$ and the lattice L. Here, our definition of semantics is slightly more general than the one used in [19]. In [19], the authors assigned to each atomic formula $P_a(x), x \le y$ or $x \in X$, respectively, the characteristic series of its associated MSO-language. Since these series take on only $0, 1$ as values, they can be viewed as "crisp" assignments. Here, in the general flavor of multi-valued logic, we wish to be more flexible. In the following, we assume that there is a function f assigning to each atomic formula φ of the form $P_a(x), x \le y$ or $x \in X$, respectively, a series $f(\varphi)$ in $L\left\langle\left\langle\Sigma_\varphi^\omega\right\rangle\right\rangle$. The assignment f is called *ω-recognizable* if the series $f(\varphi)$ is ω-recognizable for any atomic formula φ. Later on, we always require that f is an ω-recognizable assignment. As noted after Definition 3.1, then $f(\varphi)$ takes on only finitely many values, for any atomic formula φ. In general, we need and make the following assumption: We will call f a *multi-valued atomic assignment over* Σ, if $f(\varphi)$ takes on only finitely many values, for any atomic formula φ.

Definition 4.2. Let $\varphi \in MSO(L,\Sigma)$, $\mathcal{V}$ be a finite set of variables containing $Free(\varphi)$, and f be a multi-valued atomic assignment over $\Sigma_{\mathcal{V}}$. We define the *f-semantics* of φ to be an infinitary series $\|\varphi\|^f_{\mathcal{V}} \in L\langle\langle\Sigma^\omega_{\mathcal{V}}\rangle\rangle$ in the following way. Let $(w,\sigma) \in \Sigma^\omega_{\mathcal{V}}$. If σ is not a valid $(w,\mathcal{V})$-assignment, then we put $\left(\|\varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) = 0$. Otherwise, we inductively define $\left(\|\varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) \in L$ as follows:

- $\left(\|k\|^f_{\mathcal{V}}, (w,\sigma)\right) = k$
- $\left(\|\varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \big(f(\varphi), (w, \sigma|_{Free(\varphi)})\big)$ if φ is an atomic formula of the form $P_a(x), x \leq y$ or $x \in X$
- $\left(\|\neg\varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \overline{\left(\|\varphi\|^f_{\mathcal{V}}, (w,\sigma)\right)}$
- $\left(\|\varphi \vee \psi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \left(\|\varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) \vee \left(\|\psi\|^f_{\mathcal{V}}, (w,\sigma)\right)$
- $\left(\|\varphi \wedge \psi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \left(\|\varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) \wedge \left(\|\psi\|^f_{\mathcal{V}}, (w,\sigma)\right)$
- $\left(\|\exists x \,.\, \varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \bigvee_{i\in\omega} \left(\|\varphi\|^f_{\mathcal{V}\cup\{x\}}, (w,\sigma[x \to i])\right)$
- $\left(\|\exists X \,.\, \varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \bigvee_{I\subseteq\omega} \left(\|\varphi\|^f_{\mathcal{V}\cup\{X\}}, (w,\sigma[X \to I])\right)$
- $\left(\|\forall x \,.\, \varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \bigwedge_{i\in\omega} \left(\|\varphi\|^f_{\mathcal{V}\cup\{x\}}, (w,\sigma[x \to i])\right)$
- $\left(\|\forall X \,.\, \varphi\|^f_{\mathcal{V}}, (w,\sigma)\right) = \bigwedge_{I\subseteq\omega} \left(\|\varphi\|^f_{\mathcal{V}\cup\{X\}}, (w,\sigma[X \to I])\right)$.

We claim that in Definition 4.2, all the occurring infinite suprema and infima exist in L (without any further completeness assumption). For this, one can show by induction on the structure of formulas φ, that $\|\varphi\|^f_{\mathcal{V}}$ takes on only finitely many values. Indeed, for atomic formulas this is clear by assumption, and the property is preserved by negation, disjunction and conjunction. Since L is a lattice, the property is also preserved by infinite suprema and infima, proving our claim.

If the multi-valued atomic assignment is well-known, then we omit the superscript f from $\|\varphi\|^f_{\mathcal{V}}$. Furthermore, we simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. If φ has no free variables, i.e., if it is a sentence, then $\|\varphi\| \in L\langle\langle\Sigma^\omega\rangle\rangle$.

An infinitary power series $S \in L\langle\langle\Sigma^\omega\rangle\rangle$ is called *MSO-f-definable* if there is a sentence $\varphi \in MSO(L,\Sigma)$ such that $S = \|\varphi\|^f$. We let $L^{f-mso}\langle\langle\Sigma^\omega\rangle\rangle$ comprise all series from $L\langle\langle\Sigma^\omega\rangle\rangle$ which are f-definable by some sentence in $MSO(L,\Sigma)$. Our goal will be to derive a relationship between the classes $L^{f-mso}\langle\langle\Sigma^\omega\rangle\rangle$ and $L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle$.

Next, we give two examples of possible interpretations of multi-valued MSO-formulas.

Example 4.1. We consider the bounded distributive lattice $(\mathbb{N} \cup \{\infty\}, \leq, ^{-})$ (where $\mathbb{N}$ is the set of natural numbers and $^{-}$ is an arbitrary negation function). Let $\Sigma = \{a, b, c\}$ and f be the multi-valued atomic assignment over Σ, determined in the following way. For each $w \in \Sigma^\omega$ and each valid $(w, \{x\})$-assignment σ we set

- $(f(P_a(x)), (w, \sigma)) = 0$
- $(f(P_b(x)), (w, \sigma)) = \begin{cases} 1 & \text{if } w(\sigma(x)) = b \\ 0 & \text{otherwise} \end{cases}$
- $(f(P_c(x)), (w, \sigma)) = \begin{cases} 2 & \text{if } w(\sigma(x)) = c \\ 0 & \text{otherwise} \end{cases}$.

For any other atomic formula φ, $f(\varphi)$ is the constant function 0. Let $\varphi = \forall x . (P_a(x) \vee P_b(x) \vee P_c(x))$. In fact φ is a sentence, and for any word $w \in \Sigma^\omega$ the semantics $\|\varphi\|^f$ returns the value 0 if the letter a occurs at least once in w, the value 1 if no a appears in w but b occurs at least once, and it returns the value 2 if $w = c^\omega$.

Example 4.2. Let again $\Sigma = \{a, b, c\}$. Let also $(\mathcal{P}(\Sigma), \subseteq, ^{-})$ be the bounded distributive lattice of subsets of Σ, with union as supremum and intersection as infimum, and $^{-}$ any negation function. Consider the multi-valued atomic assignment f over Σ given by

- $(f(P_a(x)), (w, \sigma)) = \begin{cases} \{a\} & \text{if } w(\sigma(x)) = a \\ \emptyset & \text{otherwise} \end{cases}$
- $(f(P_b(x)), (w, \sigma)) = \begin{cases} \{b\} & \text{if } w(\sigma(x)) = b \\ \emptyset & \text{otherwise} \end{cases}$
- $(f(P_c(x)), (w, \sigma)) = \begin{cases} \Sigma & \text{if } w(\sigma(x)) = c \\ \emptyset & \text{otherwise} \end{cases}$.

For any other atomic formula φ, $f(\varphi)$ takes the constant $\emptyset$. Let $\varphi = \exists x . (P_a(x) \vee P_b(x) \vee (P_c(x) \wedge \{c\}))$. Then for any word $w \in \Sigma^\omega$ the semantics of φ on w returns the set of letters that appear in w.

Observe that the above definition of semantics is valid for each formula $\varphi \in MSO(L, \Sigma)$ and each finite set $\mathcal{V}$ of variables containing $Free(\varphi)$. As we show next, the semantics $\|\varphi\|^f_{\mathcal{V}}$ is in fact independent of the set $\mathcal{V}$; it depends only on $Free(\varphi)$. More precisely,

Proposition 4.1. For any $\varphi \in MSO(L, \Sigma)$, any finite set of variables $\mathcal{V}$ with $Free(\varphi) \subseteq \mathcal{V}$, and any multi-valued atomic assignment f over $\Sigma_{\mathcal{V}}$, it holds that

$$\left(\|\varphi\|^f_{\mathcal{V}}, (w, \sigma)\right) = \left(\|\varphi\|^f, (w, \sigma|_{Free(\varphi)})\right)$$

for each $(w, \sigma) \in \Sigma^\omega_{\mathcal{V}}$, where σ is a valid $(w, \mathcal{V})$-assignment. Furthermore, the series $\|\varphi\|^f$ is ω-recognizable iff the series $\|\varphi\|^f_{\mathcal{V}}$ is ω-recognizable.

Proof:
We establish our first claim by induction on the structure of formulas $\varphi \in MSO(L, \Sigma)$. For atomic formulas it is trivial, whereas for negation, disjunction and conjunction it follows directly by induction. For the case of quantifications and for our second claim, we can proceed almost literally in the same way as for Proposition 3.3 in [16], replacing sums by suprema and products by infima. We provide the details for the second claim for the convenience of the reader. To this end, we consider the projection $h : \Sigma_{\mathcal{V}} \to \Sigma_\varphi$ with $h(w, \sigma) = (w, \sigma|_{Free(\varphi)})$ for any $(w, \sigma) \in \Sigma_{\mathcal{V}}^\omega$. Then h is a non-deleting homomorphism. We have $\|\varphi\|_{\mathcal{V}}^f = h^{-1}(\|\varphi\|^f) \wedge 1_{N_{\mathcal{V}}^\omega}$, and if the series $\|\varphi\|^f$ is ω-recognizable, then $\|\varphi\|_{\mathcal{V}}^f$ is ω-recognizable by Propositions 3.1, 3.3 and 3.4(b). Also, we have $\|\varphi\|^f = h(\|\varphi\|_{\mathcal{V}}^f)$. Thus, if $\|\varphi\|_{\mathcal{V}}^f$ is ω-recognizable, by Proposition 3.4(b) the series $\|\varphi\|^f$ is also ω-recognizable. □

Next, we derive a further closure property of the class of ω-recognizable series.

Lemma 4.1. Let $h : \Sigma^\omega \to \Delta^\omega$ be a non-deleting homomorphism, $R \subseteq \Sigma^\omega$ be an ω-recognizable language, and $S \in L^{\omega-rec}\langle\langle \Sigma^\omega \rangle\rangle$ be an ω-recognizable series. Then the series $\bigwedge_{h,R}(S) \in L\langle\langle \Delta^\omega \rangle\rangle$ defined by $\left(\bigwedge_{h,R}(S), u\right) = \bigwedge\limits_{w \in h^{-1}(u) \cap R} (S, w)$ is ω-recognizable.

Proof:
By Theorem 3.2 the series S is an ω-recognizable step function. Now let $\left(L^d, \leq^d\right) = (L, \geq)$, the dual lattice of L, which is obtained by interchanging suprema and infima. Since S assumes only finitely many values and each value on an ω-recognizable language, S is also an ω-recognizable step function over L^d and Σ^ω. For the transformation $h_R^d : L^d\langle\langle \Sigma^\omega \rangle\rangle \to L^d\langle\langle \Delta^\omega \rangle\rangle$, by Proposition 3.4(b) we obtain $h_R^d(S) \in \left(L^d\right)^{\omega-rec}\langle\langle \Delta^\omega \rangle\rangle$, and $h_R^d(S)$ is an ω-recognizable step function over L^d, hence again also over L. Since suprema in L^d equal infima in L, we have $h_R^d(S) = \bigwedge_{h,R}(S)$ which implies the result. □

Proposition 4.2. Let $\varphi, \psi \in MSO(L, \Sigma)$ such that $\|\varphi\|_{\mathcal{V}}^f, \|\psi\|_{\mathcal{V}}^f$ are ω-recognizable series where f is a multi-valued atomic assignment, and $\mathcal{V}$ is a finite set of variables with $Free(\varphi) \cup Free(\psi) \subseteq \mathcal{V}$. Then the series $\|\varphi \vee \psi\|_{\mathcal{V}}^f, \|\varphi \wedge \psi\|_{\mathcal{V}}^f, \|\exists x \,.\, \varphi\|_{\mathcal{V}}^f, \|\exists X \,.\, \varphi\|_{\mathcal{V}}^f, \|\forall x \,.\, \varphi\|_{\mathcal{V}}^f$ and $\|\forall X \,.\, \varphi\|_{\mathcal{V}}^f$ are ω-recognizable.

Proof:
The semantics of disjunction and conjunction of φ and ψ are ω-recognizable by Proposition 3.1. Next, we deal with existential and universal quantifiers. By assumption $\|\varphi\|_{\mathcal{V}}^f$ is ω-recognizable. By Theorem 3.2, we have $\|\varphi\|_{\mathcal{V}}^f = \bigvee\limits_{1 \leq i \leq n} k_i \wedge 1_{R_i}$, with $k_i \in L$ and R_i is ω-recognizable for any $1 \leq i \leq n$. Let

$$h : \Sigma_{\mathcal{V} \cup \{x\}}^\omega \to \Sigma_{\mathcal{V}}^\omega \quad \text{and} \quad h' : \Sigma_{\mathcal{V} \cup \{X\}}^\omega \to \Sigma_{\mathcal{V}}^\omega$$

be the non-deleting homomorphisms erasing the x-row and the X-row, respectively. Then, it holds that

$$\begin{aligned} \|\exists x \,.\, \varphi\|_{\mathcal{V}}^f &= h\left(\|\varphi\|_{\mathcal{V} \cup \{x\}}^f\right) & \qquad \|\exists X \,.\, \varphi\|_{\mathcal{V}}^f &= h'\left(\|\varphi\|_{\mathcal{V} \cup \{X\}}^f\right) \\ \|\forall x \,.\, \varphi\|_{\mathcal{V}}^f &= \bigwedge\nolimits_{h, N_{\mathcal{V} \cup \{x\}}}\left(\|\varphi\|_{\mathcal{V} \cup \{x\}}^f\right) & \qquad \|\forall X \,.\, \varphi\|_{\mathcal{V}}^f &= \bigwedge\nolimits_{h', \Sigma_{\mathcal{V} \cup \{X\}}^\omega}\left(\|\varphi\|_{\mathcal{V} \cup \{X\}}^f\right). \end{aligned}$$

We conclude our proof by Proposition 3.4(b) and Lemma 4.1. □

Next we obtain the first half of our main goal, an implication between MSO-f-definable and ω-recognizable series.

Proposition 4.3. Let Σ be an alphabet and L be a bounded distributive lattice with any negation mapping. Let f be any ω-recognizable multi-valued atomic assignment. Then

$$L^{f-mso}\left\langle\left\langle\Sigma^{\omega}\right\rangle\right\rangle \subseteq L^{\omega-rec}\left\langle\left\langle\Sigma^{\omega}\right\rangle\right\rangle .$$

Proof:
We use induction on the structure of $MSO(L,\Sigma)$-formulas. First, assume that $\varphi = k \in L$. We consider the one state MVMA $\mathcal{A} = (\{q\}, in, wt, \{\{q\}\})$ with $in(q) = k$ and $wt(q,a,q) = 1$ for each $a \in \Sigma$. Obviously, $\|\mathcal{A}\| = k \wedge 1_{\Sigma^\omega}$ and thus $\|\varphi\|$ is ω-recognizable. For the remaining atomic formulas φ the semantics $\|\varphi\|^f$ are ω-recognizable by our assumption for the ω-recognizability property of f. For negation we take into account Corollary 3.1, whereas for disjunctions, conjunctions and quantifications the proof comes by Proposition 4.2. $\square$

The *crisp atomic assignment* cf for atomic formulas is defined in the following way. Let φ be an atomic formula of the form $P_a(x), x \leq y$ or $x \in X$. Then for any $(w,\sigma) \in \Sigma_\varphi^\omega$ with σ a valid assignment we set

- $(cf\left(P_a(x)\right), (w,\sigma)) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases}$,

- $(cf\left(x \leq y\right), (w,\sigma)) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$,

- $(cf\left(x \in X\right), (w,\sigma)) = \begin{cases} 1 & \text{if } \sigma(x) \in \sigma(X) \\ 0 & \text{otherwise} \end{cases}$.

Note that if φ is an atomic formula of this form then $\left(\|\neg\varphi\|^{cf}, (w,\sigma)\right) = \overline{(cf(\varphi), (w,\sigma))}$ for each $(w,\sigma) \in N_\varphi$, and by the property of $^-$ that $\overline{1} = 0$ and $\overline{0} = 1$, our semantics of $\neg\varphi$ coincides with the one given in [19].

It is not difficult to see that the crisp atomic assignment is ω-recognizable (cf. [19]). We shall denote the class $L^{cf-mso}\left\langle\left\langle\Sigma^{\omega}\right\rangle\right\rangle$ simply by $L^{mso}\left\langle\left\langle\Sigma^{\omega}\right\rangle\right\rangle$.

Proposition 4.4. Let Σ be an alphabet and L be a bounded distributive lattice with any negation function. Then

$$L^{\omega-rec}\left\langle\left\langle\Sigma^{\omega}\right\rangle\right\rangle \subseteq L^{mso}\left\langle\left\langle\Sigma^{\omega}\right\rangle\right\rangle .$$

Proof:
Let $\mathcal{A} = (Q, in, wt, \mathcal{F})$ be an MVMA over Σ and L, and L' be the finite sublattice of L generated by the set $\{in(q), wt(t) \mid q \in Q, t \in Q \times \Sigma \times Q\}$. Then, the semiring $(L', \vee, \wedge, 0, 1)$ satisfies all the technical assumptions of [19] (cf.[19], Example 1), i.e. it is totally commutative complete. Following the proof of Proposition 21 of [19], we can effectively construct an $MSO(L,\Sigma)$-formula φ such that $\|\varphi\|^{cf} = \|\mathcal{A}\|$.

$\square$

Now we are ready to state our second main result.

Theorem 4.1. Let Σ be an alphabet and L be a bounded distributive lattice with any negation function. Then

$$L^{\omega-rec}\langle\langle\Sigma^{\omega}\rangle\rangle = \bigcup_{f} L^{f-mso}\langle\langle\Sigma^{\omega}\rangle\rangle = L^{mso}\langle\langle\Sigma^{\omega}\rangle\rangle$$

where the union is taken over all ω-recognizable multi-valued atomic assignments.

Proof:
We combine Propositions 4.3 and 4.4. □

This result shows that for any formula $\varphi \in MSO(L,\Sigma)$, whose semantics is defined with any ω-recognizable multi-valued atomic assignment, we can construct an equivalent $MSO(L,\Sigma)$-formula with the crisp atomic assignment.

In the sequel, we deal with decidability results for ω-recognizable formal power series over distributive lattices with negation function.

Theorem 4.2. Let Σ be an alphabet and L be a bounded distributive lattice with any negation function. For any two sentences $\varphi, \psi \in MSO(L,\Sigma)$ the following relations are decidable:

- $\|\varphi\| \leq \|\psi\|$
- $\|\varphi\| = \|\psi\|$
- $\|\varphi\| = 0$
- $\|\varphi\| = 1$.

Proof:
We assume that the multi-valued atomic assignment is crisp for both the sentences φ and ψ or else that we are given MVMA for the series of the atomic formulas. By Theorem 3.2, then we compute for the atomic formulas occurring in φ respectively ψ, their representations as ω-recognizable step functions. By induction on φ, ψ we can effectively construct their decompositions using the proofs of Propositions 3.1, 3.4 and Lemma 4.1. So, we obtain $\|\varphi\| = \bigvee_{1\leq i\leq n} k_i \wedge 1_{R_i}$ and $\|\psi\| = \bigvee_{1\leq j\leq m} k'_j \wedge 1_{R'_i}$, where $k_i, k'_j \in L$ and R_i, R'_j are ω-recognizable languages over Σ^{ω} for any $1 \leq i \leq n$, $1 \leq j \leq m$ for which we have constructed Büchi automata. Moreover, assume that $(R_i)_{1\leq i\leq n}$ and $\left(R'_j\right)_{1\leq j\leq m}$ are partitions of Σ^{ω}. Then, in case of equality, we check that whenever $R_i \cap R'_j \neq \emptyset$, then $k_i = k'_j$. To decide whether $\|\varphi\| \leq \|\psi\|$, we check that whenever $R_i \cap R'_j \neq \emptyset$, then $k_i \leq k'_j$. □

We conclude this section with some remarks concerning the finite words case. In this paper, we considered multi-valued automata consuming only infinite words. One can define multi-valued automata over bounded distributive lattices acting on finite words, in the same manner as weighted automata over arbitrary semirings (cf. [2, 33, 36, 49, 50]). Then, by considering the same multi-valued MSO logic, we can state weighted generalizations of Büchi's and Elgot's results [11, 24], i.e., the equivalence of recognizability and definability of finitary formal power series over bounded distributive lattices with negation function. This result has been already proved in [16]. Indeed, Droste and Gastin in [16] showed that a series is recognizable iff it is definable by some sentence in their weighted MSO logic over locally finite semirings. Now one only has to observe that the bounded distributive lattice induced by the weights of a multi-valued automaton or an $MSO(L, \Sigma)$-formula is a finite semiring, hence the results of [16] apply.

5. Multi-valued tree automata and multi-valued MSO logic

In this section, we deal with multi-valued Muller tree automata and we introduce a multi-valued MSO logic over infinite trees. The main result of this section is a multi-valued version of Rabin's theorem in the setting of bounded distributive lattices with negation function. Most of the proof techniques are literally the same as the corresponding ones for words. Therefore, our presentation here is brief.

First, we recall notions concerning infinite trees. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. The *prefix relation* $\leq$ over $\mathbb{N}^*$ is a partial order defined in the usual way: for any $w, v \in \mathbb{N}^*$, $w \leq v$ iff there exists $u \in \mathbb{N}^*$ such that $wu = v$. A set $A \subseteq \mathbb{N}^*$ is called *prefix-closed* if $wu \in A$ implies $w \in A$.

A *ranked alphabet* Σ is a pair (Σ, rk) (simply denoted by Σ) where Σ is a finite set and $rk : \Sigma \to \mathbb{N}$. As usual, we set $\Sigma_k = \{\sigma \in \Sigma \mid rk(\sigma) = k\}$, $k \geq 0$ and $\deg(\Sigma) = \max\{k \in \mathbb{N} \mid \Sigma_k \neq \emptyset\}$.

A *tree t over* Σ is a partial mapping $t : \mathbb{N}_+^* \to \Sigma$ such that the domain $dom(t)$ of t is a non-empty prefix-closed set, and if $t(w) \in \Sigma_k$, $k \geq 0$ then for $i \in \mathbb{N}_+$, $wi \in dom(t)$ iff $1 \leq i \leq k$. A tree t is called *infinite* if its domain is infinite. Any element in $dom(t)$ is called a *node of* t. We shall denote by T_Σ^ω the set of all infinite trees over Σ.

Next we recall elements from classical MSO logic over trees. Let Σ be a ranked alphabet. The syntax of formulas of the MSO logic over Σ is given by:

$$\varphi := label_\sigma(x) \mid edge_i(x, y) \mid x \in X \mid \neg\varphi \mid \varphi \vee \psi \mid \exists x \,.\, \varphi \mid \exists X \,.\, \varphi$$

where $\sigma \in \Sigma$, $1 \leq i \leq \deg(\Sigma)$, x, y are first order variables and X is a second order variable.

An infinite tree $t \in T_\Sigma^\omega$ is represented by the relational structure $(dom(t), edge_1, \ldots, edge_{\deg(\Sigma)}, (R_\sigma)_{\sigma \in \Sigma})$ where $R_\sigma = \{w \in dom(t) \mid t(w) = \sigma\}$ for $\sigma \in \Sigma$, and for each $w, u \in dom(t)$, $edge_j(w, u)$ holds true iff $u = wj$, for $1 \leq j \leq \deg(\Sigma)$. Given a finite set of first and second order variables $\mathcal{V}$, a $(t, \mathcal{V})$-*assignment* ρ is a mapping assigning elements of $dom(t)$ to first order variables from $\mathcal{V}$ and subsets of $dom(t)$ to second order variables from $\mathcal{V}$. Let x be a first order variable and $w \in dom(t)$. Then $\rho[x \to w]$ denotes the $(t, \mathcal{V} \cup \{x\})$-assignment which associates w to x and acts as ρ on $\mathcal{V} \setminus \{x\}$. The notation $\rho[X \to I]$ for a second order variable X and a set $I \subseteq dom(t)$ has a similar meaning.

Now, we consider the ranked alphabet $\Sigma_\mathcal{V} = \Sigma \times \{0, 1\}^\mathcal{V}$ with $rk(\sigma, f) = rk(\sigma)$ for each $\sigma \in \Sigma$ and $f \in \{0, 1\}^\mathcal{V}$. For any $(\sigma, f) \in \Sigma_\mathcal{V}$ we denote by $(\sigma, f)_1$ and $(\sigma, f)_2$ the symbols σ and f, respectively. An infinite tree $s \in T_{\Sigma_\mathcal{V}}^\omega$ is called *valid* if for each first order variable $x \in \mathcal{V}$, there is exactly one node w of s such that $(s(w)_2)(x) = 1$. The set of all valid infinite trees over $\Sigma_\mathcal{V}$ is denoted by $T_{\Sigma_\mathcal{V}}^{\omega,v}$. Every valid tree $s \in T_{\Sigma_\mathcal{V}}^\omega$ corresponds to a pair (t, ρ) where $t \in T_\Sigma^\omega$ and ρ is a valid $(t, \mathcal{V})$-assignment, such that $t = s_1$, and for every first order variable x, second order variable X, and any node $w \in dom(s)$, we have that $\rho(x) = w$ iff $(s(w)_2)(x) = 1$, and $w \in \rho(X)$ iff $(s(w)_2)(X) = 1$. Then, we say that s and (t, ρ)

correspond to each other. In the following, we identify a valid infinite tree s with its corresponding pair (t, ρ). The following result is well-known.

Proposition 5.1. The infinitary tree language $T^{\omega,v}_{\Sigma_\mathcal{V}}$ is ω-recognizable.

Let φ be an MSO-formula over trees [53, 54]. As usual we shall write Σ_φ instead of $\Sigma_{Free(\varphi)}$. Then for $Free(\varphi) \subseteq \mathcal{V}$, the fundamental theorem of Rabin (cf. [46]) states that the infinitary tree language

$$\mathcal{L}_\mathcal{V}(\varphi) = \{(t, \rho) \in T^{\omega,v}_{\Sigma_\mathcal{V}} \mid (t, \rho) \models \varphi\}$$

is ω-recognizable; conversely, for each ω-recognizable tree language $R \subseteq T^\omega_\Sigma$ there exists an MSO-sentence φ such that $R = \mathcal{L}(\varphi)$.

Throughout this section, Σ will denote a ranked alphabet and L a bounded distributive lattice. Next, we introduce our multi-valued Muller tree automata over Σ and L.

Definition 5.1. A *multi-valued Muller tree automaton* (MVMTA for short) *over* Σ *and* L is a quadruple $\mathcal{M} = (Q, in, wt, \mathcal{F})$, where Q is the *finite state set*, $in : Q \to L$ is the *initial distribution*, $wt : \bigcup_{k \geq 0} Q \times \Sigma_k \times Q^k \to L$ is a *mapping assigning weights to the transitions* of the automaton, and $\mathcal{F} \subseteq \mathcal{P}(Q)$ is the *family of final state sets*.

Let $t \in T^\omega_\Sigma$. A *run of* $\mathcal{M}$ *over* t is a partial mapping $r_t : \mathbb{N}^*_+ \to Q$ such that $dom(r_t) = dom(t)$. The *weight of* r_t *at* $w \in dom(t)$ is the value

$$wt(r_t, w) := wt\,(r_t(w), t(w), (r_t(w1), \ldots, r_t(w \cdot rk(t(w))))).$$

Then the *weight of* r_t is defined by

$$weight(r_t) := in(r_t(\varepsilon)) \wedge \bigwedge_{w \in dom(t)} wt(r_t, w).$$

Observe that the set $\left\{ wt(t) \mid t \in \bigcup_{k \geq 0} Q \times \Sigma_k \times Q^k \right\}$ is finite; therefore the value $weight(r_t)$ is well-defined. Moreover, if L' is the (finite) sublattice of L generated by $\left\{ in(q), wt(t) \mid q \in Q, t \in \bigcup_{k \geq 0} Q \times \Sigma_k \times Q^k \right\}$ then $weight(r_t) \in L'$.
Any infinite prefix-closed chain $\pi \subseteq dom(r_t)$ is called an *infinite path of* r_t. Then, the run r_t is called *successful* if for each infinite path π of r_t, the set $In^Q(r_t|_\pi)$ of states that appear infinitely often along π constitute a final state set, i.e., $In^Q(r_t|_\pi) \in \mathcal{F}$. The *behavior* of $\mathcal{M}$ is the infinitary formal power tree series

$$\|\mathcal{M}\| : T^\omega_\Sigma \to L$$

whose coefficients are given by

$$(\|\mathcal{M}\|, t) = \bigvee_{r_t} weight(r_t)$$

for $t \in T^\omega_\Sigma$, where the supremum is taken over all successful runs r_t of $\mathcal{M}$ over t.

An infinitary tree series $S : T_\Sigma^\omega \to L$ is said to be *Muller recognizable* if there is a MVMTA $\mathcal{M}$, such that $S = \|\mathcal{M}\|$. The family of all Muller recognizable tree series over Σ and L is denoted by $L^{M-rec}\langle\langle T_\Sigma^\omega \rangle\rangle$.

In the following, we introduce our multi-valued monadic second order logic over trees.

Definition 5.2. The syntax of formulas of the multi-valued MSO logic for trees over Σ and L is given by:

$$\varphi := k \mid label_\sigma(x) \mid edge_i(x,y) \mid x = y \mid x \in X \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \\ \mid \exists x \centerdot \varphi \mid \exists X \centerdot \varphi \mid \forall x \centerdot \varphi \mid \forall X \centerdot \varphi$$

where $k \in L$, $\sigma \in \Sigma$ and $1 \leq i \leq \deg(\Sigma)$. We shall denote by $MSO(L, \Sigma)$ the set of all such multi-valued MSO-formulas φ.

Now, we represent the semantics of the formulas in $MSO(L, \Sigma)$ as infinitary tree series over the extended alphabet $\Sigma_\mathcal{V}$ and the lattice L.

Definition 5.3. Let $\varphi \in MSO(L, \Sigma)$, $\mathcal{V}$ be a finite set of variables containing $Free(\varphi)$, and f be a multi-valued atomic assignment over $\Sigma_\mathcal{V}$. The f*-semantics* of φ is an infinitary formal power tree series $\|\varphi\|_\mathcal{V}^f \in L\left\langle\left\langle T_{\Sigma_\mathcal{V}}^\omega\right\rangle\right\rangle$. Let $(t, \rho) \in T_{\Sigma_\mathcal{V}}^\omega$. If ρ is not a valid $(t, \mathcal{V})$-assignment, then we set $(\|\varphi\|_\mathcal{V}^f, (t, \rho)) = 0$. Otherwise, we inductively define $(\|\varphi\|_\mathcal{V}^f, (t, \rho)) \in L$ literally as in Definition 4.2, replacing (w, σ) by (t, ρ) and the domain ω of w by $dom(t)$.

The following example presents an interpretation of a multi-valued MSO-sentence over trees.

Example 5.1. Let Σ be a ranked alphabet with $\Sigma_3 = \{\gamma\}$, $\Sigma_2 = \{\sigma, \delta\}$ and $\Sigma_0 = \{a, b\}$. Let also $(\{0,1,2,3,4\}, \leq, ^-)$ be a bounded distributive lattice with negation mapping, where $\leq$ is the natural order and $\overline{0} = 4$, $\overline{1} = \overline{2} = 0$, $\overline{3} = 3$ and $\overline{4} = 0$. Note that this structure is not a De Morgan algebra, since $^-$ is not injective (see Section 6). We consider the multi-valued atomic assignment f as follows. For any $(t, \rho) \in T_{\Sigma_{label_\sigma(x)}}^{\omega,v}$ we let

$$(f\,(label_\sigma(x))\,, (t, \rho)) = \begin{cases} 2 & \text{if } t(\rho(x)) = \sigma \\ 0 & \text{otherwise} \end{cases},$$

for any $(t, \rho) \in T_{\Sigma_{label_\delta(x)}}^{\omega,v}$ we let

$$(f\,(label_\delta(x))\,, (t, \rho)) = \begin{cases} 3 & \text{if } t(\rho(x)) = \delta \\ 0 & \text{otherwise} \end{cases},$$

and for any other atomic formula φ and any $(t, \rho) \in T_{\Sigma_\varphi}^{\omega,v}$ we set

$(f\,(\varphi)\,, (t, \rho)) = 3.$

We define now the sentence

$$\varphi = (\forall x \centerdot label_\sigma(x)) \vee (\forall x \centerdot label_\delta(x))\,.$$

Then for any $t \in T_\Sigma^\omega$ the coefficient $\left(\|\varphi\|^f, t\right) = 2$ if t is the infinite binary σ-tree, $\left(\|\varphi\|^f, t\right) = 3$ if t is the infinite binary δ-tree, and $\left(\|\varphi\|^f, t\right) = 0$ otherwise.

An infinitary tree series $S \in L\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle$ is called *MSO-f-definable* if there is a sentence $\varphi \in MSO(L, \Sigma)$ such that $S = \|\varphi\|^f$. We let $L^{f-mso}\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle$ comprise all formal power tree series from $L\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle$, which are f-definable by some sentence in $MSO(L, \Sigma)$. If we consider the crisp atomic assignment, then we simply denote the class $L^{cf-mso}\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle$ by $L^{mso}\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle$

The next theorem states a multi-valued version of Rabin's result for infinitary tree series.

Theorem 5.1. Let Σ be a ranked alphabet and L be a bounded distributive lattice with any negation function. Then

$$L^{M-rec}\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle = \bigcup_f L^{f-mso}\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle = L^{mso}\left\langle\left\langle T_\Sigma^\omega \right\rangle\right\rangle$$

where the union is taken over all Muller recognizable multi-valued atomic assignments.

As noted before, in the proof we can proceed almost exactly as in the case of infinite words. A decidability result analogous to Theorem 4.2 also holds for $MSO(L, \Sigma)$-sentences over trees.

One can consider multi-valued tree automata over bounded distributive lattices consuming finite trees as a special case of weighted tree automata over semirings (cf. [3, 6, 7, 8, 9, 18, 21, 25, 34, 44, 51]). Then, using the same multi-valued MSO logic, but now over finite trees, we can state the results of Thatcher and Wright [52] and Doner [15] for the equivalence of tree recognizability with MSO-definability, in the framework of finitary formal power trees series over bounded distributive lattices with negation function. This result has been already established in [21]. Indeed, Droste and Vogler proved the aforementioned equivalence for locally finite semirings. Note that the weights of a multi-valued tree automaton or a $MSO(L, \Sigma)$-formula induce a finite semiring.

6. De Morgan algebras

In this section, we consider De Morgan algebras. Recently, De Morgan algebras have been investigated intensively in the literature for multi-valued model-checking, see [13, 29, 32, 37].

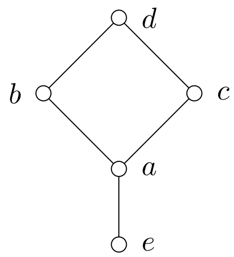
A distributive lattice $(L, \leq, ^-)$ is a *De Morgan* (or *quasi-Boolean*) *algebra*, if it is equipped with a *complement mapping* $^- : L \to L$ which satisfies the involution and De Morgan laws, i.e.,

$$\overline{\overline{a}} = a,$$
$$\overline{a \vee b} = \overline{a} \wedge \overline{b},$$
$$\overline{a \wedge b} = \overline{a} \vee \overline{b}$$

for all $a, b \in L$. Note that then $a \leq b$ implies $\overline{b} \leq \overline{a}$ for any $a, b \in L$. Furthermore, if L is bounded then $\overline{0} = 1$ and $\overline{1} = 0$. So, $^- : (L, \leq) \to (L, \geq)$ is an order-isomorphism. Hence, if $(a_i \mid i \in I) \subseteq L$ is a family of elements of L for which $\bigvee_{i \in I} a_i$ exists, then $\overline{\bigvee_{i \in I} a_i} = \bigwedge_{i \in I} \overline{a_i}$.

For instance, the lattice $(F, \leq, ^-)$ where $F = [0, 1]$, the unit interval, $\leq$ is the usual order of real numbers, and $\overline{a} = 1 - a$ for any $a \in F$ is a De Morgan algebra. We refer the reader to [43] for many more examples.

However, since any bounded distributive lattice can be endowed with a negation function, it is clear that lattices with negation function constitute a much larger class than De Morgan algebras. In particular, any bounded distributive lattice which is not anti-isomorphic to itself carries no complement operation making it a De Morgan algebra. The easiest finite example of this is provided by the four element Boolean algebra $\{a, b, c, d\}$ with an additional element e added as new 0 below it:



Note that if L is a De Morgan algebra, then clearly the collection of power series $(L\langle\langle\Sigma^\omega\rangle\rangle, \leq)$ also forms a De Morgan algebra.

In case of De Morgan algebras an alternative simpler syntax of formulas of our multi-valued MSO logic over Σ and L can be given by

$$\varphi := k \mid P_a(x) \mid x \leq y \mid x \in X \mid \neg\varphi \mid \varphi \vee \psi \mid \exists x\,.\,\varphi \mid \exists X\,.\,\varphi.$$

We define the semantics $\|\varphi\|$ of formulas φ of this syntax exactly as in Definition 4.2. Given a multi-valued atomic assignment f, let $L^{dm-f-mso}\langle\langle\Sigma^\omega\rangle\rangle$ be the collection of all series definable in this logic.

Then conjunction and universal quantifiers can be defined by letting

- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- $\forall x\,.\,\varphi = \neg(\exists x\,.\,\neg\varphi)$
- $\forall X\,.\,\varphi = \neg(\exists X\,.\,\neg\varphi)$

for each $\varphi, \psi \in MSO(L, \Sigma)$. Then, using the De Morgan laws, it is easy to see that we have the following equalities for any $(w, \sigma) \in \Sigma^\omega_\mathcal{V}$ where σ is a valid assignment:

- $\left(\|\varphi \wedge \psi\|^f_\mathcal{V}, (w, \sigma)\right) = \left(\|\varphi\|^f_\mathcal{V}, (w, \sigma)\right) \wedge \left(\|\psi\|^f_\mathcal{V}, (w, \sigma)\right)$
- $\left(\|\forall x\,.\,\varphi\|^f_\mathcal{V}, (w, \sigma)\right) = \bigwedge_{i \in \omega} \left(\|\varphi\|^f_{\mathcal{V}\cup\{x\}}, (w, \sigma[x \to i])\right)$
- $\left(\|\forall X\,.\,\varphi\|^f_\mathcal{V}, (w, \sigma)\right) = \bigwedge_{I \subseteq \omega} \left(\|\varphi\|^f_{\mathcal{V}\cup\{X\}}, (w, \sigma[X \to I])\right).$

The crisp atomic assignment cf is also defined as before, and we denote again the class $L^{dm-cf-mso}\langle\langle\Sigma^\omega\rangle\rangle$ by $L^{dm-mso}\langle\langle\Sigma^\omega\rangle\rangle$. As an immediate consequence of Theorem 4.1 and the above equalities we obtain:

Corollary 6.1. Let Σ be an alphabet and $(L, \leq, ^-)$ be a De Morgan algebra. Then

$$L^{\omega-rec}\langle\langle\Sigma^\omega\rangle\rangle = \bigcup_f L^{dm-f-mso}\langle\langle\Sigma^\omega\rangle\rangle = L^{dm-mso}\langle\langle\Sigma^\omega\rangle\rangle$$

where the union is taken over all ω-recognizable multi-valued atomic assignments.

Recall that any bounded distributive lattice $(L, \leq)$ can be considered as a semiring $(L, +, \cdot, 0, 1)$ with supremum as addition and infimum as multiplication operation. In view of our results (and the wealth of results in the literature on weighted automata having weights in arbitrary semirings) one could ask whether we could obtain our results for an arbitrary semiring with some complement function. By the above, the complement function should interconnect addition and multiplication. Therefore we consider the following concept:

Let $(A, +, \cdot, 0, 1)$ be a semiring and $f : A \to A$ be a function. We call f a *complement function*, if the following hold:

(i) f is an *involution*, i.e. $f(f(a)) = a$ for each $a \in L$,

(ii) f is a monoid morphism from $(A, +, 0)$ to $(A, \cdot, 1)$, i.e. $f(0) = 1$ and $f(a + b) = f(a) \cdot f(b)$ for any $a, b \in A$.

Note that then also $f(1) = 0$ and $f(a \cdot b) = f(a) + f(b)$ for any $a, b \in A$, and f is a monoid isomorphism from $(A, +, 0)$ to $(A, \cdot, 1)$ and from $(A, \cdot, 1)$ to $(A, +, 0)$.

Clearly, any De Morgan algebra $(L, \leq, ^-)$ constitutes a semiring with complement function, letting again addition be the supremum and multiplication be the infimum operation in L. Next we show the converse:

Proposition 6.1. Let $(A, +, \cdot, 0, 1)$ be a commutative semiring with complement function f. For any $a, b \in A$, put

$$a \leq b \quad \text{iff} \quad a + b = b$$

Then $(A, \leq, f)$ is a De Morgan algebra.

Proof:
We have $f(0) = 1$, f is an involution, $f(a+b) = f(a) \cdot f(b)$ and $f(a \cdot b) = f(a) + f(b)$. Hence, $0 \cdot 0 = 0$ implies $1 + 1 = 1$, so $(A, +, 0)$ and hence also $(A, \cdot, 1)$ are idempotent. Thus, $\leq$ is a partial order on A (cf. Proposition 20.19 in [31]) and $a + b$ is the supremum of a and b in this partial order. Also, $0 \leq a$ for any $a \in A$, and $a \cdot 0 = 0$ implies $f(a) + 1 = 1$, so $f(a) \leq 1$, showing also $a \leq 1$ for any $a \in A$.

Next note that if $a \leq b$, then by distributivity we obtain $a \cdot c \leq b \cdot c$, for any $a, b, c \in A$. We claim that $a \cdot b$ is the infimum of a and b in $(A, \leq)$, for any $a, b \in A$. Since, $a \leq 1$, the previous remark implies $a \cdot b \leq b$ and similarly $a \cdot b \leq a$. Now if $c \in A$ with $c \leq a$ and $c \leq b$, then $c = c \cdot c \leq a \cdot c \leq a \cdot b$, proving our claim. Hence $(A, \leq)$ is a distributive lattice with $+$ being the operation supremum and $\cdot$ being the infimum. Moreover, $(A, \leq)$ is bounded, and f is a complement mapping satisfying De Morgan laws. Thus, our proof is completed. □

This result shows that semirings with complement functions and De Morgan algebras provide actually the same class of structures.

7. Conclusion

We considered a multi-valued MSO logic over bounded distributive lattices and we proved a multi-valued generalization of Büchi's fundamental theorem for infinite words over bounded distributive lattices with any negation function. We showed that our methods can be extended to a multi-valued logic over infinite trees, proving a generalization of Rabin's theorem. Finally, we dealt with De Morgan algebras showing that they coincide with semirings with complement mapping. In case of De Morgan algebras our logic has a simpler syntax due to the De Morgan laws. If we consider the special case of the *fuzzy semiring* $([0,1], \vee, \wedge, 0, 1)$ (which is a De Morgan algebra with complement $\overline{a} = 1 - a$ for any $a \in [0,1]$), then we obtain a *fuzzy MSO logic*. In this case the fundamental theorem of Büchi states the expressive equivalence of ω*-recognizable fuzzy languages* (cf. [35, 43, 47]) with *fuzzy MSO-definable series*. Similar results are obtained for *fuzzy Muller recognizable tree languages* over infinite trees.

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