On the Expressiveness of TPTL and MTL

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Abstract. TPTL and MTL are two classical timed extensions of LTL. In this paper, we positively answer a 15-year-old conjecture that TPTL is strictly more expressive than MTL. But we show that, surprisingly, the TPTL formula proposed in [4] for witnessing this conjecture can be expressed in MTL. More generally, we show that TPTL formulae using only the **F** modality can be translated into MTL.

1 Introduction

Temporal logics. Temporal logics [19] are a widely used framework in the field of specification and verification of (models of) reactive systems. In particular, Linear-time Temporal Logic (LTL) allows to express properties about the executions of a model, such as the fact that *any occurrence of a problem eventually raises the alarm.* LTL has been extensively studied, both about its expressiveness [14, 11] and for model checking purposes [21, 23].

Timed temporal logics. At the beginning of the 90s, real-time constraints have naturally been added to temporal logics [15, 2], in order to add quantitative constraints to temporal logic specifications of timed models. The resulting logics allow to express, *e.g.*, that any occurrence of a problem in a system will raise the alarm *in at most 5 time units.*

When dealing with dense time, we may consider two different semantics for timed temporal logics, depending on whether the formulae are evaluated over *timed words* (*i.e.* over a discrete sequence of timed events; this is the *pointwise semantics*) or over *timed state sequences* (*i.e.*, roughly, over the continuous behavior of the system; this is the *interval-based semantics*). We refer to [6, 12] for a survey on linear-time timed temporal logics and to [20] for more recent developments on that subject.

Expressiveness of TPTL *and* MTL*.* Two interesting timed extensions of LTL are MTL (Metric Temporal Logic) [15, 7] and TPTL (Timed Propositional Temporal Logic) [8]. MTL extends LTL by adding subscripts to temporal operators: for instance, the above property can be written in MTL as

$$\mathbf{G}\,(\texttt{problem} \Rightarrow \mathbf{F}_{\leq 5}\,\texttt{alarm}).$$

TPTL is “more temporal” [8] in the sense that it uses real clocks in order to assert timed constraints. A TPTL formula can “reset” a formula clock at some point, and later compare the value of that clock to some integer. The property above would then be written as

$$\mathbf{G}\,(\texttt{problem} \Rightarrow x.\mathbf{F}\,(\texttt{alarm} \wedge x \leq 5))$$

where "$x.\varphi$" means that x is reset at the current position, before evaluating φ. This logic also allows to easily express that, for instance, within 5 t.u. after any problem, the system rings the alarm and then enters a failsafe mode:

$$\mathbf{G}\,(\texttt{problem} \Rightarrow x.\mathbf{F}\,(\texttt{alarm} \wedge \mathbf{F}\,(\texttt{failsafe} \wedge x \leq 5))). \qquad (1)$$

While it is clear that any MTL formula can be translated into an equivalent TPTL one, [6, 7] state that there is no intuitive MTL equivalent to formula (1). It has thus been conjectured that TPTL would be strictly more expressive than MTL [6, 7, 12], formula (1) being proposed as a possible witness not being expressible in MTL.

Our contributions. We consider that problem for two standard semantics (viz. the *pointwise* and the *interval-based* semantics). We prove that

- the conjecture *does* hold for both semantics;
- for the pointwise semantics, formula (1) witnesses the expressiveness gap, *i.e.* it cannot be expressed in TPTL;
- for the interval-based semantics, formula (1) *can* be expressed in MTL, but we exhibit another TPTL formula (namely, $x.\mathbf{F}\,(a \wedge x \leq 1 \wedge \mathbf{G}\,(x \leq 1 \Rightarrow \neg b))$, stating that the last atomic proposition before time point 1 is an a) and prove that it cannot be expressed in MTL.

As side results, we get that MTL is strictly more expressive under the interval-based semantics than under the pointwise one, as recently and independently proved in [10], and that, for both semantics, MTL+Past and MITL+Past (where the past-time modality "Since" is used [3]) are strictly more expressive than MTL and MITL, resp. We also get that the branching-time logic TCTL with explicit clock [13] is strictly more expressive than TCTL with subscripts [2], which had been conjectured in [1, 24].

Finally, we prove that, under the interval-based semantics, the fragment of TPTL where only the $\mathbf{F}$ modality is allowed (we call it the *existential fragment* of TPTL) can be translated into MTL. This generalizes the fact that formula (1) can be expressed in MTL.

Related work. Over the last 15 years, many researches have focused on expressiveness questions for timed temporal logics (over both integer and real time). See [5, 7, 8, 3] for original works, and [12, 20] for a survey on that topic.

MTL and TPTL have also been studied for the purpose of verification. If the underlying time domain is discrete, then MTL and TPTL have decidable verification problems [7, 8]. When considering dense time, verification problems (satisfiability, model checking) become much harder: [3] proves that the satisfiability problem for MTL is undecidable when considering the interval-based semantics. This result of course carries on for TPTL. It has recently been proved that MTL model checking and satisfiability *are* decidable over finite words under the pointwise semantics [18], while it is still undecidable for TPTL [8]. Note that our expressiveness result concerning $\mathsf{TPTL}_{\mathbf{F}}$ yields an NP decision procedure for that fragment under the pointwise semantics (see Corollary 11).

MTL and TPTL have also been studied in the scope of monitoring and path model checking. [22] proposes an (exponential) monitoring algorithm for MTL under the pointwise semantics. [17] shows that, in the interval-based semantics, MTL formulae can be verified on lasso-shaped timed state sequences in polynomial time, while TPTL formulae require at least polynomial space.

Some proofs are omitted due to lack of space. They can be found in [9].

2 Timed Linear-Time Temporal Logics

Basic definitions. In the sequel, AP represents a non-empty, countable set of atomic propositions. Let $\mathbb{R}$ denote the set of reals, $\mathbb{R}^+$ the set of nonnegative reals, $\mathbb{Q}$ the set of rationals and $\mathbb{N}$ the set of nonnegative integers. An *interval* is a convex subset of $\mathbb{R}$. Two intervals I and I' are said to be *adjacent* when $I \cap I' = \emptyset$ and $I \cup I'$ is an interval. We denote by $\mathcal{I}_\mathbb{R}$ the set of intervals, and by $\mathcal{I}_\mathbb{Q}$ the set of intervals whose bounds are in $\mathbb{Q}$.

Given a finite set X of variables called *clocks*, a *clock valuation* over X is a mapping $\alpha\colon X \to \mathbb{R}^+$ which assigns to each clock a time value in $\mathbb{R}^+$.

Timed state sequences and timed words. A *timed state sequence* over AP is a pair $\kappa = (\overline{\sigma}, \overline{I})$ where $\overline{\sigma} = \sigma_1\sigma_2\ldots$ is an infinite sequence of elements of 2^{AP} and $\overline{I} = I_1 I_2 \ldots$ is an infinite sequence of intervals satisfying the following properties:

- *(adjacency)* the intervals I_i and I_{i+1} are adjacent for all $i \geq 1$, and
- *(progress)* every time value $t \in \mathbb{R}^+$ belongs to some interval I_i.

A timed state sequence can equivalently be seen as an infinite sequence of elements of $2^{\mathsf{AP}} \times \mathcal{I}_\mathbb{R}$.

A *time sequence* over $\mathbb{R}^+$ is an infinite non-decreasing sequence $\tau = \tau_0\tau_1\ldots$ of nonnegative reals satisfying the following properties:

- *(initialization)* $\tau_0 = 0$,
- *(monotonicity)* the sequence is nondecreasing: $\forall\ i \in \mathbb{N}\ \tau_{i+1} \geq \tau_i$,
- *(progress)* every time value $t \in \mathbb{R}^+$ is eventually reached: $\forall t \in \mathbb{R}.\exists i.\ \tau_i > t$.

A *timed word* over AP is a pair $\rho = (\sigma, \tau)$, where $\sigma = \sigma_0\sigma_1\ldots$ is an infinite word over AP and $\tau = \tau_0\tau_1\ldots$ a time sequence over $\mathbb{R}^+$. It can equivalently be seen as an infinite sequence of elements $(\sigma_0, \tau_0)(\sigma_1, \tau_1)\ldots$ of $(\mathsf{AP} \times \mathbb{R})$. We force timed words to satisfy $\tau_0 = 0$ in order to have a natural way to define initial satisfiability in the semantics of MTL. This involves no loss of generality since it can be obtained by adding a special action to the alphabet.

Note that a timed word can be seen as a timed state sequence: for example the timed word $(a, 0)(a, 1.1)(b, 2)\ldots$ corresponds to the timed state sequence $(\{a\}, [0, 0])(\emptyset,]0, 1.1[)(\{a\}, [1.1, 1.1])(\emptyset, [1.1, 2[)(\{b\}, [2, 2])\ldots$

2.1 Clock Temporal Logic (TPTL)

The logic TPTL [8, 20] is a timed extension of LTL [19] which uses extra variables (clocks) explicitly in the formulae. Below, we define the syntax and semantics of TPTL+Past. The logic TPTL is the fragment of TPTL+Past not using the operator $\mathbf{S}$.

Formulae of TPTL+Past are built from atomic propositions, boolean connectives, "until" and "since" operators, clock constraints and clock resets:

$$\mathsf{TPTL{+}Past} \ni \varphi ::= p \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \varphi_1 \,\mathbf{U}\, \varphi_2 \mid \varphi_1 \,\mathbf{S}\, \varphi_2 \mid x \sim c \mid x.\varphi$$

where $p \in \mathsf{AP}$ is an atomic proposition, x is a clock variable, $c \in \mathbb{Q}$ is a rational number and $\sim\, \in \{\leq, <, =, >, \geq\}$. There are two main semantics for TPTL, the *interval-based* semantics which interprets TPTL over timed state sequences, and the *pointwise* semantics, which interprets TPTL over timed words. This last semantics is less general as (as we will see below) formulae can only be interpreted at points in time when actions occur.

In the literature, these two semantics are used interchangeably, but results highly depend on the underlying semantics. For example, a recent result [18] states that MTL (a subset of TPTL, see below) is decidable under the pointwise semantics, whereas it is known to be undecidable for finite models under the interval-based semantics [3].

Interval-based semantics. In this semantics, models are time state sequences κ, and are evaluated at a date $t \in \mathbb{R}^+$ with a valuation $\alpha : X \to \mathbb{R}^+$ (where X is the set of clocks for formulae of TPTL+Past). The satisfaction relation (denoted with $(\kappa, t, \alpha) \models_{\mathtt{i}} \varphi$) is defined inductively as follows (we omit the standard semantics of boolean operators):

$$\begin{array}{rcl}
(\kappa, t, \alpha) \models_{\mathtt{i}} p & \text{iff} & p \in \kappa(t) \\
(\kappa, t, \alpha) \models_{\mathtt{i}} x \sim c & \text{iff} & t - \alpha(x) \sim c \\
(\kappa, t, \alpha) \models_{\mathtt{i}} x.\varphi & \text{iff} & (\kappa, t, \alpha[x \mapsto t]) \models_{\mathtt{i}} \varphi \\
(\kappa, t, \alpha) \models_{\mathtt{i}} \varphi_1 \,\mathbf{U}\, \varphi_2 & \text{iff} & \exists t' > t \text{ such that } (\kappa, t', \alpha) \models_{\mathtt{i}} \varphi_2 \\
 & & \quad \text{and } \forall t < t'' < t',\ (\kappa, t'', \alpha) \models_{\mathtt{i}} \varphi_1 \vee \varphi_2{}^{1} \\
(\kappa, t, \alpha) \models_{\mathtt{i}} \varphi_1 \,\mathbf{S}\, \varphi_2 & \text{iff} & \exists t' < t \text{ such that } (\kappa, t', \alpha) \models_{\mathtt{i}} \varphi_2 \\
 & & \quad \text{and } \forall t' < t'' < t,\ (\kappa, t'', \alpha) \models_{\mathtt{i}} \varphi_1 \vee \varphi_2
\end{array}$$

We write $\kappa \models_{\mathtt{i}} \varphi$ when $(\kappa, 0, \mathbf{0}) \models_{\mathtt{i}} \varphi$ where $\mathbf{0}$ is the valuation assigning 0 to all clocks. Following [20], we interpret "$x.\varphi$" as a reset operator. Note also that the semantics of $\mathbf{U}$ is strict in the sense that, in order to satisfy $\varphi_1 \,\mathbf{U}\, \varphi_2$, a time state sequence is not required to satisfy φ_1. In the following, we use classical shorthands: $\top$ stands for $p \vee \neg p$, $\varphi_1 \Rightarrow \varphi_2$ holds for $\neg\varphi_1 \vee \varphi_2$, $\mathbf{F}\,\varphi$ holds for $\top \,\mathbf{U}\, \varphi$ (and means that φ eventually holds at a future time), and $\mathbf{G}\,\varphi$ holds for $\neg(\mathbf{F}\,\neg\varphi)$ (and means that φ always holds in the future).

[1] Following [20] we use $\varphi_1 \vee \varphi_2$ to handle open intervals in timed models.

Pointwise semantics. In this semantics, models are timed words ρ, and satisfiability is no longer interpreted at a date $t \in \mathbb{R}$ but at a position $i \in \mathbb{N}$ in the timed word. For a timed word $\rho = (\sigma, \tau)$ with $\sigma = (\sigma_i)_{i \geq 0}$ and $\tau = (\tau_i)_{i \geq 0}$, we define the satisfaction relation $(\rho, i, \alpha) \models_{\mathtt{p}} \varphi$ inductively as follows (where α is a valuation for the set X of formula clocks):

$$\begin{array}{rcl}
(\rho, i, \alpha) \models_{\mathtt{p}} p & \text{iff} & \sigma_i = p \\
(\rho, i, \alpha) \models_{\mathtt{p}} x \sim c & \text{iff} & \tau_i - \alpha(x) \sim c \\
(\rho, i, \alpha) \models_{\mathtt{p}} x.\varphi & \text{iff} & (\rho, i, \alpha[x \mapsto \tau_i]) \models_{\mathtt{p}} \varphi \\
(\rho, i, \alpha) \models_{\mathtt{p}} \varphi_1 \mathbf{U} \varphi_2 & \text{iff} & \exists j > i \text{ s.t. } (\rho, j, \alpha) \models_{\mathtt{p}} \varphi_2 \\
& & \quad \text{and } \forall i < k < j \ (\rho, k, \alpha) \models_{\mathtt{p}} \varphi_1 \\
(\rho, i, \alpha) \models_{\mathtt{p}} \varphi_1 \mathbf{S} \varphi_2 & \text{iff} & \exists j < i \text{ s.t. } (\rho, j, \alpha) \models_{\mathtt{p}} \varphi_2 \\
& & \quad \text{and } \forall j < k < i \ (\rho, k, \alpha) \models_{\mathtt{p}} \varphi_1
\end{array}$$

We write $\rho \models_{\mathtt{p}} \varphi$ whenever $(\rho, 0, \mathbf{0}) \models_{\mathtt{p}} \varphi$.

Example 1. Consider the timed word $\rho = (a, 0)(a, 1.1)(b, 2)\ldots$ which, as already mentioned, can be viewed as the time state sequence

$$\kappa = (\{a\}, [0])(\emptyset, (0, 1.1))(\{a\}, [1.1, 1.1])(\emptyset, (1.1, 2))(\{b\}, [2, 2])\ldots$$

If $\varphi = x.\mathbf{F}\,(x = 1 \wedge y.\mathbf{F}\,(y = 1 \wedge b))$, then $\rho \not\models_{\mathtt{p}} \varphi$ whereas $\kappa \models_{\mathtt{i}} \varphi$. This is due to the fact that there is no action at date 1 along ρ.

2.2 Metric Temporal Logic (MTL)

The logic MTL [15, 7] extends the logic LTL with time restrictions on "until" modalities. Here again, we first define MTL+Past:

$$\mathsf{MTL{+}Past} \ni \varphi ::= p \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \varphi_1 \mathbf{U}_I \varphi_2 \mid \varphi_1 \mathbf{S}_I \varphi_2$$

where p ranges over the set AP of atomic propositions, and I an interval in $\mathcal{I}_{\mathbb{Q}}$. MTL is the fragment of MTL+Past not using the operator $\mathbf{S}$. We also define MITL and MITL+Past as the fragments of MTL and MTL+Past in which intervals cannot be singletons.

For defining the semantics of MTL+Past, we view MTL+Past as a fragment of TPTL+Past: $\varphi_1 \mathbf{U}_I \varphi_2$ is then interpreted as $x.(\varphi_1 \mathbf{U}\,(x \in I \wedge \varphi_2))$ and $\varphi_1 \mathbf{S}_I \varphi_2$ as $x.(\varphi_1 \mathbf{S}\,(x \in I \wedge \varphi_2))$. As for TPTL, we will thus consider both the interval-based (interpreted over time state sequences) and the pointwise (interpreted over timed words) semantics. For both semantics, it is clear that TPTL is at least as expressive as MTL, which in turn is at least as expressive as MITL.

We omit the constraint on modality $\mathbf{U}$ when $[0, \infty)$ is assumed. We write $\mathbf{U}_{\sim c}$ for $\mathbf{U}_I$ when $I = \{t \mid t \sim c\}$. As previously, we use classical shorthands such as $\mathbf{F}_I$ or $\mathbf{G}_I$.

Example 2. In MTL, the formula φ of Example 1 can be expressed as $\mathbf{F}_{=1}\,\mathbf{F}_{=1}\,b$. In the interval-based semantics, this formula is equivalent to $\mathbf{F}_{=2}\,b$, and this is **not** the case in the pointwise semantics.

3 TPTL is Strictly More Expressive Than MTL

3.1 Conjecture

It has been conjectured in [6, 7, 12] that TPTL is strictly more expressive than MTL, and in particular that a TPTL formula such as

$$\mathbf{G}\,(a \Rightarrow x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge\ x \leq 2)))$$

can not be expressed in MTL. The following proposition states that this formula is not a witness for proving that TPTL is strictly more expressive than MTL.

Proposition 1. *The* TPTL *formula* $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge\ x \leq 2))$ *can be expressed in* MTL *for the interval-based semantics.*

Proof. Let Φ be the TPTL formula $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge\ x \leq 2))$. This formula expresses that, along the time state sequence, from the current point on, there is a b followed by a c, and the delay before that c is less than 2 t.u. For proving the proposition, we write an MTL formula Φ' which is equivalent to Φ over time state sequences. Formula Φ' is defined as the disjunction of three formulae $\Phi' = \Phi'_1 \vee \Phi'_2 \vee \Phi'_3$ where:

$$\begin{cases} \Phi'_1 = \mathbf{F}_{\leq 1}\, b \wedge \mathbf{F}_{[1,2]}\, c \\ \Phi'_2 = \mathbf{F}_{\leq 1}\,(b \wedge \mathbf{F}_{\leq 1}\, c) \\ \Phi'_3 = \mathbf{F}_{\leq 1}\,(\mathbf{F}_{\leq 1}\, b \wedge \mathbf{F}_{=1}\, c) \end{cases}$$

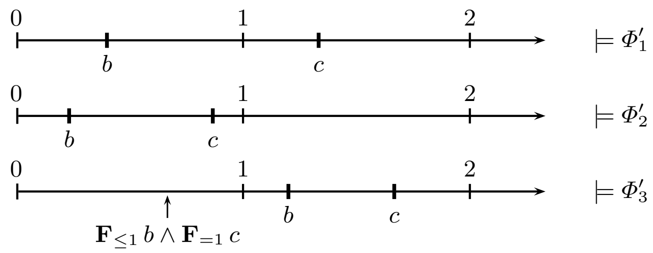


Fig. 1. Translation of TPTL formula Φ in MTL

Let κ be a time state sequence. If $\kappa \models_{\mathtt{i}} \Phi'$, it is obvious that $\kappa \models_{\mathtt{i}} \Phi$. Suppose now that $\kappa \models_{\mathtt{i}} \Phi$, then there exists $0 < t_1 < t_2 \leq 2$ such that[2] $(\kappa, t_1) \models_{\mathtt{i}} b$ and $(\kappa, t_2) \models_{\mathtt{i}} c$. If $t_1 \leq 1$ then κ satisfies Φ'_1 or Φ'_2 (or both) depending on t_2 being smaller or greater than 1. If $t_1 \in (1, 2]$ then there exists a date t' in $(0, 1]$ such that $(\kappa, t') \models_{\mathtt{i}} \mathbf{F}_{\leq 1}\, b \wedge \mathbf{F}_{=1}\, c$ which implies that $\kappa \models_{\mathtt{i}} \Phi'_3$. We illustrate the three possible cases on Fig. 1. □

[2] Here we abstract away the value for clock x as it corresponds to the date.

From the proposition above we get that the TPTL formula $\mathbf{G}\,(a \Rightarrow \Phi)$ is equivalent over time state sequences to the MTL formula $\mathbf{G}\,(a \Rightarrow \Phi')$. This does not imply that the conjecture is wrong, and we will now prove two results:

- $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge \ x \leq 2))$ can not be expressed in MTL for the pointwise semantics (thus over timed words)
- the more involved TPTL formula $x.\mathbf{F}\,(a \wedge x \leq 1 \wedge \mathbf{G}\,(x \leq 1 \Rightarrow \neg b))$ can not be expressed in MTL for the interval-based semantics.

This implies that TPTL is indeed strictly more expressive than MTL for both pointwise and interval-based semantics, which positively answers the conjecture of [6, 7, 12].

3.2 Pointwise Semantics

We now show that the formula $\Phi = x.(\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 2)))$ cannot be expressed in MTL for the pointwise semantics. This gives another proof of the strict containment of MTL with pointwise semantics in MTL with interval-based semantics [10].

We note $\mathsf{MTL}_{p,n}$ for the set of MTL formulae whose constants are multiple of p and whose temporal height (maximum number of nested modalities) is at most n. We construct two families of timed words $(\mathcal{A}_{p,n})_{p\in\mathbb{Q},n\in\mathbb{N}}$ and $(\mathcal{B}_{p,n})_{p\in\mathbb{Q},n\in\mathbb{N}}$ such that:

- $\mathcal{A}_{p,n} \models_{\mathrm{p}} \Phi$ whereas $\mathcal{B}_{p,n} \not\models_{\mathrm{p}} \Phi$ for every $p \in \mathbb{Q}$ and $n \in \mathbb{N}$,
- for all $\varphi \in \mathsf{MTL}_{p,n-3}$, $\mathcal{A}_{p,n} \models_{\mathrm{p}} \varphi \iff \mathcal{B}_{p,n} \models_{\mathrm{p}} \varphi$.

The two families of models are presented in Fig. 2. Note that there is no action between dates 0 and $2-p$. It is obvious that $\mathcal{A}_{p,n} \models_{\mathrm{p}} \Phi$ whereas $\mathcal{B}_{p,n} \not\models_{\mathrm{p}} \Phi$.

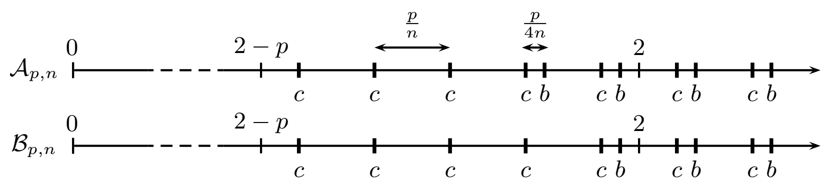


Fig. 2. Models $\mathcal{A}_{p,n}$ and $\mathcal{B}_{p,n}$

We now give a sketch of the expressiveness proof:

- we first prove that given any integer n, models $\mathcal{A}_{p,n+3}$ and $\mathcal{B}_{p,n+3}$ can not be distinguished by $\mathsf{MTL}_{p,n}$ formulae after date $2-p$. This result holds both in the pointwise and the interval-based semantics.
- we then use the fact that there are no actions between 0 and $2-p$ in the models; in the pointwise semantics, a formula cannot point a date before $2-p$. This enables us to prove that the two models $\mathcal{A}_{p,n+3}$ and $\mathcal{B}_{p,n+3}$ can not be initially distinguished by any $\mathsf{MTL}_{p,n}$ formula in the pointwise semantics. This result does not hold in the interval-based semantics.

– assume Φ has an MTL equivalent Ψ. We define the granularity P as follows: $P = \prod_{a/b \in \Psi} 1/b$. Let N be its temporal height. Then the models $\mathcal{A}_{P,N+3}$ and $\mathcal{B}_{P,N+3}$ cannot be distinguished by Ψ, according to the above result, which contradicts that Ψ is equivalent to Φ.

Theorem 2. TPTL *is strictly more expressive than* MTL *under the pointwise semantics.*

Since the MITL+Past formula $\mathbf{F}_{\leq 2}\,(c \wedge \top\,\mathbf{S}\,b)$ also distinguishes between the families of models $(\mathcal{A}_{p,n})_{p\in\mathbb{Q},n\in\mathbb{N}}$ and $(\mathcal{B}_{p,n})_{p\in\mathbb{Q},n\in\mathbb{N}}$, we get the corollary:

Corollary 3. MTL+Past *(resp.* MITL+Past*) is strictly more expressive than* MTL *(resp.* MITL*) for the pointwise semantics.*

Note that the above result is a main difference between the timed and the untimed framework where it is well-known that past does not add any expressiveness to LTL [14, 11]. This had already been proved in [6] for MITL.

3.3 Interval-Based Semantics

As we have seen, the formula which has been used for the pointwise semantics can not be used for the interval-based semantics. We will instead prove the following proposition:

Proposition 4. *The* TPTL *formula* $\Phi = x.\mathbf{F}\,(a \wedge x \leq 1 \wedge \mathbf{G}\,(x \leq 1 \Rightarrow \neg b))$ *has no equivalent* MTL *formula over time state sequences.*

Proof. Assume some formula $\Psi \in$ MTL is equivalent to Φ over time state sequences. Let P be its granularity. W.l.o.g., we may assume that Ψ only uses constraints of the form $\sim P$, with $\sim \in \{<, =, >\}$. Let N be the temporal height of this formula. We write $\mathsf{MTL}^{-}_{p,n}$ for the fragment of MTL using only $\sim p$ constraints, and with temporal height at most n. Thus $\Psi \in \mathsf{MTL}^{-}_{P,N}$.

Now, we build two different families of time state sequences $\mathcal{A}_{p,n}$ and $\mathcal{B}_{p,n}$, such that Φ holds initially in the first one but not in the second one. We will then prove that they cannot be distinguished by any formula in $\mathsf{MTL}^{-}_{p,n-3}$.

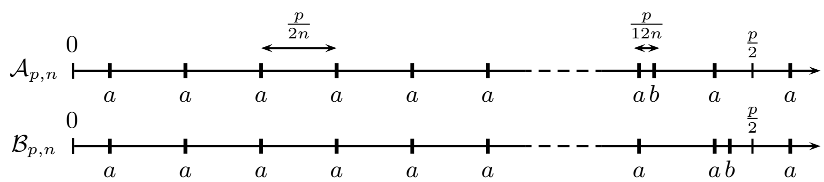


Fig. 3. Two timed paths $\mathcal{A}_{p,n}$ and $\mathcal{B}_{p,n}$

Let us first define $\mathcal{A}_{p,n}$. Along that time state sequence, atomic proposition a will be set to true exactly at time points $\frac{p}{4n} + \alpha\frac{p}{2n}$, where α may be any nonnegative integer. Atomic proposition b will hold exactly at times $(\alpha+1)\cdot\frac{p}{2} - \frac{4p}{6n}$, with $\alpha \in \mathbb{N}$. As for $\mathcal{B}_{p,n}$, it has exactly the same a's, and b holds exactly at time points $(\alpha+1)\cdot\frac{p}{2}-\frac{p}{6n}$, with $\alpha \in \mathbb{N}$. The portion between 0 and $\frac{p}{2}$ of both time state sequences is represented on Fig. 3. Both time state sequences are in fact periodic, with period $\frac{p}{2}$. The following lemma is straightforward since, for each equivalence, the suffixes of the paths are the same.

Lemma 5. *For any positive p and n, for any nonnegative real x, and for any* MTL *formula φ,*

$$\mathcal{A}_{p,n}, x \models_{\mathtt{i}} \varphi \qquad \Longleftrightarrow \qquad \mathcal{B}_{p,n}, x+\frac{p}{2n} \models_{\mathtt{i}} \varphi \tag{2}$$

$$\mathcal{A}_{p,n}, x \models_{\mathtt{i}} \varphi \qquad \Longleftrightarrow \qquad \mathcal{A}_{p,n}, x+\frac{p}{2} \models_{\mathtt{i}} \varphi \tag{3}$$

$$\mathcal{B}_{p,n}, x \models_{\mathtt{i}} \varphi \qquad \Longleftrightarrow \qquad \mathcal{B}_{p,n}, x+\frac{p}{2} \models_{\mathtt{i}} \varphi \tag{4}$$

We can then prove (by induction, see [9]) the following lemma:

Lemma 6. *For any $k \leq n$, for any $p \in \mathbb{Q}^+$, for any $\varphi \in \mathsf{MTL}^-_{p,k}$, for any $x \in \left[0, \frac{p}{2}-\frac{(k+2)p}{2(n+3)}\right)$, for any nonnegative integer α, we have*

$$\mathcal{A}_{p,n+3}, \alpha\frac{p}{2}+x \models \varphi \qquad \Longleftrightarrow \qquad \mathcal{B}_{p,n+3}, \alpha\frac{p}{2}+x \models \varphi$$

As a corollary of the lemma, when $n = N = k$, $p = P$ and $\alpha = x = 0$, we get that any formula in $\mathsf{MTL}^-_{P,N}$ cannot distinguish between models $\mathcal{A}_{P,N+3}$ and $\mathcal{B}_{P,N+3}$. This is in contradiction with the fact that Ψ is equivalent to Φ, since Ψ holds initially along $\mathcal{A}_{P,N+3}$ but fails to hold initially along $\mathcal{B}_{P,N+3}$. □

We can now state our main theorem:

Theorem 7. TPTL *is strictly more expressive than* MTL *for the interval-based semantics.*

As a side result we get that TPTL under the pointwise semantics is strictly more expressive than MTL under the interval-based semantics (assuming that the latter is restricted to timed words). Also note that the formula Φ does not use the $\mathbf{U}$ modality. However, it needs both $\mathbf{F}$ and $\mathbf{G}$, as the fragment of TPTL using only the $\mathbf{F}$ modality can be translated into MTL (see Section 4).

Since the MTL+Past formula $\mathbf{F}_{=1}(\neg b\,\mathbf{S}\,a)$ distinguishes between the two families of models $(\mathcal{A}_{p,n})_{p\in\mathbb{Q},n\in\mathbb{N}}$ and $(\mathcal{B}_{p,n})_{p\in\mathbb{Q},n\in\mathbb{N}}$, we get the following corollary:

Corollary 8. MTL+Past *is strictly more expressive than* MTL *for the interval-based semantics.*

The more involved MITL+Past formula[3] $\mathbf{F}_{\geq 1}(\neg a \wedge \mathbf{F}^{-1}_{\geq -1}(\mathbf{G}^{-1}\neg a) \wedge \neg b\,\mathbf{S}\,a)$ also distinguishes between the two families, so that we also get:

Corollary 9. MITL+Past *is strictly more expressive than* MITL *for the interval-based semantics.*

To our knowledge, these are the first expressiveness result for timed linear-time temporal logics using past modalities under the interval-based semantics.

[3] Note that $\mathbf{F}^{-1}_{\geq -1}\varphi$ holds when φ held at some point in the last time unit.

4 On the Existential Fragments of MTL and TPTL

$\mathsf{TPTL}_{\mathbf{F}}$ is the fragment of TPTL which only uses the $\mathbf{F}$ modality and which does not use the general negation but only negation of atomic propositions. Formally, $\mathsf{TPTL}_{\mathbf{F}}$ is defined by the following grammar:

$$\mathsf{TPTL}_{\mathbf{F}} \ni \varphi ::= p \mid \neg p \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mathbf{F}\,\varphi \mid x \sim c \mid x.\varphi.$$

An example of a $\mathsf{TPTL}_{\mathbf{F}}$ formula is $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 2))$ (see Subsection 3.1). Similarly we define the fragment $\mathsf{MTL}_{\mathbf{F}}$ of MTL where only $\mathbf{F}$ modalities are allowed:

$$\mathsf{MTL}_{\mathbf{F}} \ni \varphi ::= p \mid \neg p \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mathbf{F}_I\,\varphi.$$

From Subsection 3.2, we know that, under the pointwise semantics, $\mathsf{TPTL}_{\mathbf{F}}$ is strictly more expressive than $\mathsf{MTL}_{\mathbf{F}}$, since formula $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 2))$ has no equivalent in MTL (thus in $\mathsf{MTL}_{\mathbf{F}}$). On the contrary, when considering the interval-based semantics, we proved that this $\mathsf{TPTL}_{\mathbf{F}}$ formula can be expressed in $\mathsf{MTL}_{\mathbf{F}}$ (see Subsection 3.1). In this section, we generalize the construction of Subsection 3.1, and prove that $\mathsf{TPTL}_{\mathbf{F}}$ and $\mathsf{MTL}_{\mathbf{F}}$ are in fact equally expressive for the interval-based semantics.

Theorem 10. $\mathsf{TPTL}_{\mathbf{F}}$ *is as expressive as* $\mathsf{MTL}_{\mathbf{F}}$ *for the interval-based semantics.*

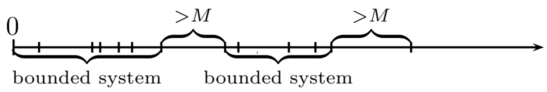
Sketch of proof. From a $\mathsf{TPTL}_{\mathbf{F}}$ formula φ, we construct a *system of difference inequations* $\mathcal{S}_\varphi$ which recognizes the same models. Such a system has a finite number of free variables corresponding to dates, these dates are constrained by difference inequations, and propositional variables must be satisfied at some dates. Here is an example of the system constructed for the formula of section 3.1.

Example 3. For the formula $x.\mathbf{F}\,(a \wedge \mathbf{F}\,(b \wedge x \leq 2))$, we obtain:

$$\mathcal{S} = \begin{cases} V\colon y_1 \mapsto a,\ y_2 \mapsto b \\ \mathcal{J} = \{y_2 \leq 2, y_2 > y_1, y_1 > 0\} \end{cases}$$

We explain now how to construct a MTL formula for such systems:

- if all variables of the system are sorted ($r_1 < y_1 < \cdots < y_p < r_2$ with $r_1, r_2 \in \mathbb{Q}$), we generalize the technique used in proposition 1 to construct a corresponding MTL formula.
- if all variables are bounded in the system, it can be obtained by an union of previous systems using a region construction.
- a general system can be decomposed in bounded systems as follows:



Each point on the line represents a variable, and a part denoted by "bounded system" gathers variables whose differences are bounded. Two variables in different bounded systems are separated by at least M t.u.

Note that this construction from $\mathsf{TPTL}_{\mathbf{F}}$ to $\mathsf{MTL}_{\mathbf{F}}$ is exponential due to the ordering of variables and the region construction.

It is known [3] that the satisfiability problem for TPTL and MTL is undecidable for the interval-based semantics, whereas it has been proved recently that the satisfiability problem for MTL is decidable but non primitive recursive for the pointwise semantics [18]. As a corollary of the previous proof, we get:

Corollary 11. *The satisfiability problem for* $\mathsf{TPTL}_{\mathbf{F}}$ *(and thus* $\mathsf{MTL}_{\mathbf{F}}$*) is NP-complete for the interval-based semantics.*

5 Conclusion

In this paper we have proved the conjecture (first proposed in [4]) that the logic TPTL is strictly more expressive than MTL. However we have also proved that the TPTL formula $\mathbf{G}\,(a \rightarrow x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 1)))$, which had been proposed as an example of formula which could not be expressed in MTL, has indeed an equivalent formula in MTL for the interval-based semantics. We have thus proposed another formula of TPTL which can not be expressed in MTL. We have also proved that the fragment of TPTL which only uses the **F** modality can be translated in MTL.

As side results, we have obtained that MTL+Past and MITL+Past are strictly more expressive than MTL and MITL, resp., which is a main difference with the untimed framework where past modalities do not add any expressive power to LTL [14, 11].

Linear models we have used for proving above expressiveness results can be viewed as special cases of branching-time models. Our results thus apply to the branching-time logic TCTL (by replacing the modality **U** with the modality **AU**), and translate as: TCTL with explicit clocks [13] is strictly more expressive than TCTL with subscripts [2], as conjectured in [1, 24].

As further developments, we would like to study automata formalisms equivalent to both logics TPTL and MTL. Several existing works may appear as interesting starting points, namely [6, 16, 18, 10].

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