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Computing Equilibria in Two-Player Timed Games *via* Turn-Based Finite Games*

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Abstract. We study two-player timed games where the objectives of the two players are not opposite. We focus on the standard notion of Nash equilibrium and propose a series of transformations that builds two finite turn-based games out of a timed game, with a precise correspondence between Nash equilibria in the original and in final games. This provides us with an algorithm to compute Nash equilibria in two-player timed games for large classes of properties.

1 Introduction

Timed games. Game theory (especially games played on graphs) has been used in computer science as a powerful framework for modelling interactions in embedded systems [15, 12]. Over the last fifteen years, games have been extended with the ability to depend on timing informations, taking advantage of the large development of timed automata [1]. Adding timing constraints allows for a more faithful representation of reactive systems, while preserving decidability of several important properties, such as the existence of a winning strategy for one of the agents to achieve her goal, whatever the other agents do [3]. Efficient algorithms exist and have been implemented, *e.g.* in the tool Uppaal-Tiga [4].

Zero sum vs. non-zero sum games. In this purely antagonist view, games can be seen as two-player games, where one agent plays against another one. Moreover, the objectives of those two agents are opposite: the aim of the second player is simply to prevent the first player from winning her own objective. More generally, a (positive or negative) payoff can be associated with each outcome of the game, which can be seen as the amount the second player will have to pay to the first player. Those games are said to be zero-sum.

In many cases, however, games can be non-zero-sum: the objectives of the two players are then no more complementary, and the aim of one player is no more to prevent the other player from winning. Such games appear *e.g.* in various problems in telecommunications, where the agents try to send data on a network [11]. Focusing only on surely-winning strategies in this setting may then be too narrow: surely-winning strategies must be winning against any behaviour of the other player, and do not consider the fact that the other player also tries to achieve her own objective.

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Nash equilibria. In the non-zero-sum game setting, it is then more interesting to look for equilibria. One of the most-famous and most-studied notion of equilibrium is that proposed by Nash in 1950 [13]: a *Nash equilibrium* is a behaviour of the players in which they act rationally, in the sense that no player can get a better payoff if she, alone, modifies her strategy [13]. Notice that a Nash equilibrium needs not exist in general, and may not be optimal, in the sense that several equilibria can coexist, and may have very different payoffs.

Our contribution. We extend the standard notion of Nash equilibria to timed games, where non-determinism naturally arises and has to be taken into account. We propose a whole chain of transformations that builds, given a two-player timed game, two turn-based finite games which, in some sense that we will make precise, preserve Nash equilibria. The first transformation consists in building a finite concurrent game with non-determinism based on the classical region abstraction; the second transformation decouples this concurrent game into two concurrent games, one per player: in each game, the preference relation of one of the players is simply dropped, but we have to consider "joint" equilibria. The last two transformations work on each of the two copies of the concurrent game: the first one solves the non-determinism by giving an advantage to the associated player, and the last one makes use of this advantage to build a turn-based game equivalent to the original concurrent game. This chain of transformations is valid for the whole class of two-player timed games, and Nash equilibria are preserved for a large class of objectives, for instance ω-regular objectives[1]. These transformations allow to recover some known results about zero-sum games, but also to get new decidability results for Nash equilibria in two-player timed games.

Related work. Nash equilibria (and other related solution concepts such as *subgame-perfect equilibria*, *secure equilibria*, ...) have recently been studied in the setting of (untimed) games played on a graph [7–9, 14, 16–19]. None of them, however, focuses on timed games. In the setting of concurrent games, mixed strategies (*i.e.*, strategies involving probabilistic choices) are arguably more relevant than pure (*i.e.*, non-randomized) strategies. However, adding probabilities to timed strategies involves several important technical issues (even in zero-sum non-probabilistic timed games), and we defer the study of mixed-strategy Nash equilibria in two-player timed games to future works.

For lack of space, proofs are omitted and can be found in the technical appendix.

2 Preliminaries

2.1 Timed games

A *valuation* over a finite set of clocks Cl is a mapping $v\colon \mathsf{Cl} \to \mathbb{R}_+$. If v is a valuation and $t \in \mathbb{R}_+$, then $v + t$ is the valuation that assigns to each $x \in \mathsf{Cl}$ the value $v(x)+t$. If v is a valuation and $Y \subseteq \mathsf{Cl}$, then $[Y \leftarrow 0]v$ is the valuation that assigns 0 to each $y \in Y$ and $v(x)$ to each $x \in \mathsf{Cl}\backslash Y$. A *clock constraint* over Cl is a formula built on the grammar: $\mathfrak{C}(\mathsf{Cl}) \ni g \ ::= \ x \sim c \ \mid \ g \wedge g$, where x ranges over Cl, $\sim \in \{<, \leq, =, \geq, >\}$, and c is an integer. The semantics of clock constraints over valuations is natural, and we omit it.

[1] In the general case, the undecidability results on (zero-sum) priced timed games entail undecidability of the existence of Nash equilibria.

Definition 1. *A* timed automaton *is a tuple* $\langle \textsf{Loc}, \textsf{Cl}, \textsf{Inv}, \textsf{Trans}\rangle$ *such that:*

- $\textsf{Loc}$ *is a finite set of locations;*
- $\textsf{Cl}$ *is a finite set of clocks;*
- $\textsf{Inv}\colon \textsf{Loc} \to \mathfrak{C}(\textsf{Cl})$ *assigns an* invariant *to each location;*
- $\textsf{Trans} \subseteq \textsf{Loc} \times \mathfrak{C}(\textsf{Cl}) \times 2^{\textsf{Cl}} \times \textsf{Loc}$ *is the set of transitions.*

We assume the reader is familiar with timed automata [1], and in particular with states (pairs $(\ell, v) \in \textsf{Loc} \times \mathbb{R}_+^X$ such that $v \models \textsf{Inv}(\ell)$), runs (seen as infinite sequences of states for our purpose), *etc*. We now define the notion of two-player timed games. The two players will be called player 1 and player 2. Our definition follows that of [10].

Definition 2. *A* (two-player) timed game *is a tuple* $\mathcal{G} = \langle \textsf{Loc}, \textsf{Cl}, \textsf{Inv}, \textsf{Trans}, \textsf{Owner}, (\preccurlyeq_1, \preccurlyeq_2)\rangle$ *where:*

- $\langle \textsf{Loc}, \textsf{Cl}, \textsf{Inv}, \textsf{Trans}\rangle$ *is a timed automaton;*
- $\textsf{Owner}\colon \textsf{Trans} \to \{1, 2\}$ *assigns a player to each transition;*
- *for each* $i \in \{1, 2\}$, $\preccurlyeq_i \subseteq \left(\textsf{Loc} \times \mathbb{R}_+^{\textsf{Cl}}\right)^\omega \times \left(\textsf{Loc} \times \mathbb{R}_+^{\textsf{Cl}}\right)^\omega$ *is a quasi-order on runs of the timed automaton, called the* preference relation *for player* i*.*

A timed game is played as follows: from each state of the underlying timed automaton (starting from an initial state $s_0 = (\ell, \mathbf{0})$, where $\mathbf{0}$ maps each clock to zero), each player chooses a nonnegative real number d and a transition δ, with the intended meaning that she wants to delay for d time units and then fire transition δ. There are several (natural) restrictions on these choices:

- spending d time units in ℓ must be allowed[2] *i.e.*, $v + d \models \textsf{Inv}(\ell)$;
- $\delta = (\ell, g, z, \ell')$ belongs to the current player (given by function $\textsf{Owner}$);
- the transition is firable after d time units (*i.e.*, $v + d \models g$), and the invariant is satisfied when entering ℓ' (*i.e.*, $[z \leftarrow 0](v + d) \models \textsf{Inv}(\ell')$).

When there is no such possible choice for a player (for instance if there is no transition from ℓ belonging to that player), she chooses a special move, denoted by $\perp$.

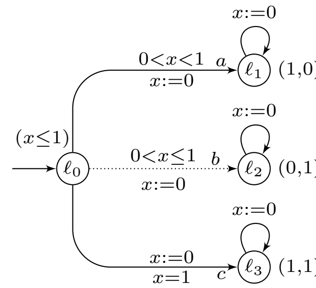
From a state (ℓ, v) and given a choice (m_1, m_2) for the two players, with $m_i \in (\mathbb{R}_+ \times \textsf{Trans}) \cup \{\perp\}$, an index i_0 such that $d_{i_0} = \min\{d_i \mid m_i = (d_i, \delta_i) \text{ and } i \in \{1, 2\}\}$ is selected (non-deterministically if both delays are identical), and the corresponding transition $\delta_{i_0} = (\ell, g, z, \ell')$ is applied, leading to a new state $(\ell', [z \leftarrow 0](v + d_{i_0}))$. To ensure well-definedness of the above semantics we assume in the sequel that timed games are *non-blocking*, that is, for any reachable state (ℓ, v), at least one player has an allowed transition (this avoids that both players play the special action $\perp$).

The outcome of such a game when players have fixed their various choices is a run of the underlying timed automaton, that is an element of $\left(\textsf{Loc} \times \mathbb{R}_+^{\textsf{Cl}}\right)^\omega$, and possible outcomes are compared by each player using their preference relations. In the examples, we will define the preference relation of a player by assigning a value (called a *payoff*) to each possible outcome of the game, and the higher the payoff, the better the run in the preference relation.

[2] Formally, this should be written $v + d' \models \textsf{Inv}(\ell)$ for all $0 \leq d' \leq d$, but this is equivalent to having only $v \models \textsf{Inv}(\ell)$ and $v + d \models \textsf{Inv}(\ell)$ since invariants are convex.

This semantics can naturally be formalized in terms of an infinite-state non-deterministic concurrent game and strategies, that we will detail in the next section.

Example 1. We give an example of a timed game, that we will use as a running example: consider the timed game $\mathcal{G}$ on the right. When relevant the name of a transition is printed on the corresponding edge. Owners of the transitions are specified as follows: player 1 plays with plain edges, whereas player 2 plays with dotted edges. On the right of these locations we indicate payoffs for the two players (if a play ends up in ℓ_1, player 1 gets payoff 1, whereas player 2 gets payoff 0). Hence player 1 will prefer runs ending in ℓ_1 or ℓ_3 than runs ending in ℓ_2.



2.2 Concurrent games

In this section we define two-player concurrent games, which we then use to encode the formal semantics of timed games. A *transition system* is a 2-tuple $\mathcal{S} = \langle \mathsf{States}, \mathsf{Edg} \rangle$ where States is a (possibly uncountable) set of states, and $\mathsf{Edg} \subseteq \mathsf{States} \times \mathsf{States}$ is the set of transitions. A *path* π in $\mathcal{S}$ is a non-empty sequence $(s_i)_{0 \le i < n}$ (where $n \in \mathbb{N} \cup \{+\infty\}$) of states of $\mathcal{S}$ such that $(s_i, s_{i+1}) \in \mathsf{Edg}$ for all $i < n-1$. The *length* of π, denoted by $|\pi|$ is $n-1$. The set of finite paths (also called *histories* in the sequel) of $\mathcal{S}$ is denoted by[3] $\mathsf{Hist}_{\mathcal{S}}$, the set of infinite paths (also called *plays*) of $\mathcal{S}$ is denoted by $\mathsf{Play}_{\mathcal{S}}$, and $\mathsf{Path}_{\mathcal{S}} = \mathsf{Hist}_{\mathcal{S}} \cup \mathsf{Play}_{\mathcal{S}}$ is the set of paths of $\mathcal{S}$. Given a path $\pi = (s_i)_{0 \le i < n}$ and an integer $j \le |\pi|$, the *j-th prefix* of π, denoted by $\pi_{\le j}$, is the finite path $(s_i)_{0 \le i < j+1}$. If $\pi = (s_i)_{0 \le i < n}$ is a history, we write $\mathsf{last}(\pi) = s_{|\pi|}$.

We extend the definition of concurrent games given *e.g.* in [2] with non-determinism:

Definition 3. *A* (two-player non-deterministic) concurrent game *is a tuple* $\mathcal{G} = \langle \mathit{States}, \mathit{Edg}, \mathit{Act}, \mathit{Mov}, \mathit{Tab}, (\preccurlyeq_1, \preccurlyeq_2) \rangle$ *in which:*

- $\langle \mathit{States}, \mathit{Edg} \rangle$ *is a transition system;*
- Act *is a (possibly uncountable) set of actions;*
- $\mathit{Mov}\colon \mathit{States} \times \{1,2\} \to 2^{\mathit{Act}} \smallsetminus \{\varnothing\}$ *is a mapping indicating the actions available to each player in a given state;*
- $\mathit{Tab}\colon \mathit{States} \times \mathit{Act}^2 \to 2^{\mathit{Edg}} \smallsetminus \{\varnothing\}$ *associates to each state and each pair of actions the set of resulting edges. It is required that if* $(s', s'') \in \mathit{Tab}(s, (m_1, m_2))$*, then* $s' = s$*.*
- *for each* $i \in \{1,2\}$*,* $\preccurlyeq_i \subseteq \mathit{States}^\omega \times \mathit{States}^\omega$ *is a quasi-order called the* preference relation *for player* i*.*

A *deterministic* concurrent game is a concurrent game where $\mathsf{Tab}(s, (m_1, m_2))$ is a singleton for every $s \in \mathsf{States}$ and $(m_1, m_2) \in \mathsf{Mov}(s,1) \times \mathsf{Mov}(s,2)$. A *turn-based* game is a concurrent game for which there exists a mapping $\mathsf{Owner}\colon \mathsf{States} \to \{1,2\}$ such that, for every state $s \in \mathsf{States}$, the set $\mathsf{Mov}(s,i)$ is a singleton unless $\mathsf{Owner}(s) = i$.

[3] For this and the following definitions, we explicitly mention the underlying transition system as a subscript. In the sequel, we may omit this subscript when the transition system is clear from the context.

In a concurrent game, from some state s, each player i selects one action m_i among its set $\mathsf{Mov}(s,i)$ of allowed actions (the resulting pair (m_1, m_2) is called a *move*). This results in a set of edges $\mathsf{Tab}(s,(m_1,m_2))$, one of which is applied and gives the next state of the game. In the sequel, we abusively write $\mathsf{Hist}_\mathcal{G}$, $\mathsf{Play}_\mathcal{G}$ and $\mathsf{Path}_\mathcal{G}$ for the corresponding set of paths in the underlying transition system of $\mathcal{G}$. We also write $\mathsf{Hist}_\mathcal{G}(s)$, $\mathsf{Play}_\mathcal{G}(s)$ and $\mathsf{Path}_\mathcal{G}(s)$ for the respective subsets of paths starting in state s.

Definition 4. *Let $\mathcal{G}$ be a concurrent game, and $i \in \{1,2\}$. A* strategy *for player i is a mapping $\sigma_i\colon \mathsf{Hist}_\mathcal{G} \to \mathsf{Act}$ such that $\sigma_i(\pi) \in \mathsf{Mov}(\mathsf{last}(\pi), i)$ for all $\pi \in \mathsf{Hist}_\mathcal{G}$. A* strategy profile *is a pair (σ_1, σ_2) where σ_i is a player-i strategy. We write $\mathsf{Strat}^i_\mathcal{G}$ for the set of strategies of player i in $\mathcal{G}$, and $\mathsf{Prof}_\mathcal{G}$ for the set of strategy profiles in $\mathcal{G}$.*

Notice that we only consider non-randomized (*pure*) strategies in this paper.

Let $\mathcal{G}$ be a concurrent game, $i \in \{1,2\}$, and σ_i be a player i-strategy. A path $\pi = (s_j)_{0\le j\le |\pi|}$ is *compatible* with the strategy σ_i if, for all $k \le |\pi|-1$, there exists a pair of actions $(m_1,m_2) \in \mathsf{Act}^2$ such that $m_j \in \mathsf{Mov}(s_k, j)$ for all $j \in \{1,2\}$, $m_i = \sigma_i(\pi_{\le k})$, and $(s_k, s_{k+1}) \in \mathsf{Tab}(s_k,(m_1,m_2))$. A path π is compatible with a strategy profile (σ_1,σ_2) whenever it is compatible with both strategies σ_1 and σ_2. We write $\mathsf{Out}_{\mathcal{G},s}(\sigma_i)$ (resp. $\mathsf{Out}_{\mathcal{G},s}(\sigma_1,\sigma_2)$) for the set of paths from s (also called *outcomes*) in $\mathcal{G}$ that are compatible with strategy σ_i (resp. strategy profile (σ_1,σ_2)). Notice that, in the case of deterministic concurrent games, a strategy profile has a single infinite outcome. This might not be the case for non-deterministic concurrent games.

Given a move (m_1,m_2) and a new action m' for player i, we write $(m_1,m_2)_{[i\mapsto m']}$ for the move (n_1,n_2) with $n_i = m'$ and $n_{3-i} = m_{3-i}$. This notation is extended to strategies in a natural way.

In the context of non-zero-sum games, several notions of equilibria have been defined. We present a refinement of *Nash equilibria* towards non-deterministic concurrent games.

Definition 5. *Let $\mathcal{G}$ be a concurrent game, and s be a state of $\mathcal{G}$. A* pseudo Nash equilibrium *in $\mathcal{G}$ from s is a tuple $((\sigma_1,\sigma_2),\pi)$ where $(\sigma_1,\sigma_2) \in \mathsf{Prof}_\mathcal{G}$, and $\pi \in \mathsf{Out}_{(\mathcal{G},s)}(\sigma_1,\sigma_2)$ is such that for all $i \in \{1,2\}$ and all $\sigma'_i \in \mathsf{Strat}^i_\mathcal{G}$, it holds:*

$$\forall \pi' \in \mathsf{Out}_{(\mathcal{G},s)}((\sigma_1,\sigma_2)_{[i\mapsto\sigma'_i]}).\ \pi' \preccurlyeq_i \pi.$$

Such an outcome π is called a best play *for the strategy profile (σ_1,σ_2).*

In the case of deterministic games, π is uniquely determined by (σ_1,σ_2), and pseudo Nash equilibria coincide with *Nash equilibria* as defined in [13]: they are strategy profiles where no player has an incentive to unilaterally deviate from her strategy.

In the case of non-deterministic games, a strategy profile for an equilibrium may give rise to several outcomes. The choice of playing the best play π is then made cooperatively by both players: once both strategies are fixed, it is the interest of both players to cooperate and play "optimally".

2.3 Back to timed games

Two comments are in order here: (i) non-determinism in timed games could be dropped by giving priority to one of the players, in case both of them play the same delay. Our algorithm could of course be adapted in this case; (ii) even if the timed game were deterministic, our transformation to region games involves some extra non-determinism. As will be seen in the sequel, the above notion of *pseudo Nash equilibria* is the notion we need for our construction to preserve equilibria.

It is easy to see the semantics of a timed game as the semantics of an infinite-state concurrent game (see Appendix A). Using that point-of-view, timed games inherit the notions of history, play, path, strategy, profile, outcome and pseudo Nash equilibrium. We illustrate some of these notions on the running example.

Example 1 (Cont'd). This game starts in configuration $(\ell_0, 0)$ (clock x is set to 0). A strategy profile is then determined by an initial choice for the first transition. If one of the players choose some delay smaller than 1, she will have payoff 1 but the other player will have payoff 0, hence the other player will be able to preempt this choice and choose a smaller delay that will improve her own payoff. Hence there will be no such pseudo Nash equilibrium. There is a single pseudo Nash equilibrium, where player 1 chooses $(1, c)$ (delay for 1 t.u. and take transition c) and player 2 chooses $(1, b)$. The best play for that strategy profile is the run taking transition c.

In this paper we will be interested in the computation of pseudo Nash equilibria in timed games. To do so we propose a sequence of transformations that will preserve equilibria (in some sense), yielding the construction of two turn-based finite games in which the initial problem will be reduced to the computation of *twin* Nash equilibria. All these transformations are presented in the next section. These transformations will also give a new point-of-view on timed games, which we will use in Section 4.2 to recover some decidability results. Many more results are expected.

3 From timed games to turn-based finite games

In this section we propose a chain of transformations of the timed game $\mathcal{G}$ into two turn-based finite games, and reduce the computation of pseudo Nash equilibria in $\mathcal{G}$ to the computation of 'twin' Nash equilibria in the two turn-based games. Notice that we will have to impose restrictions on the preference relations: indeed, price-optimal reachability is undecidable in two-player priced timed games, and these quantitative objectives can be encoded as a payoff function, see [6] for details.

3.1 From timed games to concurrent games...

We assume the reader is familiar with the region automaton abstraction for timed automata [1]. Let $\mathcal{G} = \langle \mathsf{Loc}, \mathsf{Cl}, \mathsf{Inv}, \mathsf{Trans}, \mathsf{Owner}, (\preccurlyeq_1, \preccurlyeq_2) \rangle$ be a timed game. Let $\mathfrak{R}$ be the set of regions for the timed automaton underlying $\mathcal{G}$, and $\pi_{\mathfrak{R}}$ be the projection over the regions $\mathfrak{R}$ (for configurations, runs, *etc.*) We define the *region game* $\mathcal{R} = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Act}, \mathsf{Mov}, \mathsf{Tab}, (\preccurlyeq_1^{\mathcal{R}}, \preccurlyeq_2^{\mathcal{R}}) \rangle$ as follows:

- $\mathsf{States} = \{(\ell, r) \in \mathsf{Loc} \times \mathfrak{R} \mid r \models \mathsf{Inv}(\ell)\}$;
- $\mathsf{Edg} = \{((\ell, r), (\ell', r')) \mid (\ell, r) \to (\ell', r')$ in the region automaton of $\mathcal{G}\}$;

– $\mathsf{Act} = \{\bot\} \cup \{(r, \delta) \mid r \in \mathfrak{R} \text{ and } \delta \in \mathsf{Trans}\}$;
– $\mathsf{Mov}\colon \mathsf{States} \times \{1,2\} \to 2^{\mathsf{Act}} \setminus \{\varnothing\}$ such that:

$$\mathsf{Mov}((\ell, r), i) = \{(r', \delta) \mid r' \in \mathsf{Succ}(r),\ r' \models \mathsf{Inv}(\ell), \delta = (\ell, g, Y, \ell') \text{ is s.t. } r' \models g \text{ and } [Y \leftarrow 0]r' \models \mathsf{Inv}(\ell') \text{ and } \mathsf{Owner}(\delta) = i\}$$

if this set is non-empty, and $\mathsf{Mov}((\ell, r), i) = \{\bot\}$ otherwise.
– $\mathsf{Tab}\colon \mathsf{States} \times \mathsf{Act}^2 \to 2^{\mathsf{Edg}} \setminus \{\varnothing\}$ such that for every $(\ell, r) \in \mathsf{States}$ and every $(m_1, m_2) \in \mathsf{Mov}((\ell, r), 1) \times \mathsf{Mov}((\ell, r), 2)$, if we write r' for $\min\{r_j \mid j \in \{1,2\} \text{ and } m_j = (r_j, \delta_j)\}$,[4] then we have:

$$\mathsf{Tab}((\ell, r), (m_1, m_2)) = \{((\ell, r), (\ell_j, [Y_j \leftarrow 0]r_j)) \mid j \in \{1,2\} \text{ and } m_j = (r_j, \delta_j) \text{ with } r_j = r',\ (\ell, g_j, Y_j, \ell_i) = \delta_j \text{ and } r_j \models g_j\}$$

– The preference relation $\preccurlyeq_i^{\mathcal{R}}$ for player i is defined by saying that $\gamma \preccurlyeq_i^{\mathcal{R}} \gamma'$ iff there exists ρ and ρ' such that $\pi_{\mathfrak{R}}(\rho) = \gamma$, $\pi_{\mathfrak{R}}(\rho') = \gamma'$ and $\rho \preccurlyeq_i \rho'$.

Note that the game $\mathcal{R}$ is non-deterministic, even if the original timed game is not. Indeed, non-determinism appears when players want to play delays leading to the same region. The (relative) order of the choices for the delays chosen by the two players cannot be distinguished by the region abstraction.

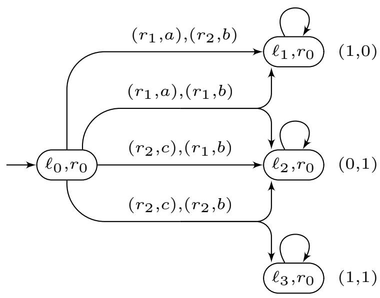
Definition 6. *A preference relation $\preccurlyeq_i$ is said to be* region-uniform *when for all plays ρ and ρ', if the sequence of regions seen in both paths are the same, then they are equivalent, i.e. $\rho \preccurlyeq_i \rho'$ and $\rho' \preccurlyeq_i \rho$.*

Proposition 7. *Let $\mathcal{G}$ be a timed game, and assume that the two preference relations of $\mathcal{G}$ are region-uniform. Let $\mathcal{R}$ be its associated region game. Then there is a pseudo Nash equilibrium in $\mathcal{G}$ from $(\ell_0, \mathbf{0})$ with best play ρ iff there is a pseudo Nash equilibrium in $\mathcal{R}$ from $(\ell_0, [\mathbf{0}]_{\mathfrak{R}})$ with best play $\pi_{\mathfrak{R}}(\rho)$. Furthermore, this equivalence is constructive.*

Example 1 (Cont'd). We illustrate the construction and the previous notions on the running example. We write r_0 (resp. r_1, r_2) for the region $x = 0$ (resp. $0 < x < 1$, $x = 1$). The region game $\mathcal{R}$ is as depicted on Fig. 1. In this region game, there are two non-deterministic transitions. First when the two players choose to wait until region r_2, in which case the game can turn to either ℓ_2 or to ℓ_3. Then when both players choose to move within the region r_1 (there is an uncertainty on whether player 1 or player 2 was faster), and depending on who was faster, the game will move to either ℓ_1 or ℓ_2. The first non-determinism is inherent to the game (and could be removed by construction assuming one player is more powerful, see Subsection 2.3 for explanations), whereas the second non-determinism is (somehow) artificial and comes from the region abstraction.

In $\mathcal{G}$, there is a single pseudo Nash equilibrium, where both players wait until $x = 1$ (region r_2), and propose to move respectively to ℓ_3 (resp. ℓ_2). The best play is then $(\ell_0, 0)(\ell_3, 0)^*$. This corresponds to the unique pseudo Nash equilibrium that we find in the region game.

[4] This is well-defined because both r_j's are time-successors of r.



The transition table from (ℓ_0, r_0)
(*i.e.*, $\mathsf{Tab}((\ell_0, r_0), (m_1, m_2))$):

	$m_2 = (r_1, b)$	$m_2 = (r_2, b)$
$m_1 = (r_1, a)$	(ℓ_1, r_0),(ℓ_2, r_0)	(ℓ_1, r_0)
$m_1 = (r_2, c)$	(ℓ_2, r_0)	(ℓ_2, r_0),(ℓ_3, r_0)

Fig. 1. The region game from our original automaton.

3.2 ... next to two twin concurrent games...

Given a concurrent non-deterministic finite game $\mathcal{R} = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Act}, \mathsf{Mov}, \mathsf{Tab}, (\preccurlyeq_1^{\mathcal{R}}, \preccurlyeq_2^{\mathcal{R}}) \rangle$, we construct two concurrent games $\mathcal{R}_1$ and $\mathcal{R}_2$ where we simply forget the preferences of one player. Formally for $i \in \{1, 2\}$, we define the game $\mathcal{R}_i = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Act}, \mathsf{Mov}, \mathsf{Tab}, (\preccurlyeq_1^i, \preccurlyeq_2^i) \rangle$, where $\preccurlyeq_i^i$ is the quasi-order $\preccurlyeq_i$, and $\preccurlyeq_{3-i}^i$ is the trivial quasi-order where all runs are equivalent.

Definition 8. *A* twin pseudo Nash equilibrium *for the two games $\mathcal{R}_1$ and $\mathcal{R}_2$ is a tuple $((\sigma_1^{\mathcal{R}_1}, \sigma_2^{\mathcal{R}_1}), (\sigma_1^{\mathcal{R}_2}, \sigma_2^{\mathcal{R}_2}), \rho)$ such that $((\sigma_1^{\mathcal{R}_1}, \sigma_2^{\mathcal{R}_1}), \rho)$ is a pseudo Nash equilibrium in the game $\mathcal{R}_1$ and $((\sigma_1^{\mathcal{R}_2}, \sigma_2^{\mathcal{R}_2}), \rho)$ is a pseudo Nash equilibrium in the game $\mathcal{R}_2$. We furthermore say that ρ is a* best play *for the twin pseudo equilibrium.*

We relate pseudo Nash equilibria in $\mathcal{R}$ with twin pseudo Nash equilibria in $\mathcal{R}_1$ and $\mathcal{R}_2$. Note that we require best plays be the same, but not strategies.

Proposition 9. *Let $\mathcal{R}$ be the region game associated with some timed game $\mathcal{G}$. Then there is a pseudo Nash equilibrium in $\mathcal{R}$ from s with best play γ if and only if there is a twin pseudo equilibrium for the corresponding games $\mathcal{R}_1$ and $\mathcal{R}_2$ from s with best play γ. Furthermore this equivalence is constructive.*

3.3 ... next to concurrent deterministic games...

We transform each game $\mathcal{R}_i$ into a concurrent deterministic game $\mathcal{C}_i$. Game $\mathcal{C}_i$ will give priority to player i, in that it will be the role of player i to solve non-determinism. The game $\mathcal{C}_i = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Act}', \mathsf{Mov}_i, \mathsf{Tab}_i, (\preccurlyeq_1^i, \preccurlyeq_2^i) \rangle$ is defined as follows:

- $\mathsf{Act}' = \mathsf{Act} \cup ((\mathsf{Act} \setminus \{\bot\}) \times \{\bullet, \circ\})$;
- $\mathsf{Mov}_i \colon \mathsf{States} \times \{1, 2\} \to 2^{\mathsf{Act}'} \setminus \{\varnothing\}$ such that:

$$\mathsf{Mov}_i(s, i) = \begin{cases} \{\bot\} & \text{if } \mathsf{Mov}(s, i) = \{\bot\} \\ \mathsf{Mov}(s, i) \times \{\bullet, \circ\} & \text{otherwise} \end{cases}$$
$$\mathsf{Mov}_i(s, 3 - i) = \mathsf{Mov}(s, 3 - i)$$

- Given $(m_1, m_2) \in \mathsf{Mov}(s,1) \times \mathsf{Mov}(s,2)$ we have that $\mathsf{Tab}(s,(m_1,m_2))$ has at least one element, and at most two elements.[5]
  - In case it has only one element, then setting $m'_{3-i} = m_{3-i}$ and picking $m'_i \in \{(m_i,\bullet),(m_i,\circ)\}$, we define: $\mathsf{Tab}_i(s,(m'_1,m'_2)) = \mathsf{Tab}(s,(m_1,m_2))$;
  - In case it has two elements, say $(s,s_\bullet)$ and $(s,s_\circ)$, one of them comes from a transition of player i in $\mathcal{G}$ and the other comes from a transition of player $3-i$ in $\mathcal{G}$. Hence w.l.o.g. we can assume that $(s,s_\bullet)$ belongs to player i. We now define $m'_{3-i} = m_{3-i}$ and for any $m'_i \in \{(m_i,\bullet),(m_i,\circ)\}$, we define:

$$\mathsf{Tab}_i(s,(m'_1,m'_2)) = \begin{cases} \{(s,s_\bullet)\} & \text{if } m'_i = (m_i,\bullet) \\ \{(s,s_\circ)\} & \text{if } m'_i = (m_i,\circ) \end{cases}$$

By construction, the two games $\mathcal{C}_1$ and $\mathcal{C}_2$ are deterministic, and they share the same structure. Only decisions on how to solve non-determinism are made by different players. Our aim will be to compute equilibria in these two similar games.

Proposition 10. *Assume $\mathcal{C}_i$ (with $i \in \{1,2\}$) is the deterministic concurrent game defined from the concurrent game $\mathcal{R}_i$. Then there is a pseudo Nash equilibrium in $\mathcal{R}_i$ from s with best play γ iff there is a Nash equilibrium in $\mathcal{C}_i$ from s with best play γ.*[6] *Furthermore this equivalence is constructive.*

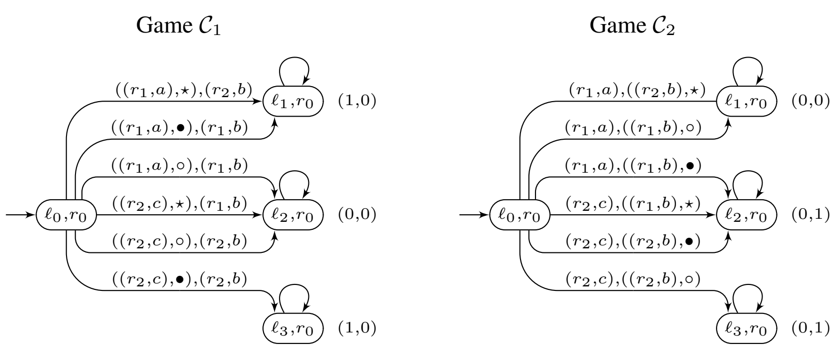


Fig. 2. Two concurrent games $\mathcal{C}_1$ and $\mathcal{C}_2$ from our original automaton.

Example 1 (Cont'd). We build on the previous example, and give the two games $\mathcal{C}_1$ and $\mathcal{C}_2$ in Fig. 2. An action $(m,\star)$ denotes either $(m,\bullet)$ or $(m,\circ)$. There are several Nash equilibria in game $\mathcal{C}_1$: one where the first player chooses $((r_2,c),\bullet)$ and the second player chooses (r_2,b), which leads to (ℓ_3,r_0) with payoff $(1,0)$; and one where both players play a pair of actions leading to (ℓ_1,r_0), in which case the payoff is also $(1,0)$.

[5] This is because the game $\mathcal{G}$ is non-blocking, and in this game, each player proposes her choice for a transition, and one of these two transitions will be chosen.

[6] Remember that $\mathcal{C}_i$'s and $\mathcal{R}_i$'s share the same structure and have the same runs.

Similarly there are several Nash equilibria in game $\mathcal{C}_2$: one where the second player chooses $((r_2, b), \circ)$ and the first player chooses (r_2, c), which leads to (ℓ_3, r_0) with payoff $(0, 1)$; the second one where both players play a pair of actions leading to (ℓ_2, r_0), in which case the payoff is also $(0, 1)$.

There is a single twin equilibrium in $\mathcal{C}_1$ and $\mathcal{C}_2$, namely the one leading to state (ℓ_3, r_0), which coincides with those equilibria already found in $\mathcal{G}$ and $\mathcal{R}$.

3.4 ... and finally to two turn-based games

In the (deterministic) concurrent game $\mathcal{C}_i$, the advantage is given to player i, who has the ability to solve non-determinism. We can give a slightly different interpretation to that mechanism, which takes into account an interpretation of the new actions. Indeed, actions have a timed interpretation in the original timed game, and can be ordered w.r.t. their delay. Taking advantage of this order on actions, we build a turn-based game $\mathcal{T}_i$.

Let $\mathcal{C}_i = \langle \mathsf{States}, s_0, \mathsf{Edg}, \mathsf{Act}', \mathsf{Mov}_i, \mathsf{Tab}_i, (\preccurlyeq_1^i, \preccurlyeq_2^i)\rangle$ be the games obtained from the previous construction. Let $s \in \mathsf{States}$. We naturally order the set $\mathsf{Mov}_i(s, 1) \cup \mathsf{Mov}_i(s, 2)$ with a relation $<_s$ so that:

- (i) if $\perp \in \mathsf{Mov}_i(s, 1) \cup \mathsf{Mov}_i(s, 2)$ then $\perp$ is maximal w.r.t. $<_s$;
- (ii) for every $m \in \mathsf{Mov}_i(s, j)$, there exists $s' \in \mathsf{States}$ such that for every $m' \in \mathsf{Mov}_i(s, 3 - j)$, $m <_s m'$ implies $\mathsf{Tab}_i(s, (m, m')) = \{(s, s')\}$.

This is possible due to the definition of game $\mathcal{C}_i$: when (r, δ_{3-i}) is allowed to player $3-i$ from s, and $((r, \delta_i), \bullet)$ and $((r, \delta_i), \circ)$ are allowed to player i from s, then the three actions are totally ordered by $<_s$ as follows:[7] $((r, \delta_i), \bullet) <_s (r, \delta_{3-i}) <_s ((r, \delta_i), \circ)$. Intuitively an action with marker $\bullet$ means that player i can play her own transition faster than player $3 - i$ can play her own transition, but also that she can decide to play more slowly (role of action with marker $\circ$).

We can also define an equivalence relation $=_s$ compatible with this order, by saying $m =_s m' \Leftrightarrow m, m' \in \mathsf{Mov}_i(s, 1) \cup \mathsf{Mov}_i(s, 2),\ m \not<_s m'$ and $m' \not<_s m$. It is worth noticing that $m =_s m'$ implies that they belong to the same player. This can be the case if two transitions are available to a player from the same region, and also if a player can only play action $\perp$. We will write $[m]_s$ for the equivalence class associated to m. We next say that $[m]_s$ belongs to player j whenever all actions in $[m]_s$ belong to player j.

Example 1 (Cont'd). Consider games $\mathcal{C}_1$ and $\mathcal{C}_2$ depicted in Fig. 2. In game $\mathcal{C}_1$, the order on actions (written simply as $<$) from (ℓ_0, r_0) is given by:

$$\begin{array}{ccccccccccc} ((r_1,a),\bullet) & < & (r_1,b) & < & ((r_1,a),\circ) & < & ((r_2,c),\bullet) & < & (r_2,b) & < & ((r_2,c),\circ) \\ \wr & & \wr & & \wr & & \wr & & \wr & & \\ (\ell_1,r_0) & & (\ell_2,r_0) & & (\ell_1,r_0) & & (\ell_3,r_0) & & (\ell_2,r_0) & & \end{array}$$

Below each action we write the target state when this action is played, provided an action smaller (for the order $<$) is not played by the other player. There is no target with action $((r_2, c), \circ)$ because it is always preempted by some 'faster' action (no $\perp$ action is available in our example).

In game $\mathcal{C}_2$, the order on actions (also written $<$) from (ℓ_0, r_0) is given by:

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[7] This is due to the fact that we have assumed edge $(s, s_\bullet)$ belong to player i, see the construction of game $\mathcal{C}_i$.

$$\begin{array}{ccccccccccc}((r_1,b),\bullet) & < & (r_1,a) & < & ((r_1,b),\circ) & < & ((r_2,b),\bullet) & < & (r_2,c) & < & ((r_2,b),\circ)\\ \wr & & \wr & & \wr & & \wr & & \wr & & \\ (\ell_2,r_0) & & (\ell_1,r_0) & & (\ell_2,r_0) & & (\ell_2,r_0) & & (\ell_3,r_0) & & \end{array}$$

We will take advantage of this order on actions to build turn-based games that will in some sense be equivalent with the previous concurrent (deterministic) games. The idea will be to take the smallest action(s) in the order, and ask the corresponding player whether or not she wants to play that action; if yes, we proceed with this action in the game, otherwise we do the same with the second action in the order until one of the players plays her action; The meaning in the context of timed games is actually also the following: we see that if the two players want to play in the same region, then in game $\mathcal{C}_i$ the advantage of player i is that we first ask her whether she wishes to play her action (role of action labelled with $\bullet$), then if not, the other player will be asked to decide whether she wants to play her own action, and finally, if not, we ask a last time player i whether she wants to play her action (now she has the additional knowledge that the other player didn't choose her own action).

Formally we define the turn-based game $\mathcal{T}_i$ as follows: $\mathcal{T}_i = \langle \mathsf{States}_i, \mathsf{Edg}_i, \mathsf{Act}' \cup \{\mathsf{del}\}, \mathsf{Mov}'_i, \mathsf{Tab}'_i, (\preccurlyeq'^i_1, \preccurlyeq'^i_2)\rangle$ where:

- $\mathsf{States}_i = \{(s,[m]_s) \mid s \in \mathsf{States}$ and $m \in (\mathsf{Mov}_i(s,1) \cup \mathsf{Mov}_i(s,2)) \setminus \{\bot\}\}$;
- The set Edg_i is defined as follows:

$$\begin{aligned}\mathsf{Edg}_i = \; & \{((s,[m]_s),(s,[m']_s)) \mid m' \neq \bot \text{ is next after } m \text{ w.r.t. } <_s\}\\ & \cup \{((s,[m]_s),(s',[m']_{s'})) \mid \{(s,s')\} = \mathsf{Tab}_i(s,(m,m''))\\ & \quad \text{for every } m <_s m'' \text{ and } m' \text{ is minimal w.r.t. } <_{s'} \text{ from } s'\};\end{aligned}$$

- The set of available actions is defined as follows:
  - if $[m]_s$ belongs to player j, then we use the new action del (for *delay*):

$$\mathsf{Mov}'_i((s,[m]_s),j) = \begin{cases}\mathsf{Mov}_i(s,j) \cap [m]_s & \text{if } m \text{ is maximal w.r.t. } <_s \text{ in } \mathsf{Mov}_i(s,j)\\ (\mathsf{Mov}_i(s,j) \cap [m]_s) \cup \{\mathsf{del}\} & \text{otherwise}\end{cases}$$

  - if $[m]_s$ belongs to player $3-j$, then $\mathsf{Mov}'_i((s,[m]_s),j) = \{\bot\}$.
- The transition table is defined as follows:
  - if $[m]_s$ belongs to player 1:

$$\begin{cases}\mathsf{Tab}'_i((s,[m]_s),(m,\bot)) = \{((s,[m]_s),(s',[m']_s)) \mid \{(s,s')\} = \mathsf{Tab}_i(s,(m,m''))\\ \qquad \text{for every } m <_s m'' \text{ and } m' \text{ is minimal w.r.t. } <_{s'} \text{ from } s'\}\\ \mathsf{Tab}'_i((s,[m]_s),(\mathsf{del},\bot)) = \{((s,[m]_s),(s,[m']_s)) \mid m' \text{ is next after } m \text{ w.r.t. } <_s\}\end{cases}$$

  - the second case ($[m]_s$ belongs to player 2) is similar, just swap m or del with $\bot$.
- In order to define the preference relations we first define a projection from plays in the turn-based game $\mathcal{T}_i$ onto plays in the concurrent game $\mathcal{C}_i$. Pick a run ν in $\mathcal{T}_i$, and define its projection $\psi_i(\nu)$ in $\mathcal{C}_i$ as follows: if

$$\nu = (s_1,\mathfrak{m}_1^1)(s_1,\mathfrak{m}_1^2)...(s_1,\mathfrak{m}_1^{k_1})(s_2,\mathfrak{m}_2^1)...(s_2,\mathfrak{m}_2^{k_2})...(s_p,\mathfrak{m}_p^1)...(s_p,\mathfrak{m}_p^{k_p})...$$

  with $\mathfrak{m}_i^1$ minimal w.r.t. $<_{s_i}$ for every $1 \leq i$, then $\psi(\nu) = s_1 s_2 \ldots s_p \ldots$. The preference relations are then defined according to this projection:

$$\nu \preccurlyeq'^i_j \nu' \Leftrightarrow \psi_i(\nu) \preccurlyeq^i_j \psi_i(\nu')$$

Note that the game $\mathcal{T}_i$ is turn-based[8] and that a state $(s, [m]_s)$ *belongs* to player j such that $m \in \mathsf{Mov}_i(s, j)$ (as already mentioned this is independent of the choice of m in $[m]_s$). The structure of the turn-based games $\mathcal{T}_1$ and $\mathcal{T}_2$ are now slightly different from that of the previous concurrent deterministic games $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proposition 11. *Let $\mathcal{C}_i$ (with $i \in \{1, 2\}$) be the previous deterministic concurrent game, and let $\mathcal{T}_i$ be the associated turn-based game. There is a Nash equilibrium in $\mathcal{C}_i$ from s with best play $\psi_i(\nu)$ iff there is a Nash equilibrium in $\mathcal{T}_i$ from $(s, [m]_s)$ with best play ν, where m is a minimal action w.r.t. $<_s$. Furthermore this equivalence is constructive.*

Example 1 (Cont'd). We build on our running example, and compute the corresponding games $\mathcal{T}_1$ and $\mathcal{T}_2$. They are displayed on Fig. 3. Plain states and plain edges belong to

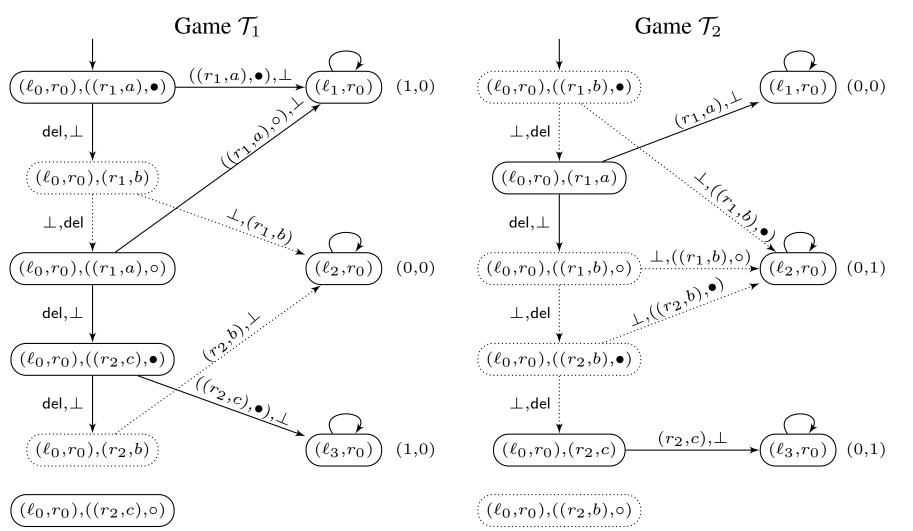


Fig. 3. Final turn-based games from our original timed game

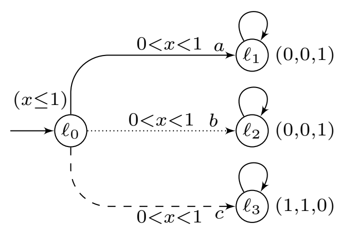
player 1 whereas dotted states and dotted edges belong to player 2. We do recognize here the various Nash equilibria that we described in the concurrent deterministic games, and only one is "common" to both games, namely the one leading to (ℓ_3, r_0).

3.5 Summary of the construction

The following theorem summarizes our construction:

[8] By construction, in any state $(s, [m]_s)$, one of $\mathsf{Mov}'_i((s, [m]_s), j)$ with $j \in \{1, 2\}$ equals $\{\bot\}$.

Theorem 12. *Let $\mathcal{G}$ be a timed game with region-uniform preference relations. Assume $\mathcal{T}_1$ and $\mathcal{T}_2$ are the two turn-based (deterministic) games constructed in this section. Then, there is a pseudo Nash equilibrium in $\mathcal{G}$ from $(\ell_0, \mathbf{0})$ with best play ρ iff there are two Nash equilibria in $\mathcal{T}_1$ and $\mathcal{T}_2$ from $((\ell_0, \mathbf{0}), [m]_{(\ell_0,\mathbf{0})})$ with best plays ν_1 and ν_2 respectively, where m is a minimal action w.r.t. $<_{(\ell_0,\mathbf{0})}$, such that $\psi_1(\nu_1) = \psi_2(\nu_2) = \pi_{\mathfrak{R}}(\rho)$. Furthermore this equivalence is constructive.*



Remark 1. The three-player game on the right has several Nash equilibria, for instance player 1 (plain arrows) chooses her transition at time 0.6, player 2 (dotted arrows) chooses her transition at time 0.7, and player 3 (dashed arrows) chooses her transition at time 0.8. If we build the region abstraction, each player will have a single possible move (play her transition in the region $0 < x < 1$), and the game will proceed by selecting non-deterministically one of them. There would be several ways to extend the method developed in this paper to three players: have a copy of the game for each player, assuming she plays against a coalition of the other players, or have a copy of the game for each priority order given to the players. It is not hard to be convinced that none of these choices will be correct on this example.

4 Decidability results

4.1 Some general decidability results

We first need a representation for the preference relations (which must be region-uniform) of both players. Let $\mathcal{G} = \langle \mathsf{Loc}, \mathsf{Cl}, \mathsf{Inv}, \mathsf{Trans}, \mathsf{Owner}, (\preccurlyeq_1, \preccurlyeq_2)\rangle$ be a two-player timed game. We assume the preference relation for player i is given by a (possibly infinite) sequence of linear-time objectives $(\Omega^i_j)_{j\geq 1}$ where it is better for a run to satisfy Ω^i_j than Ω^i_k as soon as $k > j$ (w.l.o.g. we assume that Ω^i_{j+1} implies $\neg\Omega^i_l$ for all $l \leq j$). In other terms, the aim of player i is to minimize the index j for which the play belongs to Ω^i_j. These objectives include ω-regular or LTL-definable objectives, and also more *quantitative* objectives (for instance, given a distinguished goal state $\mathsf{Goal}_i \in \mathsf{Loc}$ for player i, by defining Ω^i_j to be the set of traces visiting Goal_i in less than j steps.

We first need to (be able to) transfer objectives (and preference relations) to the two turn-based games $\mathcal{T}_1$ and $\mathcal{T}_2$: a linear-time objective Ω in $\mathcal{G}$ is said to be *transferable to game* $\mathcal{T}_i$ whenever we can construct an objective $\widehat{\Omega}$ such that for every run ν in $\mathcal{T}_i$, $\nu \models \widehat{\Omega}$ iff for all ρ with $\pi_{\mathfrak{R}}(\rho) = \psi_i(\nu)$, $\rho \models \Omega$. It is said *transferable* whenever it is transferable to both $\mathcal{T}_1$ and $\mathcal{T}_2$. For example, notice that (sequences of) stutter-free region-uniform objectives are transferable.

Nash equilibria in game $\mathcal{T}_i$ will be rather easy to characterize since player $3-i$ will never be enclined to deviate from her strategy (all runs are equivalent for her preference relation). We assume all objectives Ω^i_j are transferable, and we write $W^{3-i}_i(j)$ for the set of winning states in game $\mathcal{T}_i$ for player $3-i$ with the objective $\bigwedge_{1\leq k<j}\left(\neg\widehat{\Omega}^i_k\right)$. Those sets are computable for many classes of objectives. Then:

Theorem 13. *Let $\mathcal{G}$ be a timed game with preference relations given as transferable, region-uniform, prefix-independent sequences $(\Omega^i_j)_j$ of objectives. There is a pseudo Nash equilibrium in $\mathcal{G}$ with payoff (Ω^j_1, Ω^k_2) iff there are two runs ν_1 in $\mathcal{T}_1$ and ν_2 in $\mathcal{T}_2$ s.t.*[9] *(i) $\nu_1 \models \big(\mathbf{G}\, W^2_1(j)\big) \wedge \widehat{\Omega}^1_j$, (ii) $\nu_2 \models \big(\mathbf{G}\, W^1_2(k)\big) \wedge \widehat{\Omega}^2_k$, and (iii) $\psi_1(\nu_1) = \psi_2(\nu_2)$.*

Notice that this allows to handle ω-regular (and LTL-definable) objectives by considering the product of the game with a suitable (deterministic) automaton.

The sequence of states $(W^{3-i}_i(j))_{j\geq 1}$ in game $\mathcal{T}_i$ is non-increasing and hence stationary (because $\mathcal{T}_i$ is finite-state). Hence there exist indices $h_0 = 1 < h_1 < h_2 < \dots < h_l$ such that the function $j \mapsto W^{3-i}_i(j)$ is constant on all intervals $[h_p, h_{p+1})$ and on $[h_l, +\infty)$. Those indices can be computed together with the corresponding sets of winning states. Then the only possible equilibria in $\mathcal{T}_i$ are those such that there is a run satisfying $\widehat{\Omega}^i_j$ that stays furthermore within the set $W^{3-i}_i(h_p)$ if $h_p \leq j < h_{p+1}$, or within $W^{3-i}_i(h_l)$ if $j \geq h_l$. This can be done for instance if each player is given a goal state Goal_i, and Ω^i_j is "reach Goal_i in j steps". In that case, $W^{3-i}_i(h_l)$ is the set of states from which player $3-i$ can avoid Goal_i. Hence we can compute Nash equilibria in two-player timed games where each player tries to minimize the number of steps to the goal state. This allows to recover part of the results of [7] for two-player games.

4.2 Zero-sum games

Our chain of transformations also yields a new point-of-view on classical two-player timed games with zero-sum objectives. In that case the preference relation of player 1 is characterized by the sequence $(\Omega, \neg\Omega)$ whereas that of player 2 is characterized by the sequence $(\neg\Omega, \Omega)$. In that case we say that the objective of player 1 is Ω.

Theorem 14. *Let $\mathcal{G}$ be a zero-sum timed game where player 1's objective is Ω, and is assumed to be transferable. Then player 1 has a winning strategy in $\mathcal{G}$ from $(\ell, \mathbf{0})$ iff player 1 has a winning strategy in game $\mathcal{T}_2$ from $(\ell, [\mathbf{0}])$ for the objective $\widehat{\Omega}$.*

5 Conclusion

We have proposed a series of transformations of two-player timed games into two turn-based finite games. These transformations reduce the computation of Nash equilibria in timed games for a large class of objectives (the so-called region-uniform objectives) to the computation of "twin" equilibria for related objectives in the two turn-based finite games. We give an example on how this can be used to compute Nash equilibria in timed games. In turn our transformations give a nice and new point-of-view on zero-sum timed games, which can then be interpreted as a turn-based finite game.

Our method does not extend to n players. In [5], we have developed a completely new approach that allows to compute Nash equilibria in timed games with an arbitrary number of players but only for reachability objectives. We plan to continue working on the computation of Nash equilibria in timed games with an arbitrary number of players.

[9] $\mathbf{G}$ is the LTL modality for "*always*".

Another interesting research direction is the computation of other kinds of equilibria in timed games (secure equilibria, subgame-perfect equilibria, *etc*). We believe that the transformations that we have made in this paper are correct also for these other notions, and that we can for instance reduce the computation of subgame-perfect equilibria to the computation of subgame-perfect equilibria in the two turn-based finite games. A major difference is that Theorem 13 has to be refined. Tree automata could be the adequate tool for this problem [16].

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A Semantics of timed games

With a timed game $\mathcal{G} = \langle \mathsf{Loc}, \mathsf{Cl}, \mathsf{Inv}, \mathsf{Trans}, \mathsf{Owner}, (\preccurlyeq_i)_{i=1,2}\rangle$, we associate the infinite concurrent game $\mathcal{G}' = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Act}, \mathsf{Mov}, \mathsf{Tab}, (\preccurlyeq_i)_{i=1,2}\rangle$ such that

- the set of states is the set of configurations of the timed game: $\mathsf{States} = \{(\ell, v) \mid \ell \in \mathsf{Loc}, v\colon \mathsf{Cl} \to \mathbb{R}_+ \text{ such that } v \models \mathsf{Inv}(\ell)\}$;
- $s_0 = (\ell_0, \mathbf{0})$ is the initial state;
- transitions give rise to set of edges Edg as follows: for each $d \in \mathbb{R}_+$ and each $\delta = (\ell, g, z, \ell')$ in Trans, for each $(\ell, v) \in \mathsf{States}$ such that $v + d \models \mathsf{Inv}(\ell) \wedge g$, there is an edge $((\ell, v), (\ell', [z \leftarrow 0](v + d)))$;
- the set of actions is $\mathsf{Act} = \{(d, \delta) \mid d \in \mathbb{R}_+, \delta \in \mathsf{Trans}\} \cup \{\bot\}$;
- an action (d, δ) is allowed to player i in state (ℓ, v) iff the following three conditions hold:
  - $(\ell, v + d) \in \mathsf{States}$;
  - $\delta = (\ell, g, z, \ell')$ is such that $\mathsf{Owner}(\delta) = i$;
  - $v + d \models g$ and $[z \leftarrow 0](v + d) \models \mathsf{Inv}(\ell')$.

  Then $\mathsf{Mov}((\ell, v), i)$ is the set of actions allowed to player i when this set is non empty, and it is $\{\bot\}$ otherwise;
- finally, given a state (ℓ, v) and moves (m_1, m_2) allowed from this state, $\mathsf{Tab}((\ell, v), (m_1, m_2))$ is the set

$$\Big\{((\ell, v), (\ell', v')) \;\Big|\; \exists i.\; d_i = \min\{d_j \mid j \in \{1, 2\} \text{ s.t. } m_j = (d_j, \delta_j)\} \text{ and } \delta_i = (\ell, g_i, z_i, \ell') \text{ and } v' = [z_i \leftarrow 0](v + d_i)\Big\}.$$

Remark 2. In the following proofs, we write Tab for the transition table of the (infinite-state) concurrent game which gives the semantic of the timed game $\mathcal{G}$.

B Proof of Section 3.1

In this appendix we prove the correctness of the region game $\mathcal{R}$ w.r.t. timed game $\mathcal{G}$:

Proposition 7. *Let $\mathcal{G}$ be a timed game, and assume that the two preference relations of $\mathcal{G}$ are region-uniform. Let $\mathcal{R}$ be its associated region game. Then there is a pseudo Nash equilibrium in $\mathcal{G}$ from $(\ell_0, \mathbf{0})$ with best play ρ iff there is a pseudo Nash equilibrium in $\mathcal{R}$ from $(\ell_0, [\mathbf{0}]_{\mathfrak{R}})$ with best play $\pi_{\mathfrak{R}}(\rho)$. Furthermore, this equivalence is constructive.*

To that aim, we define transformers of strategies in both directions and prove that they preserve pseudo Nash equilibria.

We first extend the function Owner to each edge in $\mathcal{G}$ and in $\mathcal{R}$ in a natural way: if $\delta = (\ell, g, Y, \ell') \in \mathsf{Trans}$ and $\mathsf{Owner}(\delta) = j$, then for every edge $e = ((\ell, v), (\ell', v'))$ (resp. $e = ((\ell, r), (\ell', r'))$) in $\mathcal{G}$ (resp. $\mathcal{R}$), we set $\mathsf{Owner}(e) = j$.[10]

We define two partial functions $f_1, f_2\colon \mathbb{R}_+^{\mathsf{Cl}} \times \mathfrak{R} \to \mathbb{R}_+$ such that for every $v \in \mathbb{R}_+^X$ and $r \in \mathfrak{R}$, it holds $v + f_i(v, r) \in r$ for all i and $f_1(v, r) < f_2(v, r)$ in case this is possible, otherwise $f_1(v, r) = f_2(v, r)$.

[10] This is possible due to the assumption on $\mathcal{G}$ that there is at most one transition between two locations in $\mathcal{G}$.

From timed game $\mathcal{G}$ to region games $\mathcal{R}$. In this section, we prove that a pseudo Nash equilibrium in $\mathcal{G}$ gives rise to a pseudo Nash equilibrium in $\mathcal{R}$.

Pick a play ρ in $\mathcal{G}$, and first define a reciprocal function to π_R: for every history γ of $\mathcal{R}$, $\pi_R^{-1}(\gamma)$ is a history h in $\mathcal{G}$ such that $\pi_R(h) = \gamma$; furthermore, if γ is a prefix of $\pi_R(\rho)$, then we require h to be a prefix of ρ. Notice that we have $\pi_R(\pi_R^{-1}(\gamma)) = \gamma$ but it may be the case that $\pi_R^{-1}(\pi_R(h)) \neq h$, unless h is a prefix of ρ. Note also that this choice depends on ρ, but we omit any indication of ρ in our notation.

If σ_i is a player-i strategy in $\mathcal{G}$, we define the player-i strategy $\lambda_\rho(\sigma_i)$ in $\mathcal{R}$ as follows: for every history γ in $\mathcal{R}$,

- if $\sigma_i(\pi_R^{-1}(\gamma)) = \bot$, then set $\lambda_\rho(\sigma_i)(\gamma) = \bot$;
- if $\sigma_i(\pi_R^{-1}(\gamma)) = (d, \delta)$, then set $\lambda_\rho(\sigma_i)(\gamma) = (r, \delta)$, where r is the region corresponding to valuation $v + d$ if v is the clock valuation at the end of $\pi_R^{-1}(\gamma)$.

Lemma 15. *Let (σ_1, σ_2) be a strategy profile in game $\mathcal{G}$. Then*

$$\pi_R(\mathsf{Out}_{\mathcal{G}}(\sigma_1, \sigma_2)) \subseteq \mathsf{Out}_{\mathcal{R}}(\lambda_\rho(\sigma_1), \lambda_\rho(\sigma_2))$$

Proof. Let $\rho \in \mathsf{Out}_{\mathcal{G}}(\sigma_1, \sigma_2)$, we want to prove that the projection $\pi_R(\rho)$ is in the set $\mathsf{Out}_{\mathcal{R}}(\lambda_\rho(\sigma_1), \lambda_\rho(\sigma_2))$. To that aim we prove that $\pi_R(\rho)$ follows the rules of the strategy profile $(\lambda_\rho(\sigma_1), \lambda_\rho(\sigma_2))$.

We have that[11] $e = (\rho_{=p}, \rho_{=p+1}) \in \mathsf{Tab}(\rho_{=p}, (m_1, m_2))$ with $m_i = \sigma_i(\rho_{\le p})$, which is equal to $\bot$ or to some (d_i, δ_i). Let $j = \mathsf{Owner}(e)$. We have that $d_j = \min\{d_i \mid m_i \neq \bot\}$, and if, for every player i, we write r_i for the region reached after delaying d_i time units at the end of $\rho_{\le p}$, we have that $r_j \le r_{3-j}$.

We have already noticed that $\pi_R^{-1}(\pi_R(\rho)_{\le p}) = \rho_{\le p}$ because it is a prefix of ρ. Hence, for every player i, $\sigma_i(\pi_R^{-1}(\pi_R(\rho)_{\le p})) = \sigma_i(\rho_{\le p}) = m_i$. Then, applying the definition of $\lambda_\rho(\sigma_i)$, we get that:

$$m'_i \stackrel{\text{def}}{=} \lambda_\rho(\sigma_i)(\pi_R(\rho)_{\le p}) = \begin{cases} (r_i, \delta_i) & \text{if } m_i \neq \bot \\ \bot & \text{if } m_i = \bot \end{cases}$$

The region r_j is minimal among the regions proposed by the two players. We deduce that

$$(\pi_R(\rho)_{=p}, \pi_R(\rho)_{=p+1}) \in \mathsf{Tab}(\pi_R(\rho)_{=p}, (m'_1, m'_2))$$

□

Lemma 16. *If σ_i is a strategy of player i, then*

$$\mathsf{Out}_{\mathcal{R}}(\lambda_\rho(\sigma_i)) \subseteq \pi_R(\mathsf{Out}_{\mathcal{G}}(\sigma_i))$$

Proof. For a play γ in $\mathcal{R}$, and a history h in $\mathcal{G}$ such that $\pi_R(h) = \gamma_{\le p}$, define v the valuation of the clocks at the end of h, and define d_i such that $\sigma_i(\rho_{\le p}) = (d_i, \delta_i)$ and r_i the region corresponding to valuation $v + d_i$. $(\gamma_{=p}, \gamma_{=p+1}) \in \mathsf{Tab}(\gamma_{=p}, \lambda(\sigma_i)(\gamma_{\le p}))$ so there is $m_j = (r_j, \delta_j)$ or $\bot$ such that $(\gamma_{=p}, \gamma_{=p+1}) \in \mathsf{Tab}(\gamma_{=p}, ((r_i, \delta_i), m_j))$

[11] We write $\rho_{=p}$ for the p-th state of ρ.

– If $m_j = \bot$, then $(h_{=p}, s) \in \mathsf{Tab}_{\mathcal{G}}(h_{=p}, ((d_i, \delta_i), \bot))$ with $\pi_R(s) = \gamma_{=p+1}$.
– Else
  - assume that $r_i \neq r_j$. Then $(h_{=p}, s) \in \mathsf{Tab}_{\mathcal{G}}(h_{=p}, ((d_i, \delta_i), (f_1(v, r_j), \delta_j)))$ with $\pi_R(s) = \gamma_{=p+1}$.
  - assume that $r_i = r_j$. Then $(h_{=p}, s) \in \mathsf{Tab}_{\mathcal{G}}(h_{=p}, ((d_i, \delta_i), (d_i, \delta_j)))$ with $\pi_R(s) = \gamma_{=p+1}$.

In all cases $(h_{=p}, s) \in \mathsf{Tab}_{\mathcal{G}}(h_{=p}, (d_i, \delta_i))$ with $\pi_R(s) = \gamma_{=p+1}$. There is a run ρ in $\mathsf{Out}_{\mathcal{G}}(\sigma_i)$ such that $\pi_R(\rho) = \gamma$ which is sufficient to get the result. □

The following lemma will imply one direction of Proposition 7:

Lemma 17. *If $((\sigma_1, \sigma_2), \rho)$ is a pseudo Nash equilibrium in the timed game $\mathcal{G}$, then $((\lambda_\rho(\sigma_1), \lambda_\rho(\sigma_2)), \pi_R(\rho))$ is a pseudo Nash equilibrium in $\mathcal{R}$.*

Proof. Assume that $((\sigma_1, \sigma_2), \rho)$ is a pseudo Nash equilibrium in game $\mathcal{G}$. Thanks to Lemma 15, we know that $\pi_R(\rho)$ is a possible outcome of the profile $(\lambda_\rho(\sigma_1), \lambda_\rho(\sigma_2))$. We prove that $((\lambda_\rho(\sigma_1), \lambda_\rho(\sigma_2)), \pi_R(\rho))$ is a pseudo Nash equilibrium in game $\mathcal{R}$.

Towards a contradiction assume that there is some player i which can improve her strategy. Then there exists some outcome $\gamma \in \mathsf{Out}_{\mathcal{R}}(\lambda_\rho(\sigma_{3-i}))$ with $\gamma \succ_i \pi_R(\rho)$. Now due to Lemma 16 it is the case that $\gamma \in \pi_R(\mathsf{Out}_{\mathcal{G}}(\sigma_{3-i}))$. There exists some $\rho' \in \mathsf{Out}_{\mathcal{G}}(\sigma_{3-i})$ such that $\pi_R(\rho') = \gamma$. The preferences are invariant by region (and by projection π_R), which implies that $\rho' \succ_i \rho$. This contradicts the fact that $(\sigma_1, \sigma_2, \rho)$ is a pseudo Nash equilibrium.

Hence $((\lambda_\rho(\sigma_1), \lambda_\rho(\sigma_2)), \pi_R(\rho))$ is a pseudo Nash equilibrium in $\mathcal{R}$. □

From region games $\mathcal{R}$ to timed game $\mathcal{G}$. Let γ be a play in $\mathcal{R}$. If α_i is a player-i strategy in $\mathcal{R}$, we define the player-i strategy $\lambda_\gamma^{-1}(\alpha_i)$ in $\mathcal{G}$ as follows: for every history h in $\mathcal{G}$, letting (ℓ, v) be the last configuration of h, we define:

– if $\alpha_i(\pi_R(h)) = \bot$, then $\lambda_\gamma^{-1}(\alpha_i)(h) = \bot$;
– if $\alpha_i(\pi_R(h)) = (r, \delta)$,
  - if $\pi_R(h) = \gamma_{\leq j}$ is a prefix of γ, $e = (\gamma_{=j}, \gamma_{=j+1})$ and $\mathsf{Owner}(e) \neq i$, we set $\lambda_\gamma^{-1}(\alpha_i)(h) = (f_2(v, r), \delta)$;
  - otherwise we set $\lambda_\gamma^{-1}(\alpha_i)(h) = (f_1(v, r), \delta)$.

Lemma 18. *Let (α_1, α_2) be a strategy profile in game $\mathcal{R}$.*

$$\text{If } \gamma \in \mathsf{Out}_{\mathcal{R}}(\alpha_1, \alpha_2) \text{ then } \gamma \in \pi_R(\mathsf{Out}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_1), \lambda_\gamma^{-1}(\alpha_2)))$$

Proof. By hypothesis we have that $e = (\gamma_{=p}, \gamma_{=p+1}) \in \mathsf{Tab}(\gamma_{=p}, (\alpha_1(\gamma_{\leq p}), \alpha_2(\gamma_{\leq p})))$. Let $i = \mathsf{Owner}(e)$. Then it is the case that $\alpha_i(\gamma_{\leq p}) = (r_i, \delta_i)$ for some appropriate r_i and δ_i, and that $\alpha_{3-i}(\gamma_{\leq p}) = (r_{3-i}, \delta_{3-i})$ with $\delta_{3-i} > \delta_i$ or $\alpha_{3-i}(\gamma_{\leq p}) = \bot$.

By definition of λ_γ^{-1}, if v is the last valuation in h, we have that $\lambda_\gamma^{-1}(\alpha_i)(h) = (f_1(v, r_i), \delta_i)$, and either $\lambda_\gamma^{-1}(\alpha_{3-i})(h) = (f_k(v, r_{3-i}), \delta_{3-i})$ for some $k \in \{1; 2\}$, or $\lambda_\gamma^{-1}(\alpha_{3-i})(h) = \bot$. In all cases the transition δ_i is one that is selected. So there is a transition e in $\mathsf{Tab}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_1)(h), \lambda_\gamma^{-1}(\alpha_1)(h))$ such that $\pi_R(e) = \delta_i$. This is enough to get the result. □

Lemma 19. *Let α_i be a player-i strategy in game $\mathcal{R}$. Then,*

$$\pi_R(\mathit{Out}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_i)) \subseteq \mathit{Out}_{\mathcal{R}}(\alpha_i)$$

Proof. Take $\rho \in \mathsf{Out}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_i))$. $m_i = \alpha_i(\pi_R(\rho_{\leq p})) = (r_i, \delta_i)$ or $\bot$, we have that $\lambda_\gamma^{-1}(\alpha_i)(\rho_{\leq p}) = (d_i, \delta_i)$ or $\bot$ where $v + d_i \in r_i$ if v is the clock valuation at the end of $\rho_{\leq p}$. There is $m_{3-i} = (d_{3-i}, \delta_{3-i})$ such that $v + d_{3-i}$ satisfies the constraints on δ_{3-i} or in the case where $\alpha_i(\pi_R(\rho_{\leq p})) \neq \bot$ it is possible that there is no allowed action for player $3-i$, in which case $m_{3-i} = \bot$. Either $d_i \leq d_{3-i}$ (or $m_{3-i} = \bot$) and $\rho_{=p+1} = (l_i, [z_i \leftarrow 0](v + d_i))$ where $\delta_i = (l, g_i, z_i, l_i)$, or $d_i \geq d_{3-i}$ (or $m_i = \bot$) and $\rho_{=p+1} = (l_{3-i}, [z_{3-i} \leftarrow 0](v + d_{3-i}))$ where $\delta_{3-i} = (l, g_{3-i}, z_{3-i}, l_{3-i})$. We will write r_{3-i} the region corresponding to the valuation $v + d_{3-i}$, we have that $(r_{3-i}, \delta_{3-i}) \in \mathsf{Mov}_{3-i}(\pi_R(\rho_{=p}))$. There are three cases :

- If $d_i < d_{3-i}$ or $m_{3-i} = \bot$, then $r_i \leq r_{3-i}$ and $((l, r), (l_i, [z_i \leftarrow 0]r_i)) \in \mathsf{Tab}((l, r), ((r_i, \delta_i), m_{3-i}))$.
- If $d_{3-i} < d_i$ or $m_i = \bot$, then $r_{3-i} \leq r_i$ and $((l, r), (l_{3-i}, [z_{3-i} \leftarrow 0]r_{3-i})) \in \mathsf{Tab}((l, r), (m_i, (r_{3-i}, \delta_{3-i})))$.
- Else $d_j = d_i$, $r_j = r_i$ and both transitions belongs to $\mathsf{Tab}((l, r), ((r_i, \delta_i), (r_j, \delta_j)))$.

In all the cases we get that $\pi_R(\rho_{=p}, \rho_{=p+1}) \in \mathsf{Tab}(\pi_R(\rho_{=p}), (m_i, m_{3-i}))$. □

We are now ready to prove this next lemma, which will imply the second implication of Proposition 7.

Lemma 20. *If $((\alpha_1, \alpha_2), \gamma)$ is a pseudo Nash equilibrium in $\mathcal{R}$, then $((\lambda_\gamma^{-1}(\alpha_1), \lambda_\gamma^{-1}(\alpha_2)), \pi_R^{-1}(\gamma))$ is a pseudo Nash equilibrium in $\mathcal{G}$.*

Proof. Assume that $(\alpha_1, \alpha_2, \gamma)$ is a pseudo Nash equilibrium in $\mathcal{R}$. Assume however that $(\lambda_\gamma^{-1}(\alpha_1), \lambda_\gamma^{-1}(\alpha_2))$ is not a pseudo Nash equilibrium in $\mathcal{G}$. This means that for every play ρ in $\mathsf{Out}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_1), \lambda_\gamma^{-1}(\alpha_2))$, there is $i \in \{1, 2\}$ such that there is σ_i a strategy for i in $\mathcal{G}$ and some $\rho' \in \mathsf{Out}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_{3-i}), \sigma_i)$ with $\rho' \succ_i \rho$.

We have that $\gamma \in \pi_R(\mathsf{Out}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_1), \lambda_\gamma^{-1}(\alpha_2))$ (due to Lemma 18). Assume however that $(\lambda_\gamma^{-1}(\alpha_1), \lambda_\gamma^{-1}(\alpha_2), \pi_R^{-1}(\gamma))$ is not a pseudo-Nash equilibrium in $\mathcal{G}$. This means that there is $i \in \{1, 2\}$ and some $\rho' \in \mathsf{Out}_{\mathcal{G}}(\lambda_\gamma^{-1}(\alpha_{3-i}))$ with $\rho' \succ_i \pi_R^{-1}(\gamma)$. Now applying Lemma 19, it is the case that $\pi_R(\rho') \in \mathsf{Out}_{\mathcal{R}}(\alpha_{3-i})$. As the preference relation is invariant by region, we get that $\pi_R(\rho') \succ_i \pi_R(\rho)$. This contradicts the fact that $(\alpha_1, \alpha_2, \gamma)$ is a pseudo-Nash equilibrium in $\mathcal{R}$.

Thus, $(\lambda_\gamma^{-1}(\alpha_1), \lambda_\gamma^{-1}(\alpha_2), \pi_R^{-1}(\gamma))$ is a pseudo Nash equilibrium in $\mathcal{G}$. □

C Proof of Section 3.2

In this section, we prove the correspondence bitween pseudo Nash equilibria in $\mathcal{R}$ and twin pseudo Nash equilibria in $\mathcal{R}_1$ and $\mathcal{R}_2$:

Proposition 9. *Let $\mathcal{R}$ be the region game associated with some timed game $\mathcal{G}$. Then there is a pseudo Nash equilibrium in $\mathcal{R}$ from s with best play γ if and only if there is a twin pseudo equilibrium for the corresponding games $\mathcal{R}_1$ and $\mathcal{R}_2$ from s with best play γ. Furthermore this equivalence is constructive.*

To establish this result, we first prove the following property on game $\mathcal{R}$:

Property $(\star)$: If $(s,s') \in \mathsf{Tab}(s,(m_1,m_2)) \cap \mathsf{Tab}(s,(m'_1,m'_2))$, then there exists $(m''_1,m''_2) \in \mathsf{Mov}(s,1) \times \mathsf{Mov}(s,2)$ such that:

$$\begin{cases} (s,s') \in \mathsf{Tab}(s,(m''_1,m''_2)) \\ \mathsf{Tab}(s,m''_1) \subseteq \mathsf{Tab}(s,m'_1) \\ \mathsf{Tab}(s,m''_2) \subseteq \mathsf{Tab}(s,m_2) \end{cases}$$

where $\mathsf{Tab}(s,m_j) = \{(s,s') \mid \exists m_{3-j} \in \mathsf{Mov}(s,3-j).\ (s,s') \in \mathsf{Tab}(s,(m_j,m_{3-j}))\}$.

This property expresses that if two different moves lead to the same state, we can choose one that is more restrictive in terms of possible outcomes. In timed game, this corresponds to choose the shortest delay.

Lemma 21. *The region game constructed in subsection 3.1 satisfies Property* $(\star)$*.*

Proof. First notice that there is a partial order on the set of actions Act defined by: $(r,\delta) \le (r',\delta')$ is $r = r'$ or r' is a time successor of r, and $(r,\delta) \le \bot$ (we will also say that $(r,\delta) < (r',\delta')$ if $(r,\delta) \le (r',\delta')$ and $r \ne r'$). We will write $t((l,r),(r_1,\delta_1))$ for the transition $((l,r),(l',r'))$ where $\delta_1 = (l,g_1,Y_1,l')$, r_1 is a region succussor of r satisfying the constraint g_1 and $r' = [Y_1 \leftarrow 0]r_1$. We can notice that if $m_i, m'_i \in \mathsf{Mov}(s,i)$ is such that $m_i \le m'_i$, then $\mathsf{Tab}(s,m_i) \subseteq \mathsf{Tab}(s,m'_i) \cup t(s,m_i)$.

Now assume that there is $(s,s') \in \mathsf{Tab}(s,(m_1,m_2)) \cap \mathsf{Tab}(s,(m'_1,m'_2))$. If $r_1 \le r'_1$ and $t(s,m_1) = (s,s')$ then take $m''_1 = m_1$, otherwise take $m''_1 = m'_1$. This insure that $\mathsf{Tab}(s,m''_1) \subseteq \mathsf{Tab}(s,m'_1)$. We do the same for m''_2, . With this choice we have that the minimum of the actions among $\{m \in \{m_1,m_2,m'_1,m'_2\} \mid t(s,m) = (s,s')\}$ get selected, therefore we have that $(s,s') \in \mathsf{Tab}(s,m''_1,m''_2)$. □

For each (s,s',m_1,m_2,m'_1,m'_2), we fix a pair (m''_1,m''_2) satisfying the conditions of Property $(\star)$, which we denote by $\mu((s,s'),(m_1,m_2),(m'_1,m'_2))$.

First, it is obvious that if $((\alpha_1,\alpha_2),\gamma)$ is a pseudo Nash equilibrium in $\mathcal{R}$ then it is also a pseudo Nash equilibrium in $\mathcal{R}_1$ and $\mathcal{R}_2$, thus a twin equilibrium.

Assume now that we have a twin equilibrium $((\alpha_1^{\mathcal{R}_1},\alpha_2^{\mathcal{R}_1}),(\alpha_1^{\mathcal{R}_2},\alpha_2^{\mathcal{R}_2}),\gamma)$. We construct the strategy profile (α_1,α_2) as follows :

- if h is a prefix of γ then

  $$(\alpha_1(h),\alpha_2(h)) = \mu(\mathsf{last}(h),\gamma_{=|h|+1},\alpha_1^{\mathcal{R}_1}(h),\alpha_2^{\mathcal{R}_1}(h),\alpha_1^{\mathcal{R}_2}(h),\alpha_2^{\mathcal{R}_2}(h))$$

  This is correctly defined because $(\gamma_{=|h|},\gamma_{=|h+1|})$ is in the intersection of $\mathsf{Tab}(s,(\alpha_1^{\mathcal{R}_1}(h),\alpha_2^{\mathcal{R}_1}(h)))$ and $\mathsf{Tab}(s,(\alpha_1^{\mathcal{R}_2}(h),\alpha_2^{\mathcal{R}_2}(h)))$.
- otherwise $\alpha_1(h) = \alpha_1^{\mathcal{R}_2}(h)$ and $\alpha_2(h) = \alpha_2^{\mathcal{R}_1}(h)$

Lemma 22. $\gamma \in Out_{\mathcal{R}}(\alpha_1,\alpha_2)$.

Proof. Let k be a natural integer, (γ_k,γ_{k+1}) belongs to $\mathsf{Tab}(\gamma_{=k},(\alpha_1^{\mathcal{R}_1}(\gamma_{\le k}),\alpha_2^{\mathcal{R}_1}(\gamma_{\le k})))$ and to $\mathsf{Tab}(\gamma_{=k},(\alpha_1^{\mathcal{R}_2}(\gamma_{\le k}),\alpha_2^{\mathcal{R}_2}(\gamma_{\le k})))$. So, obviously

$$(\gamma_{=k},\gamma_{=k+1}) \in \mathsf{Tab}(\gamma_{=k},\mu(\gamma_{=k},\gamma_{=k+1},\alpha_1^{\mathcal{R}_1}(\gamma_{\le k}),\alpha_2^{\mathcal{R}_1}(\gamma_{\le k}),\alpha_1^{\mathcal{R}_2}(\gamma_{\le k}),\alpha_2^{\mathcal{R}_2}(\gamma_{\le k}))).$$

This proves that $\gamma \in Out_{\mathcal{R}}(\alpha_1,\alpha_2)$. □

Lemma 23. $Out_{\mathcal{R}}(\alpha_1) \subseteq \mathsf{Out}_{\mathcal{R}_2}(\alpha_1^{\mathcal{R}_2})$

Proof. On the path γ this is true because γ is a possible outcome of $\alpha_1^{\mathcal{R}_2}$, and outside this is obvious by the definition of α_1. □

Assume $((\alpha_1^{\mathcal{R}_2}, \alpha_2^{\mathcal{R}_2}), \gamma)$ is a pseudo Nash equilibrium in the game $\mathcal{R}_2$. By Lemma 22 γ is a possible outcome of the profile (α_1, α_2). From Lemma 23 and the fact that $\forall \gamma' \in Out_{\mathcal{R}_2}(\alpha_1^{\mathcal{R}_2}).\ \gamma' \preccurlyeq_2 \gamma$, we can deduce that $\forall \gamma' \in Out_{\mathcal{R}}(\alpha_1).\ \gamma' \preccurlyeq_2 \gamma$. Thus player 2 can not improve her strategy. Symmetrically for player 1.

This shows that if $((\alpha_1^{\mathcal{R}_1}, \alpha_2^{\mathcal{R}_1}), (\alpha_1^{\mathcal{R}_2}, \alpha_2^{\mathcal{R}_2}), \gamma)$ is a twin equilibrium then $((\alpha_1, \alpha_2), \gamma)$ is a pseudo Nash equilibrium in the game $\mathcal{R}$.

D Proof of Section 3.3

In this appendix we prove the correctness of the deterministic concurrent game $\mathcal{C}_i$ w.r.t. game $\mathcal{R}_i$.

Proposition 10. *Assume $\mathcal{C}_i$ (with $i \in \{1, 2\}$) is the deterministic concurrent game defined from the concurrent game $\mathcal{R}_i$. Then there is a pseudo Nash equilibrium in $\mathcal{R}_i$ from s with best play γ iff there is a Nash equilibrium in $\mathcal{C}_i$ from s with best play γ. Furthermore this equivalence is constructive.*

We prove this result for $\mathcal{C}_1$ w.r.t. game $\mathcal{R}_1$, the other case being symmetric.

We first give a transformation of the strategies in games $\mathcal{C}_i$ to strategies in the original game $\mathcal{R}_i$. Formally, we define the projection $\lambda^{\mathcal{R}} \colon \mathsf{Act}' \to \mathsf{Act}$ by $\lambda^{\mathcal{R}}(m) = m$, and $\lambda^{\mathcal{R}}(m, \star) = m$ where $\star \in \{\bullet, \circ\}$, and extend it to strategies in a straightforward way: if ι is a strategy, the strategy $\lambda^{\mathcal{R}}(\iota)$ is defined by $\lambda^{\mathcal{R}}(\iota)(\gamma) = \lambda^{\mathcal{R}}(\iota(\gamma))$ for every history γ.

We now give a transformation in the other direction. We let γ be a play in $\mathcal{R}_1$. Given two strategies α_1 and α_2 (for both players) in game $\mathcal{R}_1$, we define the strategies $\lambda^{\mathcal{C}}_{\gamma}(\alpha_j)$ as follows:

– for a prefix $\gamma_{\leq p}$ of γ:

$$\lambda^{\mathcal{C}}_{\gamma}(\alpha_j)(\gamma_{\leq p}) = \begin{cases} (m_j, \star_j) & \text{if } j = 1 \\ m_j & \text{if } j = 2 \end{cases}$$

where $\mathsf{Tab}_1(\gamma_{=p}, ((m_1, \star_1), m_2)) = \{(\gamma_{=p}, \gamma_{=p+1})\}$.

– for a history γ' which is not a prefix of γ:

$$\lambda^{\mathcal{C}}_{\gamma}(\alpha_j)(\gamma') = \begin{cases} (\alpha_j(\gamma'), \bullet)^{12} & \text{if } j = 1 \\ \alpha_j(\gamma') & \text{if } j = 2 \end{cases}$$

Lemma 24. – *For every strategy α in game $\mathcal{R}_1$, $\lambda^{\mathcal{R}} \circ \lambda^{\mathcal{C}}_{\gamma}(\alpha) = \alpha$.*

– *Furthermore, for every player-2 strategy α_2 in game $\mathcal{R}_1$, $\lambda^{\mathcal{C}}_{\gamma}(\alpha_2) = \alpha_2$. In particular, the strategy $\lambda^{\mathcal{C}}_{\gamma}(\alpha_2)$ is independent on γ.*

[12] This choice is arbitrary, and we could have chosen ∘ instead of •.

– *For every player-2 strategy ι_2 in game $\mathcal{C}_1$, $\lambda^{\mathcal{R}}(\iota_2) = \iota_2$.*

Proof. The two first points are obvious by definition of $\lambda^{\mathcal{R}}$ and $\lambda^{\mathcal{C}}_\gamma$, and the last one is a consequence of the two first points. $\square$

Lemma 25. *If $\gamma \in \mathsf{Out}_{\mathcal{C}_1}(\iota_1, \iota_2)$ then*

$$\gamma \in \mathsf{Out}_{\mathcal{R}_1}(\lambda^{\mathcal{R}}(\iota_1), \lambda^{\mathcal{R}}(\iota_2))$$

Proof. Let γ be an outcome in $\mathsf{Out}_{\mathcal{C}_1}(\iota_1, \iota_2)$, assume that $\iota_1(\gamma_{\leq p}) = (m_1, \star_1)$ and that $\iota_2(\gamma_{\leq p}) = m_2$. It means that $\mathsf{Tab}_1(\gamma_{=p}, ((m_1, \star_1), m_2) = \{(\gamma_{=p}, \gamma_{=p+1})\}$. In particular, $(\gamma_{=p}, \gamma_{=p+1}) \in \mathsf{Tab}(\gamma_{=p}, (m_1, m_2))$, which is precisely what we were meant at proving, because $\lambda^{\mathcal{R}}(\iota_j)(\gamma_{\leq p}) = m_j$ (for $j \in \{1, 2\}$). $\square$

Lemma 26. *If $\gamma \in \mathsf{Out}_{\mathcal{R}_1}(\alpha_1, \alpha_2)$ then*

$$\gamma \in \mathsf{Out}_{\mathcal{C}_1}(\lambda^{\mathcal{C}}_\gamma(\alpha_1), \lambda^{\mathcal{C}}_\gamma(\alpha_2))$$

Proof. Examine the definition of $\lambda^{\mathcal{C}}_\gamma$: $\lambda^{\mathcal{C}}_\gamma(\alpha_1)(\gamma_{\leq p}) = (m_1, \star_1)$ and $\lambda^{\mathcal{C}}_\gamma(\alpha_2(\gamma_{\leq p}) = m_2$, where $\mathsf{Tab}_1(\gamma_{=p}, ((m_1, \star_1), m_2) = \{(\gamma_{=p}, \gamma_{=p+1})\}$. Therefore $\gamma \in \mathsf{Out}_{\mathcal{C}_1}(\lambda^{\mathcal{C}}_\gamma(\alpha_1), \lambda^{\mathcal{C}}_\gamma(\alpha_2))$. $\square$

Lemma 27. $\mathsf{Out}_{\mathcal{R}_1}(\lambda^{\mathcal{R}}(\iota_2)) \subseteq \mathsf{Out}_{\mathcal{C}_1}(\iota_2)$.

Proof. Using Lemma 26 $\mathsf{Out}_{\mathcal{R}_1}(\lambda^{\mathcal{R}}(\iota_2)) \subseteq \mathsf{Out}_{\mathcal{C}_1}(\lambda^{\mathcal{C}}_\gamma(\lambda^{\mathcal{R}}(\iota_2)))$. And as $\lambda^{\mathcal{C}}_\gamma$ and $\lambda^{\mathcal{R}}$ are just the identity function for player 2 strategies, $\mathsf{Out}_{\mathcal{C}_1}(\lambda^{\mathcal{C}}_\gamma(\lambda^{\mathcal{R}}(\iota_2))) = \mathsf{Out}_{\mathcal{C}_1}(\iota_2)$. $\square$

Lemma 28. *If $\gamma \in \mathsf{Out}_{\mathcal{R}_1}(\alpha_1, \alpha_2)$ then*

$$\mathsf{Out}_{\mathcal{C}_1}(\lambda^{\mathcal{C}}_\gamma(\alpha_2)) \subseteq \mathsf{Out}_{\mathcal{R}_1}(\alpha_2)$$

Proof. Using Lemma 25 with the strategy $\lambda^{\mathcal{C}}_\gamma(\alpha_2)$, $\mathsf{Out}_{\mathcal{C}_1}(\lambda^{\mathcal{C}}_\gamma(\alpha_2)) \subseteq \mathsf{Out}_{\mathcal{R}_1}(\lambda^{\mathcal{R}}(\lambda^{\mathcal{C}}_\gamma(\alpha_2)))$. And as $\lambda^{\mathcal{R}} \circ \lambda^{\mathcal{C}}_\gamma(\alpha) = \alpha$, $\mathsf{Out}_{\mathcal{R}_1}(\lambda^{\mathcal{R}}(\lambda^{\mathcal{C}}_\gamma(\alpha_2))) = \mathsf{Out}_{\mathcal{R}_1}(\alpha_2)$. $\square$

Assume that $((\alpha_1, \alpha_2), \gamma)$ is a pseudo Nash equilibrium in the game $\mathcal{R}_1$, then $\gamma \in \mathsf{Out}_{\mathcal{C}_1}(\lambda^{\mathcal{C}}_\gamma(\alpha_1), \lambda^{\mathcal{C}}_\gamma(\alpha_2))$ because of Lemma 26 and player 1 can not improve her strategy because of Lemma 28. Therefore $((\lambda^{\mathcal{C}}_\gamma(\alpha_1), \lambda^{\mathcal{C}}_\gamma(\alpha_2)), \gamma)$ is a Nash equilibrium in game $\mathcal{C}_1$. The proof in the other direction is the same using Lemma 25 and Lemma 27.

E Proof of Section 3.4

In this appendix we prove the correctness of the transformation into two turn-based games:

Proposition 11. *Let $\mathcal{C}_i$ (with $i \in \{1, 2\}$) be the previous deterministic concurrent game, and let $\mathcal{T}_i$ be the associated turn-based game. There is a Nash equilibrium in $\mathcal{C}_i$ from s with best play $\psi_i(\nu)$ iff there is a Nash equilibrium in $\mathcal{T}_i$ from $(s, [m]_s)$ with best play ν, where m is a minimal action w.r.t. $<_s$. Furthermore this equivalence is constructive.*

As for the two previous transformations we define transformations from strategies in game $\mathcal{C}_i$ to and from strategies in game $\mathcal{T}_i$.

From concurrent to turn-based games. Let α_j be a player-j strategy in game $\mathcal{C}_i$. We consider a history ν such that

$$\nu = (s_1, \mathfrak{m}_1^1)(s_1, \mathfrak{m}_1^2) \dots (s_1, \mathfrak{m}_1^{k_1})(s_2, \mathfrak{m}_2^1) \dots (s_2, \mathfrak{m}_2^{k_2}) \dots (s_p, \mathfrak{m}_p^1) \dots (s_p, \mathfrak{m}_p^k)$$

Note that by construction of the transition table Tab'_i, for every $1 \leq h \leq p$, $\mathfrak{m}_h^1$ is minimal w.r.t. $<_{s_h}$ and the sequence $(\mathfrak{m}_h^k)_{1 \leq k \leq k_h}$ is increasing (w.r.t. $<_{s_h}$): $\mathfrak{m}_h^{k+1}$ is next after $\mathfrak{m}_h^k$ w.r.t. $<_{s_h}$. Then define

$$\tau(\alpha_j)(\nu) = \begin{cases} \bot & \text{if } \mathfrak{m}_p^k \cap \mathsf{Mov}_i(s_p, j) = \varnothing \\ \mathsf{del} & \text{if } \mathfrak{m}_p^k <_{s_p} [\alpha_j(\psi(\nu))]_{s_p} \\ \alpha_j(\psi(\nu)) & \text{if } \alpha_j(\psi(\nu)) \in \mathfrak{m}_p^k \\ m & \text{if } \mathfrak{m}_p^k >_{s_p} [\alpha_j(\psi(\nu))]_{s_p} \text{ with } m \in \mathfrak{m}_p^k \text{ chosen arbitrarly} \end{cases}$$

Lemma 29. *Assume $(\alpha_j)_{j=1,2}$ are player-j strategies in game $\mathcal{C}_i$. Then*

$$\psi(\mathit{Out}_{\mathcal{T}_i}(\tau(\alpha_1), \tau(\alpha_2))) = \mathit{Out}_{\mathcal{C}_i}(\alpha_1, \alpha_2).$$

Proof. Let ν be the outcome of the strategy profile $(\tau(\alpha_1), \tau(\alpha_2))$. We consider a position q in the history where the next state will be made of a minimal action : $(\nu_{=q}, \nu_{=q+1}) = ((s_p, \mathfrak{m}_p^{k_p}), (s_{p+1}, \mathfrak{m}_{p+1}^1))$. We have that $\tau(\alpha_j)(\nu_{\leq q'}) = \mathsf{del}$ or $\bot$ for all q' among the k_p last positions of $\nu_{\leq q}$. This imply that $\mathfrak{m}_p^k \leq \alpha_j(\psi(\nu_{\leq q}))$ for $j \in \{1, 2\}$. We also have that for one $j \in \{1, 2\}$, $\tau(\alpha_j)(\nu_{\leq q}) = m_j \in \mathfrak{m}_p^{k_p}$, by definition of τ it has to be equal to $\alpha_j(\psi(\nu_{\leq q}))$. By definition of Tab'_i, s_{p+1} is such that $(s_p, s_{p+1}) \in \mathsf{Tab}'_i(s, (m_j, m''))$ for all $m'' >_{s_p} m_j$, which is the case for $m'' = \alpha_{3-j}(\psi(\nu_{\leq q}))$. Therefore, $(s_p, s_{p+1}) \in \mathsf{Tab}_i(s_p, (\alpha_1(s_{\leq p}), \alpha_2(s_{\leq p})))$. As the games are deterministic, a strategy profile only allow one outcome, we can conclude with the equality $\psi(\mathsf{Out}_{\mathcal{T}_i}(\tau(\alpha_1), \tau(\alpha_2))) = \mathsf{Out}_{\mathcal{C}_i}(\alpha_1, \alpha_2)$. □

Lemma 30. *Assume α_j is a player-j strategy in game $\mathcal{C}_i$. Then*

$$\psi(\mathit{Out}_{\mathcal{T}_i}(\tau(\alpha_j))) \subseteq \mathit{Out}_{\mathcal{C}_i}(\alpha_j).$$

Proof. Once again we considere a position before a state with a minimal action : $(\nu_{=q}, \nu_{=q+1}) = ((s_p, \mathfrak{m}_p^{k_p}), (s_{p+1}, \mathfrak{m}_{p+1}^1))$. We know that we are in one of this three cases :

- $\alpha_j(\psi(\nu_{\leq q})) = \bot$ and $\mathfrak{m}_p^{k_p} \subseteq \mathsf{Mov}(s_p, 3-j)$.
- $\alpha_j(\psi(\nu_{\leq q})) = m_j \in \mathfrak{m}_p^{k_p}$ and their exists an action $m' \in \mathsf{Mov}(s_p, 3-j)$ such that $m' >_s m_j$.
- $\alpha_j(\psi(\nu_{\leq q})) >_s \mathfrak{m}_p^{k_p}$ and $\mathfrak{m}_p^{k_p} \subseteq \mathsf{Mov}(s_p, 3-j)$.

In all these cases $(s_p, s_{p+1}) \in \mathsf{Tab}(s_p, \alpha_j(s_{\leq p}))$, therefore we can conclude that $\psi(\nu) \in \mathsf{Out}_{\mathcal{C}_i}(\alpha_j)$. □

Proposition 31. *If (α_1, α_2) is a Nash equilibrium in game $\mathcal{C}_i$, then $(\tau(\alpha_1), \tau(\alpha_2))$ is a Nash equilibrium in game $\mathcal{T}_i$.*

Proof. Write ν for the play such that $\{\nu\} = \mathsf{Out}_{\mathcal{T}_i}(\tau(\alpha_1), \tau(\alpha_2))$. Then applying Lemma 29 we have that $\{\psi^{-1}(\nu)\} = \mathsf{Out}_{\mathcal{C}_i}(\alpha_1, \alpha_2)$. No player can improve her strategy because of Lemma 30. We thus get the expected result: $(\tau(\alpha_1), \tau(\alpha_2))$ is a Nash equilibrium in game $\mathcal{T}_i$ as soon as (α_1, α_2) is a Nash equilibrium in $\mathcal{C}_i$. □

From turn-based to concurrent games. Let β be a player-j strategy in game $\mathcal{T}_i$, and fix a play ν in game $\mathcal{T}_i$. We will write

$$\nu = (s_1, \mathfrak{m}_1^1)(s_1, \mathfrak{m}_1^2) \dots (s_1, \mathfrak{m}_1^{k_1})(s_2, \mathfrak{m}_2^1) \dots (s_2, \mathfrak{m}_2^{k_2}) \dots (s_p, \mathfrak{m}_p^1) \dots (s_p, \mathfrak{m}_p^{k_p}) \dots$$

We define the player-j strategy $\tau_\nu^{-1}(\beta)$ in game $\mathcal{C}_i$ as follows. Pick a history γ in $\mathcal{C}_i$. We distinguish between two cases:

- either γ is a prefix of $\psi(\nu)$, in which case we take p such that $\gamma = s_1 s_2 \dots s_p$ and define $\tau_\nu^{-1}(\beta)(\gamma)$ to be the minimum (with respect to the order $<_{s_p}$) of the set

$$\left\{\beta((s_1, \mathfrak{m}_1^1) \dots (s_p, \mathfrak{m}_p^1) \dots (s_p, [m]_{s_p})) \mid m \in \mathsf{Mov}_i(s_p, j)\right\} \smallsetminus \{\mathsf{del}\}$$

- or γ is not a prefix of $\psi(\nu)$, in which case we take a history $\nu' = (s'_1, \mathfrak{m'}_1^1) \dots (s'_p, \mathfrak{m'}_p^1)$ in $\mathsf{Out}_{\mathcal{T}_i}(\beta)$, such that $\psi(\nu') = \gamma$, then define $\tau_\nu^{-1}(\beta)(\gamma)$ as the minimum of

$$\left\{\beta((s'_1, \mathfrak{m'}_1^1) \dots (s'_p, \mathfrak{m'}_p^1) \dots (s'_p, [m]_{s'_p})) \mid m \in \mathsf{Mov}_i(s'_p, j)\right\} \smallsetminus \{\mathsf{del}\}$$

Lemma 32. *Assume β_j are player-j strategies in game $\mathcal{T}_i$. If $\mathsf{Out}_{\mathcal{T}_i}(\beta_1, \beta_2) = \{\nu\}$, then $\mathsf{Out}_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_1), \tau_\nu^{-1}(\beta_2)) = \{\psi(\nu)\}$.*

Proof. We prove this result by showing that the transition $(\psi(\nu)_{=p}, \psi(\nu)_{=p+1})$ belongs to $\mathsf{Tab}_i(\psi(\nu)_{=p}, \tau_\nu^{-1}(\beta_1)(\psi(\nu)_{\leq p}), \tau_\nu^{-1}(\beta_2)(\psi(\nu)_{\leq p}))$. We write:

$$\nu = \nu_{\mathsf{init}} \cdot (\psi(\nu)_{=p}, \mathfrak{m}_p^1) \dots (\psi(\nu)_{=p}, \mathfrak{m}_p^{k_p})(\psi(\nu)_{=p+1}), \mathfrak{m}_{p+1}^1) \dots$$

With these notations, we have that there is $j \in \{1, 2\}$ such that:

$$\begin{cases} \beta_j(\nu_{\mathsf{init}} \cdot (\psi(\nu)_{=p}, \mathfrak{m}_p^1) \dots (\psi(\nu)_{=p}, \mathfrak{m}_p^h)) \in \{\perp, \mathsf{del}\} \text{ if } h < k_p \\ \beta_j(\nu_{\mathsf{init}} \cdot (\psi(\nu)_{=p}, \mathfrak{m}_p^1) \dots (\psi(\nu)_{=p}, \mathfrak{m}_p^{k_p})) = m_j \end{cases}$$

And we also have that $\mathsf{Tab}'_i((\psi(\nu)_{=p}, \mathfrak{m}_p^{k_p}), (m_j, \perp)) = \{(\psi(\nu)_{=p+1}, \mathfrak{m}_{p+1}^1)\}$, and

$$\begin{cases} \beta_{3-j}(\nu_{\mathsf{init}} \cdot (\psi(\nu)_{=p}, \mathfrak{m}_p^1) \dots (\psi(\nu)_{=p}, \mathfrak{m}_p^h)) \in \{\perp, \mathsf{del}\} \text{ if } h < k_p \\ \beta_{3-j}(\nu_{\mathsf{init}} \cdot (\psi(\nu)_{=p}, \mathfrak{m}_p^1) \dots (\psi(\nu)_{\leq p}, \mathfrak{m}_p^{k_p})) = \perp \end{cases}$$

Hence we get that

$$\begin{cases} \tau_\nu^{-1}(\beta_j)(\psi(\nu)_{\leq p}) = m_j \in \mathfrak{m}_p^{k_p} \\ \tau_\nu^{-1}(\beta_{3-j})(\psi(\nu)_{\leq p}) = \perp \text{ or } m \text{ with } m >_{\psi^{-1}(\nu)_{=p}} \mathfrak{m}_p^{k_p} \end{cases}$$

Thus $\mathsf{Tab}_i(\psi(\nu)_{=p}, \tau_\nu^{-1}(\beta_j)(\psi(\nu)_{\leq p}), \tau_\nu^{-1}(\beta_{3-j})(\psi(\nu)_{\leq p}))$ contains the transition $(\psi(\nu)_{=p}, \psi(\nu)_{=p+1})$. This is enough to show that $\psi(\nu) \in \mathsf{Out}_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_1), \tau_\nu^{-1}(\beta_2))$. □

Lemma 33. *If β_j is a strategy in game $\mathcal{T}_i$ and ν a play in $\mathcal{T}_i$, then $\mathsf{Out}_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_j)) \subseteq \psi(\mathsf{Out}_{\mathcal{T}_i}(\beta_j))$.*

Proof. We prove this result by induction on the length of the finite outcomes of β. Assume we have proven the results for the outcomes of length no more than p in $\mathsf{Out}^f_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_j))$. Assume that $\gamma \cdot s \in \mathsf{Out}^f_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_j))$ has length $p+1$ (hence $\gamma \in \mathsf{Out}^f_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_j))$ has length p).

Assume that $\tau_\nu^{-1}(\beta)(\gamma) = m$ then there exists a move m' in $\mathsf{Mov}_i(\mathsf{last}(\gamma), 3-j)$ such that $\mathsf{Tab}_i(\mathsf{last}(\gamma), (m, m')) = \{(\mathsf{last}(\gamma), s)\}$. We also have that there is a path $\nu' \in \mathsf{Out}^f_i(\beta_j)$ such that $\psi(\nu') = \gamma$ and $\tau_\nu^{-1}(\beta)(\gamma)$ is the minimum of the set

$$\Big\{\beta(\nu' \cdot (\mathsf{last}(\nu'), \mathfrak{m}'^2_{\mathsf{last}(\nu')}) \dots (\mathsf{last}(\nu'), [m]_{\mathsf{last}(\nu')})) \mid m \in \mathsf{Mov}_i(\mathsf{last}(\nu'), j)\Big\} \smallsetminus \{\mathsf{del}\}$$

Assume that $m < m'$, then for all classes $\mathfrak{m}^k_{s_p} <_{s_p} [m]_{s_p}$ we have $\beta(\nu' \dots (s_p, \mathfrak{m}^k_{s_p})) \in \{\perp, \mathsf{del}\}$. We also have that

$$\begin{cases} ((s_p, \mathfrak{m}^k_{s_p}), (s_p, \mathfrak{m}^k_{s_p})) \in \mathsf{Tab}'_i((s_p, \mathfrak{m}^k_{s_p}), (\mathsf{del}, \perp)) \\ ((s_p, [m]_{s_p}), (s, \mathfrak{m}^1_s)) \in \mathsf{Tab}'_i((s_p, [m]_{s_p}), (m, \perp)) \end{cases}$$

Therefore there is an history $\nu'' \in \mathsf{Out}^f_{\mathcal{T}_i}(\beta)$ such that $\psi(\nu'') = \gamma \cdot s$.

If $m > m'$, then $((s_p, [m']_{s_p}), (s, \mathfrak{m}^1_s)) \in \mathsf{Tab}'_i((s_p, [m']_{s_p}), (\perp, m'))$ and we can show the same result. ☐

Proposition 34. *Let (β_1, β_2) be a Nash equilibrium in game $\mathcal{T}_i$, and write ν for the unique outcome in $\mathsf{Out}_{\mathcal{T}_i}(\beta_1, \beta_2)$. Then $(\tau_\nu^{-1}(\beta_1), \tau_\nu^{-1}(\beta_2))$ is a Nash equilibrium in game $\mathcal{C}_i$.*

Proof. Write γ for $\{\gamma\} = \mathsf{Out}_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_1), \tau_\nu^{-1}(\beta_2))$. Applying Lemma 32, we have that $\gamma = \psi^{-1}(\nu)$.

Towards a contradiction assume that $(\tau_\nu^{-1}(\beta_1), \tau_\nu^{-1}(\beta_2))$ is not a Nash equilibrium in game $\mathcal{C}_i$. W.l.o.g. we assume that α_2 is a player-2 strategy in game $\mathcal{C}_i$ such that writing γ' for the unique outcome in $\mathsf{Out}_{\mathcal{C}_i}(\tau_\nu^{-1}(\beta_1), \alpha_2)$, we have that $\gamma <_2 \gamma'$. In particular, applying Lemma 33, we have that $\gamma' \in \psi^{-1}(\mathsf{Out}_{\mathcal{T}_i}(\beta_1))$. There exists $\nu' \in \mathsf{Out}_{\mathcal{T}_i}(\beta_1)$ with $\psi^{-1}(\nu') = \gamma'$. In particular, $\gamma =_2 \nu <_2 \nu' =_2 \gamma'$, which contradicts the fact that (β_1, β_2) is a Nash equilibrium in game $\mathcal{T}_i$.

Thus, $(\tau_\nu^{-1}(\beta_1), \tau_\nu^{-1}(\beta_2))$ is a Nash equilibrium in game $\mathcal{C}_i$ as soon as (β_1, β_2) is a Nash equilibrium in game $\mathcal{T}_i$. ☐

F Proofs of Section 4.1

Lemma 35. *Assume all the objectives are ω-regular and prefix-independent. A run ν_i is the outcome of a Nash equilibrium in $\mathcal{T}_i$ with payoff $\widehat{\Omega}^i_j$ iff ν_i satisfies the formula*[13] $\big(\mathbf{G}\, W_i^{3-i}(j)\big) \wedge \widehat{\Omega}^i_j$.

Proof. Let (β_1, β_2) be a Nash equilibrium in $\mathcal{T}_i$ with payoff $\widehat{\Omega}^i_j$ for player i. It means that for every player-i strategy β'_i, the single outcome $\nu'_i \in \mathsf{Out}_{\mathcal{T}_i}(\beta'_i, \beta_{3-i})$ satisfies $\bigwedge_{1 \le k < j}(\neg\widehat{\Omega}^i_k)$; that is, every outcome $\nu'_i \in \mathsf{Out}_{\mathcal{T}_i}(\beta_{3-i})$ satisfies $\bigwedge_{1 \le k < j}(\neg\widehat{\Omega}^i_k)$.

[13] In this formula, $\mathbf{G}$ is the LTL modality meaning "*always*".

Hence β_{3-i} is a winning strategy for player $3-i$ with objective $\bigwedge_{1\le k<j}(\neg\widehat{\Omega}^i_k)$. Now, as all objectives Ω^i_k's are prefix-independent, so are the objectives $\widehat{\Omega}^i_k$'s, and hence any outcome $\nu'_i \in \mathsf{Out}_{\mathcal{T}_i}(\beta_{3-i})$ satisfies $\mathbf{G}\, W_i^{3-i}(j)$. Letting ν_i be the unique outcome in $\mathsf{Out}_{\mathcal{T}_i}(\beta_1,\beta_2)$, we finally get that ν_i satisfies $\big(\mathbf{G}\, W_i^{3-i}(j)\big) \wedge \widehat{\Omega}^i_j$.

Conversely assume that a run ν_i satisfies the formula $\big(\mathbf{G}\, W_i^{3-i}(j)\big) \wedge \widehat{\Omega}^i_j$. Then it means that from every state s along ν_i, player $3-i$ has a winning strategy, say β^s_{3-i}, to enforce $\bigwedge_{1\le k<j}(\neg\widehat{\Omega}^i_k)$. Then build a Nash equilibrium with outcome ν_i as follows:

- if h is a finite prefix of ν_i ending a state belonging to player k, then define $\beta_k(h)$ as the next transition along h;
- otherwise, if $h = h' \cdot h''$ is such that h' is the longest common prefix of h and ν_i, write s as the last state of h'.
  - If h (or equivalently h'') ends in a state of player $3-i$, then define $\beta_{3-i}(h)$ as $\beta^s_{3-i}(h'')$;
  - If h ends in a state of player i, then define $\beta_i(h)$ as any possible continuation of h (the choice will not matter).

With that definition, it is not hard to be convinced that (β_1,β_2) is a Nash equilibrium with unique outcome ν_i. Furthermore, if player i deviates from this outcome, then player $3-i$ will play in such a way that player i will not satisfy any of the conditions $(\widehat{\Omega}^i_k)_{1\le k<j}$. □

Note that thanks to a product construction, we can handle more general properties such as LTL-defined properties, and in particular bounded reachability.

As a direct consequence of Lemma 35 and Theorem 12, we get:

Theorem 13. *Let $\mathcal{G}$ be a timed game with preference relations given as transferable, region-uniform, prefix-independent sequences $(\Omega^i_j)_j$ of objectives. There is a pseudo Nash equilibrium in $\mathcal{G}$ with payoff (Ω^j_1, Ω^k_2) iff there are two runs ν_1 in $\mathcal{T}_1$ and ν_2 in $\mathcal{T}_2$ s.t.*[14] *(i) $\nu_1 \models \big(\mathbf{G}\, W_1^2(j)\big) \wedge \widehat{\Omega}^1_j$, (ii) $\nu_2 \models \big(\mathbf{G}\, W_2^1(k)\big) \wedge \widehat{\Omega}^2_k$, and (iii) $\psi_1(\nu_1) = \psi_2(\nu_2)$.*

G Proofs of Section 4.2

Theorem 14. *Let $\mathcal{G}$ be a zero-sum timed game where player 1's objective is Ω, and is assumed to be transferable. Then player 1 has a winning strategy in $\mathcal{G}$ from $(\ell, \mathbf{0})$ iff player 1 has a winning strategy in game $\mathcal{T}_2$ from $(\ell, [\mathbf{0}])$ for the objective $\widehat{\Omega}$.*

Proof. Assume σ_1 is a player 1 winning strategy in $\mathcal{G}$ for the objective Ω. Take any strategy σ_2 for player 2. We will argue that for any $\rho \in \mathsf{Out}(\sigma_1,\sigma_2)$, $((\sigma_1,\sigma_2),\rho)$ is a pseudo Nash equilibrium in $\mathcal{G}$ (with payoff $(1,-1)$). Indeed, player 1 has maximal payoff in that case, and player 2 has minimal payoff but cannot improve because σ_1 is winning for player 1. Then, this means that there is a Nash equilibrium with payoff $(0,0)$ in game $\mathcal{T}_2$, which means precisely that player 1 has a strategy so that whatever is the

[14] **G** is the LTL modality for "*always*".

strategy of player 2, player 2 loses (hence does not satisfy her objective, which is $\neg\widehat{\Omega}$). Hence player 1 has a winning strategy for the objective $\widehat{\Omega}$ in game $\mathcal{T}_2$.

Conversely, if player 1 has a winning strategy for the objective $\widehat{\Omega}$ in game $\mathcal{T}_2$, then it can be transferred to a winning strategy in game $\mathcal{G}$. □