Nash Equilibria for Reachability Objectives in Multi-player Timed Games

Patricia Bouyer, Romain Brenguier,
Nicolas Markey

June 2010

Research report LSV-10-12

Laboratoire Spécification & Vérification

École Normale Supérieure de Cachan
61, avenue du Président Wilson
94235 Cachan Cedex France

Nash Equilibria for Reachability Objectives in Multi-player Timed Games*

Patricia Bouyer, Romain Brenguier, and Nicolas Markey

LSV, ENS Cachan & CNRS, France
{bouyer,brenguie,markey}@lsv.ens-cachan.fr

Abstract. We propose a procedure for computing Nash equilibria in multi-player timed games with reachability objectives. Our procedure is based on the construction of a finite concurrent game, and on a generic characterization of Nash equilibria in (possibly infinite) concurrent games. Along the way, we use our characterization to compute Nash equilibria in finite concurrent games.

1 Introduction

Timed games. Game theory (especially games played on graphs) has been used in computer science as a powerful framework for modelling interactions in embedded systems [14, 9]. Over the last fifteen years, games have been extended with the ability to depend on timing informations. Timed games allows for a more faithful representation of reactive systems, while preserving decidability of several important properties, such as the existence of a winning strategy for one of the agents to achieve her goal, whatever the other agents do [3]. Efficient algorithms exist and have been implemented, *e.g.* in the tool Uppaal Tiga [4].

Zero-sum vs. non-zero-sum games. In this purely antagonist view, games can be seen as two-player games, where one agent plays against another one. Moreover, the objectives of those two agents are opposite: the aim of the second player is simply to prevent the first player from winning her own objective. More generally, a (positive or negative) *payoff* can be associated with each outcome of the game, which can be seen as the amount the second player will have to pay to the first player. Those games are said to be *zero-sum*.

In many cases, however, games can be *non-zero-sum*, especially when they involve more than two agents, whose objectives may not be complementary. Such games appear *e.g.* in various problems in telecommunications, where several agents try to send data on a network [8]. Focusing only on surely-winning strategies in this setting may then be too narrow: surely-winning strategies must be winning against any behaviour of the other agents, and does not consider the fact that the other agents also try to achieve their own objectives.

Nash equilibria. In the non-zero-sum game setting, it is then more interesting to look for *equilibria*. For instance, a Nash equilibrium is a behaviour of the

* This work is partly supported by projects DOTS (ANR-06-SETI-003), QUASIMODO (FP7-ICT-STREP-214755) and GASICS (ESF-EUROCORES LogiCCC).

agents in which they play rationally, in the sense that no agent can get a better payoff if she, alone, modifies her strategy [12]. This corresponds to stable states of the game. Notice that a Nash equilibrium need not exist in general, and is not necessarily optimal: several equilibria can coexist, possibly with very different payoffs.

Our contribution. We extend the study of Nash equilibria for reachability objectives (where the payoff is 1 when the objective is reached, and 0 otherwise) in the setting of timed games, as defined in [7] (but extended to n players in the obvious way). Since timed games are non-deterministic, we introduce the notion of *pseudo-Nash equilibrium*, in which non-determinism is solved "optimally" (*i.e.*, only the best outcome is considered). This corresponds to letting the players "optimally" solve non-determinism, in such a way that they have no incentive to change their choice.

As is usual in the timed setting, we rely on a region-based abstraction, which in our context is a finite concurrent game. In order to prove that the abstraction preserves Nash equilibria, we define a characterization of Nash equilibria in (possibly infinite-state) concurrent games. This characterization is built on the new concept of *repellor sets*: the repellor set for a subset A of agents is, roughly, the set of states from which players in A will not be winning in any Nash equilibrium. We explain how to compute those sets, and how they can be used to characterize Nash equilibria.

We also use repellor sets to effectively compute Nash equilibria in finite games, which solves open problems in the setting of equilibria in finite games and gives a complete solution to our original problem.

Related work. Nash equilibria (and other related solution concepts such as subgame-perfect equilibria, secure equilibria, ...) have recently been studied in the setting of (untimed) games played on a graph [5, 6, 13, 15–18]. Most of them, however, focus on turn-based games. In the setting of concurrent games, mixed strategies (*i.e.*, strategies involving probabilistic choices) are arguably more relevant than pure (*i.e.*, non-randomized) strategies. However, adding probabilities to timed strategies (over both delays and actions) involves several important technical issues (even in zero-sum non-probabilistic timed games), and we defer the study of mixed-strategy Nash equilibria in timed games to future works.

For lack of space, only sketches of proofs are given in the paper. Full proofs can be found in the technical appendix.

2 Preliminaries

We begin with defining concurrent games and Nash equilibria.

2.1 Concurrent Games

A *transition system* is a 2-tuple $\mathcal{S} = \langle \mathsf{States}, \mathsf{Edg} \rangle$ where States is a (possibly uncountable) set of states and $\mathsf{Edg} \subseteq \mathsf{States} \times \mathsf{States}$ is the set of transitions.

Given a transition system $\mathcal{S}$, a *path* π in $\mathcal{S}$ is a non-empty sequence $(s_i)_{0\le i<n}$ (where $n \in \mathbb{N} \cup \{+\infty\}$) of states of $\mathcal{S}$ such that $(s_i, s_{i+1}) \in \mathsf{Edg}$ for all $i < n-1$. The *length* of π, denoted by $|\pi|$, is $n-1$. The set of finite paths (also called *histories* in the sequel) of $\mathcal{S}$ is denoted by[1] $\mathsf{Hist}_{\mathcal{S}}$, the set of infinite paths (also called *plays*) of $\mathcal{S}$ is denoted by $\mathsf{Play}_{\mathcal{S}}$, and $\mathsf{Path}_{\mathcal{S}} = \mathsf{Hist}_{\mathcal{S}} \cup \mathsf{Play}_{\mathcal{S}}$ is the set of paths of $\mathcal{S}$. Given a path $\pi = (s_i)_{0\le i<n}$ and an integer $j < n$, the *j-th prefix* of π, denoted by $\pi_{\le j}$, is the finite path $(s_i)_{0\le i<j+1}$. If $\pi = (s_i)_{0\le i<n}$ is a history, we write $\mathsf{last}(\pi) = s_{|\pi|}$.

We extend the definition of concurrent games given *e.g.* in [2] with non-determinism:

Definition 1. *A* non-deterministic concurrent game *is a 7-tuple* $\mathcal{G} = \langle \mathit{States}, \mathit{Edg}, \mathit{Agt}, \mathit{Act}, \mathit{Mov}, \mathit{Tab}, \preccurlyeq\rangle$ *in which:*

- $\langle \mathit{States}, \mathit{Edg}\rangle$ *is a transition system;*
- Agt *is a finite set of players (or agents);*
- Act *is a (possibly uncountable) set of actions;*
- $\mathit{Mov}\colon \mathit{States} \times \mathit{Agt} \to 2^{\mathit{Act}} \smallsetminus \{\varnothing\}$ *is a mapping indicating the actions available to a given player in a given state;*
- $\mathit{Tab}\colon \mathit{States} \times \mathit{Act}^{\mathit{Agt}} \to 2^{\mathit{Edg}} \smallsetminus \{\varnothing\}$ *associates, in a given state, a set of moves of the players with the resulting set of transitions. It is required that if* $(s', s'') \in \mathit{Tab}(s, (m_A)_{A\in\mathit{Agt}})$*, then* $s' = s$*.*
- $\preccurlyeq\colon \mathit{Agt} \to 2^{\mathit{States}^\omega \times \mathit{States}^\omega}$ *defines, for each player, a quasi-ordering on the runs of* $\mathcal{G}$*, called* preference relation*. We simply write* $\preccurlyeq_A$ *for* $\preccurlyeq(A)$*.*

In the rest of this paper, we restrict to simple qualitative preference relations given by reachability conditions for each player. Formally, we assume that the preference relation is given as a tuple $(\Omega_A)_{A\in\mathsf{Agt}}$ of sets of states, and is defined as follows: if a path π visits Ω_A, then we let $v_A(\pi) = 1$, otherwise $v_A(\pi) = 0$; we then say that path π' is preferred by Player A over path π, which is written $\pi \preccurlyeq_A \pi'$, whenever either $\pi = \pi'$ or $v_A(\pi) < v_A(\pi')$.

A *deterministic* concurrent game is a concurrent game where $\mathsf{Tab}(s, (m_A)_{A\in\mathsf{Agt}})$ is a singleton for every $s \in \mathsf{States}$ and $(m_A)_{A\in\mathsf{Agt}}$ with $m_A \in \mathsf{Mov}(s, A)$. A *turn-based* game is a concurrent game for which there exists a mapping $\mathsf{Owner}\colon \mathsf{States} \to \mathsf{Agt}$ such that, for every state $s \in \mathsf{States}$, the set $\mathsf{Mov}(s, A)$ is a singleton unless $A = \mathsf{Owner}(s)$.

In a non-deterministic concurrent game, from some state s, each player A selects one action m_A among its set $\mathsf{Mov}(s, A)$ of allowed actions (the resulting tuple $(m_A)_{A\in\mathsf{Agt}}$, which we may also write m_{Agt} in the sequel, is called a *move*). This corresponds to a set of transitions $\mathsf{Tab}(s, (m_A)_{A\in\mathsf{Agt}})$, one of which is applied and gives the next state of the game. In the sequel, we abusively write $\mathsf{Hist}_{\mathcal{G}}$, $\mathsf{Play}_{\mathcal{G}}$ and $\mathsf{Path}_{\mathcal{G}}$ for the corresponding set of paths in the underlying transition system of $\mathcal{G}$. We also write $\mathsf{Hist}_{\mathcal{G}}(s)$, $\mathsf{Play}_{\mathcal{G}}(s)$ and $\mathsf{Path}_{\mathcal{G}}(s)$ for the respective subsets of paths starting in state s.

[1] For this and the coming definitions, we indicate the underlying transition system as a subscript. This may be omitted in the sequel if no ambiguity may arise.

Definition 2. *Let $\mathcal{G}$ be a concurrent game, and $A \in \mathsf{Agt}$. A* strategy *for A is a mapping $\sigma_A \colon \mathsf{Hist} \to \mathsf{Act}$ such that for any $\pi \in \mathsf{Hist}$ it holds $\sigma_A(\pi) \in \mathsf{Mov}(\mathsf{last}(\pi), A)$.*

Given a coalition (i.e., a subset of agents) $P \subseteq \mathsf{Agt}$, a strategy σ_P for coalition P is a tuple of strategies, one for each player in P. We write $\sigma_P = (\sigma_A)_{A \in P}$ for such a strategy. A strategy profile *is a strategy for the coalition Agt. We write $\mathsf{Strat}^A_{\mathcal{G}}$ for the set of strategies of player A in $\mathcal{G}$, and $\mathsf{Prof}_{\mathcal{G}}$ for the set of strategy profiles in $\mathcal{G}$.*

Notice that we only consider non-randomized (*pure*) strategies in this paper.

Let $\mathcal{G}$ be a concurrent game, P be a coalition, and σ_P be a strategy for P. A path $\pi = (s_j)_{0 \le j \le |\pi|}$ is *compatible* with the strategy σ_P if, for all $k \le |\pi| - 1$, there exists a move m_{Agt} such that:

- $m_A \in \mathsf{Mov}(s_k, A)$ for all $A \in \mathsf{Agt}$,
- $m_A = \sigma_A(\pi_{\le k})$ for all $A \in P$,
- $(s_k, s_{k+1}) \in \mathsf{Tab}(s_k, m_{\mathsf{Agt}})$.

We write $\mathsf{Out}_{\mathcal{G}}(\sigma_P)$ for the set of paths (also called *outcomes*) in $\mathcal{G}$ that are compatible with strategy σ_P of coalition P. We write $\mathsf{Out}^f_{\mathcal{G}}$ (resp. $\mathsf{Out}^\infty_{\mathcal{G}}$) for the finite (resp. infinite) outcomes, and $\mathsf{Out}_{\mathcal{G}}(s, \sigma_P)$, $\mathsf{Out}_{\mathcal{G}}(s, \sigma_P)$ and $\mathsf{Out}_{\mathcal{G}}(s, \sigma_P)$ for the respective sets of outcomes of σ_P with initial state s.

Notice that, in the case of deterministic concurrent games, any strategy profile has a single infinite outcome. This might not be the case for non-deterministic concurrent games.

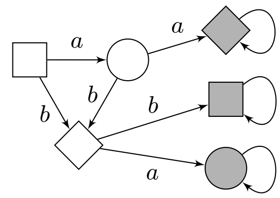

Fig. 1. A 3-player turn-based game

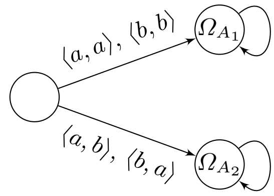

Fig. 2. A 2-player concurrent game

Example 1. Figure 1 displays an example of a three-player turn-based game. The shape of a node indicates its owner, and the goal states are those marked in grey: for instance, Player □ controls square states, and her objective is to reach ■. She cannot achieve this on her own (she has no *winning* strategy), but can achieve it with the help of Player ◇ (both should play action b).

Figure 2 is a two-player concurrent game: from the left-most state, both players choose between actions a and b, and the game goes to the top state (which is a goal state for player A_1) if they play the same action, and to the bottom state otherwise (which is a goal state for player A_2).

Given a move m_{Agt} and an action m' for some player B, we write $m_{\mathsf{Agt}}[B \mapsto m']$ for the move n_{Agt} with $n_A = m_A$ when $A \neq B$ and $n_B = m'$. This notation is extended to strategies in the natural way. In the context of non-zero-sum games, several notions of equilibria have been defined. We present here a refinement of Nash equilibria towards non-deterministic concurrent games.

Definition 3. *Let $\mathcal{G}$ be a non-deterministic concurrent game, and s be a state of $\mathcal{G}$. A* pseudo-Nash equilibrium *in $\mathcal{G}$ from s is a pair $(\sigma_{\mathsf{Agt}}, \pi)$ where $\sigma_{\mathsf{Agt}} \in \mathsf{Prof}_{\mathcal{G}}$ and $\pi \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}})$ is such that for all $B \in \mathsf{Agt}$ and all $\sigma' \in \mathsf{Strat}^B$, it holds:*

$$\forall \pi' \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}}[B \mapsto \sigma']).\ \pi' \preccurlyeq_B \pi.$$

Such an outcome π is called an optimal play *for the strategy profile σ_{Agt}. The* payoff *of a pseudo-Nash equilibrium $(\sigma_{\mathsf{Agt}}, \pi)$ is the function $\nu\colon \mathsf{Agt} \to \{0,1\}$ where $\nu(A) = 1$ if π visits Ω_A (the objective of Player A), and $\nu(A) = 0$ otherwise.*

In the case of deterministic games, π is uniquely determined by σ_{Agt}, and pseudo-Nash equilibria coincide with *Nash equilibria* as defined in [12]: they are strategy profiles where no player has an incentive to unilaterally deviate from her strategy.

In the case of non-deterministic games, a strategy profile for an equilibrium may give rise to several outcomes. The choice of playing the optimal play π is then made cooperatively by all players: once a strategy profile is fixed, non-determinism is resolved by all players choosing one of the possible outcomes in such a way that each player has no incentive to unilaterally changing her choice (nor her strategy). To our knowledge, this cannot be encoded by adding an extra player for solving non-determinism. Notice that solution concepts involving an extra player for solving non-determinism can be handled by our algorithm since it yields a deterministic game (leading to *real* Nash equilibria).

Example 1 (cont'd). In the (deterministic) game of Fig. 1, the strategy profile where all players play a is *not* a Nash equilibrium from □, since player ○ would better play b and reach her winning state. The profile where they all play b is a Nash equilibrium. Actually, deterministic turn-based games such as this one always admit a Nash equilibrium [6].

Now, consider the same game as depicted in Fig. 1, but in which player ◇ has only one action available, say a, which non-deterministically leads to either ■ or ●. Then none of the two outcomes □→◇→■ and □→◇→● is globally better than the other one, hence they do not correspond to a pseudo-Nash equilibrium. The reader can check that, for any strategy profile, there never exists an optimal play, so that this modified, non-deterministic turn-based game does not admit any pseudo-Nash equilibrium.

Regarding the concurrent game of Fig. 2, it is easily seen that it also does not admit a (non-randomized) Nash equilibrium.

2.2 Decision problems

In this paper we are interested in several decision problems related to the existence of pseudo-Nash equilibria. Let $\mathcal{S}$ be a class of concurrent games. In the sequel, we consider the following problems: given $\mathcal{G} = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Agt}, \mathsf{Act}, \mathsf{Mov}, \mathsf{Tab}, \preccurlyeq \rangle$ a concurrent game in class $\mathcal{S}$ with reachability objectives $\Omega_A \subseteq \mathsf{States}$ for every player A, and a state $s \in \mathsf{States}$:

Problem 1 (Existence). Does there exists a pseudo-Nash-equilibrium in $\mathcal{G}$ from s?

Problem 2 (Verification). Given a payoff $\nu\colon \mathsf{Agt} \to \{0,1\}$, does there exists a pseudo-Nash-equilibrium in $\mathcal{G}$ from s with payoff ν?

Problem 3 (Constrained-Existence). Given a constraint (given as a subset $P \subseteq \mathsf{Agt}$ and a function $\varpi\colon P \to \{0,1\}$), does there exists a pseudo-Nash-equilibrium in $\mathcal{G}$ from s with some payoff ν satisfying the constraint (*i.e.*, s.t. $\nu(A) = \varpi(A)$ for all $A \in P$)?

Notice that Problems 1 and 2 are trivially logspace-reducible to Problem 3. Together with these problems, we also consider the corresponding *function problems*: for the *verification* problem ("does the given payoff vector correspond to some equilibrium?"), the function problem asks to build a strategy profile that is an equilibrium for this payoff. For the other two problems, the function problem asks to compute a possible payoff function, and a corresponding strategy profile.

3 Qualitative Nash Equilibria

We now explain a procedure to describe pseudo-Nash equilibria in our setting. To this aim, we introduce the notion of *repellor sets*.

3.1 The repellor sets

Definition 4. *We define the set of* suspect *players for an edge $e = (s, s')$ given a move m_{Agt}, which we denote with $\mathsf{Susp}_{\mathcal{G}}(e, m_{\mathsf{Agt}})$, as the set:*

$$\{B \in \mathsf{Agt} \mid \exists m' \in \mathsf{Mov}(s, B) \text{ s.t. } e \in \mathsf{Tab}(s, m_{\mathsf{Agt}}[B \mapsto m'])\}.$$

We extend this notion to a finite path $\pi = (s_p)_{p \leq |\pi|}$ given strategies σ_{Agt} as follows:

$$\mathsf{Susp}(\pi, \sigma_{\mathsf{Agt}}) = \bigcap_{p < |\pi|} \mathsf{Susp}((s_p, s_{p+1}), (\sigma_A(\pi_{\leq p}))_{A \in \mathsf{Agt}}).$$

Intuitively, Player B is suspect for an edge e, given a move m_{Agt}, whenever she can unilaterally change her action (while the other actions are unchanged) and take edge e. Notice that if $e \in \mathsf{Tab}(s, m_{\mathsf{Agt}})$, then $\mathsf{Susp}(e, m_{\mathsf{Agt}}) = \mathsf{Agt}$. Player B is then suspect for a finite path π, given a tuple of strategies σ_{Agt}, whenever she has a strategy to enforce path π under the strategies $(\sigma_A)_{A \in \mathsf{Agt} \smallsetminus \{B\}}$ of the other players.

Lemma 5. *Given $\sigma_{\mathsf{Agt}} \in \mathsf{Prof}$ and $\pi \in \mathsf{Hist}$, the following three propositions are equivalent:*

(i) $B \in \mathit{Susp}(\pi, \sigma_{\mathit{Agt}})$
(ii) $\exists \sigma' \in \mathit{Strat}^B.\ \pi \in \mathit{Out}^f(\sigma_{\mathit{Agt}}[B \mapsto \sigma'])$
(iii) $\pi \in \mathit{Out}^f((\sigma_A)_{A \in \mathit{Agt} \smallsetminus \{B\}})$

We now define the central notion of this paper, namely the *repellor sets.*

Definition 6. *Let $\mathcal{G}$ be a non-deterministic concurrent game. Given a subset $P \subseteq \mathit{Agt}$, the* repellor set of P, *denoted by $\mathit{Rep}_{\mathcal{G}}(P)$, is defined inductively on P as follows: as the base case, $\mathit{Rep}_{\mathcal{G}}(\varnothing) = \mathit{States}$; Then, assuming that $\mathit{Rep}_{\mathcal{G}}(P')$ has been defined for all $P' \subsetneq P$, we let $\mathit{Rep}_{\mathcal{G}}(P)$ be the largest set[2] satisfying the following two conditions:*

- $\forall A \in P.\ \mathit{Rep}_{\mathcal{G}}(P) \cap \Omega_A = \varnothing$ (1)
- $\forall s \in \mathit{Rep}_{\mathcal{G}}(P).\ \exists m_{\mathit{Agt}} \in \mathit{Act}^{\mathit{Agt}}.\ \forall s' \in \mathit{States}.$
  $s' \in \mathit{Rep}_{\mathcal{G}}(P \cap \mathit{Susp}_{\mathcal{G}}((s, s'), m_{\mathit{Agt}}))$ (2)

Intuitively, from a state in $\mathsf{Rep}_{\mathcal{G}}(P)$, the players can cooperate in order to stay in this repellor set (thus never satisfying the objectives of players in P) in such a way that breaking the cooperation does not help fulfilling one's objective.

Lemma 7. *If $P \subseteq P'$, then $\mathit{Rep}(P') \subseteq \mathit{Rep}(P)$.*

Remark 8. Because deterministic turn-based games are determined, they enjoy the property that $\mathsf{Rep}(\{A\}) = \mathsf{States} \smallsetminus \mathsf{Win}(\{A\})$, where $\mathsf{Win}(\{A\})$ is the set of states from which player A has a winning strategy for reaching her objective against the coalition $\mathsf{Agt} \smallsetminus \{A\}$. Notice that this does not hold in concurrent games: in the game depicted on Fig. 2, the initial state is neither in the repellor set nor in the winning set of any player.

The sets of *secure moves* for staying in $\mathsf{Rep}(P)$ is defined as:

$$\mathsf{Secure}_{\mathcal{G}}(s, P) = \{(m_i)_{A_i \in \mathsf{Agt}} \in \mathsf{Act}^{\mathsf{Agt}} \mid \forall s' \in \mathsf{States}.\ s' \in \mathsf{Rep}(P \cap \mathsf{Susp}((s, s'), m_{\mathsf{Agt}}))$$

We define the transition system $\mathcal{S}_{\mathcal{G}}(P) = (\mathsf{States}, \mathsf{Edg}')$ as follows: $(s, s') \in \mathsf{Edg}'$ iff there exists some $m_{\mathsf{Agt}} \in \mathsf{Secure}(s, P)$ such that $(s, s') \in \mathsf{Tab}(s, m_{\mathsf{Agt}})$. Note in particular that any $s \in \mathsf{Rep}(P)$ has an outgoing transition in $\mathcal{S}_{\mathcal{G}}(P)$.

Example 2. In the game of Fig. 1, state ◇ is in the repellor set of {□, ◇} and of {○, ◇} but not in that of {□, ○}. Intuitively, from that state, Player ◇ can prevent one of the other two players to reach her objective, but not both of them at the same time. It can be checked that Rep({□, ◇}) = {□; ○; ◇; ●}.

Looking now at the same game but with non-determinism in state ◇, the repellor sets are different; in particular, state ◇ is no longer in Rep({□}) nor in Rep({○}).

[2] This is uniquely defined since if two sets satisfy both conditions, then so does their union.

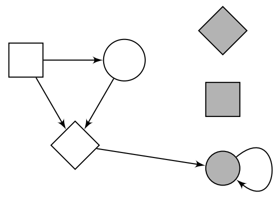

Fig. 3. $\mathcal{S}(\{\square, \diamond\})$ for the det. game

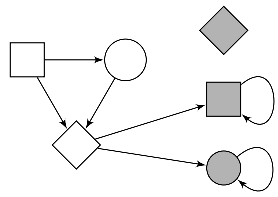

Fig. 4. $\mathcal{S}(\{\diamond\})$ for the non-det. game

3.2 Using the repellor to characterize (pseudo-)Nash equilibria

We now draw the link between the repellor sets and (pseudo-)Nash equilibria.

Lemma 9. *Let $P \subseteq \mathsf{Agt}$, and $s \in \mathsf{States}$. Then $s \in \mathsf{Rep}(P)$ if and only if there exists an infinite path π in $\mathcal{S}(P)$ starting from s.*

Repellor sets characterize those states from which one can find equilibria that avoid the reachability objectives of players in P:

Proposition 10. *Let $P \subseteq \mathsf{Agt}$, and $\pi \in \mathsf{Play}(s)$ be an infinite play with initial state s. Then π is a path in $\mathcal{S}(P)$ if and only if there exists $\sigma_{\mathsf{Agt}} \in \mathsf{Prof}$ such that $\pi \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}})$ and for all $B \in P$ and all $\sigma' \in \mathsf{Strat}^B$ it holds:*

$$\forall \pi \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}}[B \mapsto \sigma']).\ \pi \textit{ does not visit } \Omega_B.$$

Theorem 11. *Let $\mathcal{G} = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Agt}, \mathsf{Act}, \mathsf{Mov}, \mathsf{Tab}, \preccurlyeq \rangle$ be a concurrent game, with reachability objectives $\Omega_A \subseteq \mathsf{States}$ for each player $A \in \mathsf{Agt}$, and $s \in \mathsf{States}$. There is a pseudo-Nash equilibrium from s with payoff ν iff, letting P be the set $\{A \in \mathsf{Agt} \mid \nu(A) = 0\}$, there is an infinite path π in $\mathcal{S}(P)$ which starts in s and which visits Ω_A for every A not in P. Furthermore, π is the optimal play of some pseudo-Nash equilibrium.*

This gives a generic procedure to decide the existence of pseudo-Nash equilibria in non-deterministic concurrent games. It is not effective yet (remember that we allow uncountably-infinite games), but will yield algorithms when instantiated on finite games and timed games in the forthcoming sections.

Proof (of Theorem 11). ($\Rightarrow$) Let $(\sigma_{\mathsf{Agt}}, \pi)$ be a pseudo-Nash equilibrium: σ_{Agt} is a strategy profile, and $\pi \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}})$ is such that for any player B and any strategy σ' for B, it holds

$$\forall \pi' \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}}[B \mapsto \sigma']).\ \pi' \text{ visits } \Omega_B \Rightarrow \pi \text{ visits } \Omega_B.$$

Moreover, π visits Ω_B iff $B \notin P$. According to Proposition 10, π must be a path in the transition system $\mathcal{S}(P)$.

($\Leftarrow$) Let π be an infinite path in $\mathcal{S}(P)$ such that for every $B \notin P$, π visits some state in Ω_B. According to Proposition 10, there is a strategy profile such that π is one of its outcomes and if any player $A \in P$ deviates, no outcome visits Ω_A. Together with π, this forms a pseudo-Nash equilibrium. $\square$

Theorem 11 gives a (necessary and sufficient) condition for the existence of a pseudo-Nash equilibrium in a game. In case an equilibrium exists, repellor sets (and the corresponding transition systems) also contain all the necessary information for effectively computing a pseudo-Nash equilibrium:

Proposition 12. *If π is an infinite path in $\mathcal{S}(P)$ from s visiting Ω_B for every $B \notin P$, then there is a pseudo-Nash equilibrium $(\sigma_{\mathsf{Agt}}, \pi)$ where strategies σ_{Agt} consist in playing secure moves in the transition system $\mathcal{S}(P \cap P')$, for some P'.*

Proof (Sketch). The strategy profile should contain π as one of its outcomes, which can be done by selecting the relevant moves from $\mathcal{S}(P)$. Now, if the play ever gets outside of π but still in $\mathsf{Rep}(P)$ (which may occur because of non-determinism, or because some player, who would have won, has deviated from her strategy), then the strategy profile should select secure moves to stay in this set. Finally, if the history exits $\mathsf{Rep}(P)$, this indicates that (at least) one player in P is trying to deviate from her selected strategy. The strategy profile must ensure that she cannot win: this is achieved by detecting the set P' of players who may be responsible for the deviation, and play secure moves in $\mathcal{S}(P \cap P')$. $\square$

Example 3. For the game of Fig. 1, consider for instance the transition system $\mathcal{S}(\{\square, \diamond\})$, which is depicted on Fig. 3. There are two infinite paths from state $\square$; they correspond to the outcomes of the two Nash equilibria in the game of Fig. 1, both of which have payoff ($\square \mapsto 0, \bigcirc \mapsto 1, \Diamond \mapsto 0$).

In the same game with non-determinism in state $\Diamond$, the transition system $\mathcal{S}(\{\square, \diamond\})$ can be checked to contain no edges, so that there is no pseudo-Nash equilibria with payoff ($\square \mapsto 0, \bigcirc \mapsto 1, \Diamond \mapsto 0$). Now, if we look at $\mathcal{S}(\{\diamond\})$, which is depicted at Fig. 4, there are four possible infinite paths in this transition system, but none of them visits both $\blacksquare$ and $\bullet$. It does not give us a pseudo-Nash equilibrium and in fact there is none in this game.

4 Application to finite games

In this section, we apply the previous generic procedure to finite concurrent games. We consider four classes of finite concurrent games: $\mathfrak{C}^{\mathrm{nd}}$ is the whole class of finite concurrent non-deterministic games, $\mathfrak{C}^{\mathrm{d}}$ is the restriction to deterministic games, $\mathfrak{TB}^{\mathrm{nd}}$ is the restriction to turn-based games, and $\mathfrak{TB}^{\mathrm{d}}$ is the intersection of $\mathfrak{C}^{\mathrm{d}}$ and $\mathfrak{TB}^{\mathrm{nd}}$.

We also consider subclasses where the number of players is bounded *a priori*, and thus is not taken into account in the complexity. Our results can be summarized as follows (in grey are previously known results [6, Corollary 1][3]):

[3] The results in [16] concern parity objectives, and do not encompass reachability objectives.

	$\mathfrak{C}^{\mathrm{nd}}$, $\mathfrak{C}^{\mathrm{d}}$, $\mathfrak{TB}^{\mathrm{nd}}$		$\mathfrak{TB}^{\mathrm{d}}$	
	bounded	general	bounded	general
Existence	P-c.	NP-c.	True	True
Verification	P-c.	NP-c.	P-c.	NP-c.
Constr. Ex.	P-c.	NP-c.	P-c.	NP-c.

These results all follow from the following Proposition:

Proposition 13. *1. The following problems are P-hard with bounded number of players and NP-hard in the general case:*
(a) checking that a payoff ν corresponds to a Nash equilibrium in $\mathfrak{TB}^d$;
(b) deciding the existence of a pseudo-Nash equilibrium in $\mathfrak{TB}^{nd}$;
(c) deciding the existence of a Nash equilibrium in $\mathfrak{C}^d$.
2. Solving the constrained-existence problem in $\mathfrak{C}^{nd}$ is in P for a bounded number of players, and in NP in the general case.

Proof (Sketch of proof). P- and NP-hardness results are obtained by straightforward encodings of the CIRCUIT-VALUE and 3SAT problems, respectively.

The NP algorithm for the constrained existence problem is obtained by first guessing the payoff function, and then checking that Theorem 11 holds. This is achieved by guessing a sequence of states in $\mathcal{S}(P)$, and checking that it is indeed a path in $\mathcal{S}(P)$ and that it visits the correct sets in Ω_{Agt}. A naive implementation of this procedure runs in exponential time (because computing $\mathcal{S}(P)$ may require the computation of intermediate sets $\mathsf{Rep}(P \cap P')$ for many subsets P' of Agt, which may result in up to $2^{|P|}$ computation steps), but using non-determinism, we can select polynomially many intermediate repellor sets that must be computed. The procedure thus runs in non-deterministic polynomial time.

In the case where the number of agents is bounded, the naive approach above is already polynomial, and the number of payoff functions is also polynomial. We can then enumerate all payoff functions, build the transition system $\mathcal{S}(P)$ for each of them, and check the existence of a "witness" path in this transition system. □

Remark 14. In the case of turn-based games, the set of suspects is always either empty, or a singleton, or the whole set of players. As a consequence, the naive implementation of the procedure above will not result in computing $2^{|P|}$ repellor sets, but only $|P|$. The global algorithm still runs in NP, because finding a path in $\mathcal{S}(P)$ with several reachability constraints is NP-complete.

5 Application to timed games

5.1 Definitions of timed games

A *valuation* over a finite set of clocks X is an application $v\colon X \to \mathbb{R}_+$. If v is a valuation and $t \in \mathbb{R}_+$, then $v + t$ is the valuation that assigns to each $x \in X$ the value $v(x) + t$. If v is a valuation and $Y \subseteq X$, then $[Y \leftarrow 0]v$ is the valuation that assigns 0 to each $y \in Y$ and $v(x)$ to each $x \in X \setminus Y$. A *clock constraint* over X is a formula built on the grammar: $\mathfrak{C}(X) \ni g ::= x \sim c \mid g \wedge g$, where x ranges over X, $\sim \in \{<, \leq, =, \geq, >\}$, and c is an integer. The semantics of clock constraints over valuations is natural, and we omit it.

We now define the notion of timed games that we will use in this paper. Our definition follows that of [7].

Definition 15. *A* timed game *is a* 7*-tuple* $\mathcal{G} = \langle \mathsf{Loc}, X, \mathsf{Inv}, \mathsf{Trans}, \mathsf{Agt}, \mathsf{Owner}, \preccurlyeq \rangle$ *where:*

- Loc *is a finite set of locations;*
- X *is a finite set of* clocks*;*
- $\mathsf{Inv}\colon \mathsf{Loc} \to \mathfrak{C}(X)$ *assigns an* invariant *to each location;*
- $\mathsf{Trans} \subseteq \mathsf{Loc} \times \mathfrak{C}(clocks) \times 2^X \times \mathsf{Loc}$ *is the set of transitions;*
- Agt *is a finite set of agents (or players);*
- $\mathsf{Owner}\colon \mathsf{Trans} \to \mathsf{Agt}$ *assigns an agent to each transition;*
- $\preccurlyeq\colon \mathsf{Agt} \to 2^{(\mathsf{States}\times\mathbb{R}_+)^\omega \times (\mathsf{States}\times\mathbb{R}_+)^\omega}$ *defines, for each player, a quasi-ordering on the runs of* $\mathcal{G}$*, called* preference relation.

As in the previous sections, we restrict here to the case where $\preccurlyeq$ is given in terms of reachability objectives $(\Omega_A)_{A\in\mathsf{Agt}}$, with $\Omega_A \subseteq \mathsf{Loc}$ for each $A \in \mathsf{Agt}$.

A timed game is played as follows: a state of the game is a pair (ℓ, v) where ℓ is a location and v is a clock valuation, provided that $v \models \mathsf{Inv}(\ell)$. From each state (starting from an initial state $s_0 = (\ell, \mathbf{0})$, where $\mathbf{0}$ maps each clock to zero and is assumed to satisfy $\mathsf{Inv}(\ell)$), each player A chooses a nonnegative real number d and a transition δ, with the intended meaning that she wants to delay for d time units and then fire transition δ. There are several (natural) restrictions on these choices:

- spending d time units in ℓ must be allowed[4] *i.e.*, $v + d \models \mathsf{Inv}(\ell)$;
- $\delta = (\ell, g, z, \ell')$ belongs to player A, *i.e.*, $\mathsf{Owner}(\delta) = A$;
- the transition is firable after d time units (*i.e.*, $v + d \models g$), and the invariant is satisfied when entering ℓ' (*i.e.*, $[z \leftarrow 0](v + d) \models \mathsf{Inv}(\ell')$).

If (and only if) there is no such possible choice for some player A (*e.g.* if no transition from ℓ belongs to A), then she chooses a special move, denoted by $\perp$.

Given the set of choices m_{Agt} of all the players, with $m_A \in (\mathbb{R}_+ \times \mathsf{Trans}) \cup \{\perp\}$, a player B such that $d_B = \min\{d_A \mid A \in \mathsf{Agt} \text{ and } m_A = (d_A, \delta_A)\}$ is selected (non-deterministically), and the corresponding transition δ_B is applied, leading to a new state $(\ell', [z \leftarrow 0](v + d))$.

This semantics can naturally be expressed in terms of an infinite-state non-deterministic concurrent game. Timed games inherit the notions of history, play, path, strategy, profile, outcome and (pseudo-)Nash equilibrium *via* this correspondence.

[4] Formally, this should be written $v + d' \models \mathsf{Inv}(\ell)$ for all $0 \leq d' \leq d$, but this is equivalent to having only $v \models \mathsf{Inv}(\ell)$ and $v + d \models \mathsf{Inv}(\ell)$ since invariants are convex.

In the sequel, we consider only *non-blocking* timed games, *i.e.*, timed games in which, for any reachable state (ℓ, v), at least one player has an allowed action:

$$\prod_{A \in \mathsf{Agt}} \mathsf{Mov}((\ell, v), A) \neq \{(\bot)_{A \in \mathsf{Agt}}\}.$$

5.2 Computing pseudo-Nash equilibria in timed games

Let $\mathcal{G} = \langle \mathsf{Loc}, X, \mathsf{Inv}, \mathsf{Trans}, \mathsf{Agt}, \mathsf{Owner}, \preccurlyeq \rangle$ be a timed game, where $\preccurlyeq$ is given in terms of reachability objectives $(\Omega_A)_{A \in \mathsf{Agt}}$. In this section, we explain how pseudo-Nash equilibria can be computed in such reachability timed games, using Theorem 11. This relies on the classical notion of *regions* [1], which we assume the reader is familiar with.

We define the region game $\mathcal{R} = \langle \mathsf{States}_\mathcal{R}, \mathsf{Edg}_\mathcal{R}, \mathsf{Agt}, \mathsf{Act}_\mathcal{R}, \mathsf{Mov}_\mathcal{R}, \mathsf{Tab}_\mathcal{R}, \preccurlyeq_\mathcal{R} \rangle$ as follows:

- $\mathsf{States}_\mathcal{R} = \{(\ell, r) \in \mathsf{Loc} \times \mathfrak{R} \mid r \models \mathsf{Inv}(\ell)\}$, where $\mathfrak{R}$ is the set of clock regions;
- $\mathsf{Edg}_\mathcal{R}$ is the set of transitions of the region automaton underlying $\mathcal{G}$;
- $\mathsf{Act}_\mathcal{R} = \{(r, p, \delta) \mid r \in \mathfrak{R},\ p \in \{1; 2; 3\} \text{ and } \delta \in \mathsf{Trans}\} \cup \{\bot\}$;
- $\mathsf{Mov}_\mathcal{R} \colon \mathsf{States}_\mathcal{R} \times \mathsf{Agt} \to 2^{\mathsf{Act}_\mathcal{R}} \setminus \{\varnothing\}$ is such that

$$\begin{aligned}\mathsf{Mov}_\mathcal{R}((\ell, r), A) = \{(r', p, \delta) \mid\ & r' \in \mathsf{Succ}(r),\ r' \models \mathsf{Inv}(\ell),\\ & p \in \{1; 2; 3\} \text{ if } r' \text{ is time-elapsing, else } p = 1,\\ & \delta = (\ell, g, z, \ell') \in \mathsf{Trans} \text{ is such that } r' \models g\\ & \qquad \text{and } [z \leftarrow 0]r' \models \mathsf{Inv}(\ell') \text{and } \mathsf{Owner}(\delta) = A\}\end{aligned}$$

  if it is non-empty, and $\mathsf{Mov}_\mathcal{R}((\ell, r), A) = \{\bot\}$ otherwise. Roughly, the index p allows the players to say if they want to play first, second or later if their region is selected.
- $\mathsf{Tab}_\mathcal{R} \colon \mathsf{States}_\mathcal{R} \times {\mathsf{Act}_\mathcal{R}}^{\mathsf{Agt}} \to 2^{\mathsf{Edg}_\mathcal{R}} \setminus \{\varnothing\}$ is such that for every $(\ell, r) \in \mathsf{States}_\mathcal{R}$ and every $m_\mathsf{Agt} \in \prod_{A \in \mathsf{Agt}} \mathsf{Mov}_\mathcal{R}((\ell, r), A)$, if we write r' for[5] $\min\{r_A \mid m_A = (r_A, p_A, \delta_A)\}$ and p' for $\min\{p_A \mid m_A = (r', p_A, \delta_A)\}$,

$$\begin{aligned}\mathsf{Tab}_\mathcal{R}((\ell, r), m_\mathsf{Agt}) = \{((\ell, r), (\ell_B, [z_B \leftarrow 0]r_B)) \mid\\ m_B = (r_B, p_B, \delta_B) \text{ with } r_B = r',\ p_B = p' \text{ and } \delta_B = (\ell, g_B, z_B, \ell_B)\}.\end{aligned}$$

- The preference relation $\preccurlyeq_\mathcal{R}$ is defined in terms of reachability objectives for each player, where the set of objectives $(\Omega'_i)_{A_i \in \mathsf{Agt}}$ $(\Omega'_A)_{A \in \mathsf{Agt}}$ is defined, for each $A \in \mathsf{Agt}$, as $\Omega'_A = \{(\ell, r) \mid \ell \in \Omega_A, r \in \mathfrak{R}\}$.

Proposition 16. *Let $\mathcal{G}$ be a timed game, and $\mathcal{R}$ its associated region game. Then there is a pseudo-Nash equilibrium in $\mathcal{G}$ from $(s, \mathbf{0})$ iff there is a pseudo-Nash equilibria in $\mathcal{R}$ from $(s, [\mathbf{0}])$, where $[\mathbf{0}]$ is the region associated to $\mathbf{0}$. Furthermore, this equivalence is constructive.*

[5] This is well-defined for two reasons: first, not all m_i's may be $\bot$, since we consider non-blocking games; second, the set of regions appearing in a move from (ℓ, r) only contains successors of r, and is then totally ordered.

Proof (Sketch of proof). The proof is in three steps: we first define a kind of generic *simulation* relation between games, which gives information on their respective repellor sets and transition systems:

Lemma 17. *Consider two games $\mathcal{G}$ and $\mathcal{G}'$ involving the same set of agents, with preference relations defined in terms of reachability conditions $(\Omega_A)_{A\in\mathsf{Agt}}$ and $(\Omega'_A)_{A\in\mathsf{Agt}}$, respectively. Assume that there exists a binary relation $\lhd$ between states of $\mathcal{G}$ and states of $\mathcal{G}'$ such that, if $s \lhd s'$, then:*

1. *if $s' \in \Omega'_A$ then $s \in \Omega_A$ for any $A \in \mathsf{Agt}$;*
2. *for all move m_{Agt} in $\mathcal{G}$, there exists a move m'_{Agt} in $\mathcal{G}'$ such that:*
   - *for any t' in $\mathcal{G}'$, there is $t \lhd t'$ in $\mathcal{G}$ s.t. $\mathsf{Susp}((s',t'),m'_{\mathsf{Agt}}) \subseteq \mathsf{Susp}((s,t),m_{\mathsf{Agt}})$;*
   - *for any (s,t) in $\mathsf{Tab}(s,m_{\mathsf{Agt}})$, there is a (s',t') in $\mathsf{Tab}(s',m'_{\mathsf{Agt}})$ s.t. $t \lhd t'$.*

Then for any $P \subseteq \mathsf{Agt}$ and for any s and s' such that $s \lhd s'$, it holds:

1. *if s is in $\mathsf{Rep}_{\mathcal{G}}(P)$, then s' is in $\mathsf{Rep}_{\mathcal{G}'}(P)$;*
2. *for any $(s,t) \in \mathsf{Edg}_{\mathsf{Rep}}$, there exists (s',t') in $\mathsf{Edg}'_{\mathsf{Rep}}$ s.t. $t \lhd t'$, where $\mathsf{Edg}_{\mathsf{Rep}}$ and $\mathsf{Edg}'_{\mathsf{Rep}}$ are the set of edges in the transition systems $\mathcal{S}_{\mathcal{G}}(P)$ and $\mathcal{S}_{\mathcal{G}'}(P)$, respectively.*

It remains to show that a timed game and its associated region game simulate one another in the sense of Lemma 17, which entail that they have the same sets of repellors. This is achieved by defining two functions λ, mapping moves in $\mathcal{G}$ to *equivalent* moves in $\mathcal{R}$, and μ, mapping moves in $\mathcal{R}$ to *equivalent* moves in $\mathcal{G}$, in such a way that $v \lhd r$ iff r is the region containing v. Theorem 11 concludes the proof. ◻

Because the region game $\mathcal{R}$ has size exponential in the size of $\mathcal{G}$, we get:

Theorem 18. *The constrained existence problem (and thus the existence- and verification problems) in timed game can be solved in* EXPTIME.

Remark 19. Given a pseudo-Nash equilibrium $(\alpha_{\mathsf{Agt}}, \pi)$ in the region game, we can obtain one in the timed game for the same payoff vector. Assume that $(\alpha_{\mathsf{Agt}}, \pi)$ is a pseudo-Nash equilibrium in $\mathcal{R}$. Given a history h in $\mathcal{G}$ and its projection $\mathsf{proj}(h)$ in $\mathcal{R}$, if $(\alpha_A(\mathsf{proj}(h)))_{A\in\mathsf{Agt}} = (r_A, p_A, \delta_A)_{A\in\mathsf{Agt}}$ is a secure move in $\mathcal{R}$, then so is $(\mu_A)_{A\in\mathsf{Agt}} = \mu(\mathsf{last}(h), (\alpha_A(\mathsf{proj}(h)))_{A\in\mathsf{Agt}})$, where μ is the function used in the proof of Proposition 16 to simulate moves from $\mathcal{R}$ in $\mathcal{G}$. Moreover, there exists a play $\pi' \in \mathsf{Out}((\ell, v), (\mu_A)_{A\in\mathsf{Agt}})$ such that $\mathsf{proj}(\pi') = \pi$, therefore the payoff function for these two plays is the same. Hence $((\mu_A)_{A\in\mathsf{Agt}}), \pi')$ is a pseudo-Nash equilibrium in the timed game.

Our algorithm is optimal, as we prove EXPTIME-hardness of our problems:

Proposition 20. *The constrained-existence and verification problems for deterministic turn-based timed games with at least two clocks and two players is* EXPTIME*-hard. The existence problem is* EXPTIME*-hard for concurrent timed games (with at least two clocks and two players).*

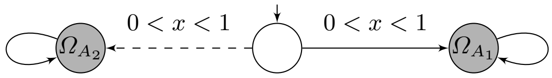

Fig. 5. A timed game with no equilibria (the solid transition belongs to the first player, the dashed one to the second one).

This is proved by encoding countdown games [10]. The second part of the Proposition requires the use of a timed game with no equilibria; an example of such a game is depicted on Fig. 5.

Remark 21. Since deterministic turn-based timed games yield deterministic turn-based region games, they always admit a Nash equilibrium.

6 Conclusion

In this paper we have described a procedure to compute qualitative pseudo-Nash equilibria in multi-player concurrent (possibly non-deterministic) games with reachability objectives. The development of this procedure has required technical tools as the repellor sets, which can be seen as an alternative to the classical attractor sets for computing equilibria in games. We have applied this procedure to finite concurrent games and to timed games, yielding concrete algorithms to compute equilibria in those games. We have furthermore proved that those algorithms have optimal complexities.

Multiple extensions of this work are rather natural:

- First we would like to apply the generic procedure to other classes of systems, for instance to pushdown games [19]. Note that we are not aware of any result on the computation of equilibria in pushdown games.
- Then our procedure only applies to reachability objectives for every player. It would be interesting to adapt it to other ω-regular winning objectives. This is *a priori* non-trivial as this will require developing new tools (the repellor sets are dedicated to reachability objectives).
- We have applied our procedure to concurrent games as defined *e.g.* in [2], where the transition table of the game is given in extensive form (for each tuple of possible actions, there is an entry in a table). In [11], a more compact way of representing concurrent game is proposed, which assumes a symbolic representation of the transition table. It would be interesting to study how this does impact on the complexity of the computation of Nash equilibria. In particular the argument for having an NP algorithm (Proposition 13) does not hold anymore.
- Finally other notions of equilibria (subgame-perfect equilibria, secure equilibria, *etc*) could be investigated, and extensions of concurrent games with probabilities could also be investigated.

References

1. R. Alur and D. L. Dill. A theory of timed automata. *TCS*, 126(2):183–235, 1994.
2. R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *J. ACM*, 49:672–713, 2002.
3. E. Asarin, O. Maler, A. Pnueli, and J. Sifakis. Controller synthesis for timed automata. In *Proc. IFAC Symp. System Structure and Control*, pages 469–474. Elsevier, 1998.
4. G. Behrmann, A. Cougnard, A. David, E. Fleury, K. G. Larsen, and D. Lime. UPPAAL-Tiga: Time for playing games! In *CAV'07*, volume 4590 of *LNCS*, pages 121–125. Springer, 2007.
5. K. Chatterjee, T. A. Henzinger, and M. Jurdziński. Games with secure equilibria. In *LICS'06*, pages 160–169. IEEE Comp. Soc. Press, 2006.
6. K. Chatterjee, R. Majumdar, and M. Jurdziński. On Nash equilibria in stochastic games. In *CSL'04*, volume 3210 of *LNCS*, pages 26–40. Springer, 2004.
7. L. de Alfaro, M. Faella, T. A. Henzinger, R. Majumdar, and M. Stoelinga. The element of surprise in timed games. In *CONCUR'03*, volume 2761 of *LNCS*, pages 142–156. Springer, 2003.
8. M. Félegyházi, J.-P. Hubaux, and L. Buttyán. Nash equilibria of packet forwarding strategies in wireless ad hoc networks. *IEEE Trans. Mobile Computing*, 5(5):463–476, 2006.
9. T. A. Henzinger. Games in system design and verification. In *TARK'05*, pages 1–4. Nat. Univ. Singapore, 2005.
10. M. Jurdziński, F. Laroussinie, and J. Sproston. Model checking probabilistic timed automata with one or two clocks. In *TACAS'07*, volume 4424 of *LNCS*, pages 170–184. Springer, 2007.
11. F. Laroussinie, N. Markey, and G. Oreiby. On the expressiveness and complexity of ATL. *LMCS*, 4(2:7), 2008.
12. J. F. Nash. Equilibrium points in n-person games. *Proc. National Academy of Sciences of the USA*, 36(1):48–49, 1950.
13. S. Paul and S. Simon. Nash equilibrium in generalised Muller games. In *FSTTCS'09*, volume 4 of *LIPIcs*, pages 335–346. LZI, 2009.
14. W. Thomas. Infinite games and verification (extended abstract of a tutoral). In *CAV'02*, volume 2404 of *LNCS*, pages 58–64. Springer, 2002.
15. M. Ummels. Rational behaviour and strategy construction in infinite multiplayer games. In *FSTTCS'06*, volume 4337 of *LNCS*, pages 212–223. Springer, 2006.
16. M. Ummels. The complexity of Nash equilibria in infinite multiplayer games. In *FoSSaCS'08*, volume 4962 of *LNCS*, pages 20–34. Springer, 2008.
17. M. Ummels and D. Wojtczak. The complexity of Nash equilibria in simple stochastic multiplayer games. In *ICALP'09*, volume 5556 of *LNCS*, pages 297–308. Springer, 2009.
18. M. Ummels and D. Wojtczak. Decision problems for Nash equilibria in stochastic games. In *CSL'09*, volume 5771 of *LNCS*, pages 515–530. Springer, 2009.
19. I. Walukiewicz. Pushdown processes: Games and model checking. In *CAV'96*, volume 1102 of *LNCS*, pages 234–263. Springer, 1996.

A Proofs of Section 3

Lemma 5. *Given $\sigma_{\mathsf{Agt}} \in \mathsf{Prof}$ and $\pi \in \mathsf{Hist}$, the following three propositions are equivalent:*

(i) $B \in \mathsf{Susp}(\pi, \sigma_{\mathsf{Agt}})$
(ii) $\exists \sigma' \in \mathsf{Strat}^B.\ \pi \in \mathsf{Out}^f(\sigma_{\mathsf{Agt}}[B \mapsto \sigma'])$
(iii) $\pi \in \mathsf{Out}^f((\sigma_A)_{A \in \mathsf{Agt} \smallsetminus \{B\}})$

Proof. Sentences (ii) and (iii) are obviously equivalent, and we now prove the equivalence of sentences (i) and (ii).

$(i) \Rightarrow (ii)$: Assume that $\pi = (s_p)_{p \leq |\pi|}$ and that $B \in \mathsf{Susp}(\pi, \sigma_{\mathsf{Agt}})$. By definition, for each $p < |\pi|$, there exists an action $m' \in \mathsf{Mov}(s_p, B)$ s.t. $(s_p, s_{p+1}) \in \mathsf{Tab}(s_p, m_{\mathsf{Agt}}[B \mapsto m'])$ Letting $\sigma'(\pi_{\leq p}) = m'$, we have that $\pi \in \mathsf{Out}^f(\sigma_{\mathsf{Agt}}[B \mapsto \sigma'])$, yielding (ii).

$(i) \Leftarrow (ii)$: Let $p < |\pi|$, $m_A = \sigma_A(\pi_{\leq p})$, and $m' = \sigma'(\pi_{\leq p})$. From (ii), we have that

$$(s_p, s_{p+1}) \in \mathsf{Tab}(s_p, m_{\mathsf{Agt}}[B \mapsto m']).$$

It follows that $B \in \mathsf{Susp}((s_p, s_{p+1}), (\sigma_A(\pi_{\leq p}))_{A \in \mathsf{Agt}})$. Being true for all $p < |\pi|$, this entails $B \in \mathsf{Susp}(\pi, \sigma_{\mathsf{Agt}})$.

Lemma 7. *If $P \subseteq P'$, then $\mathsf{Rep}(P') \subseteq \mathsf{Rep}(P)$.*

Proof. By induction on P', we prove that for any subset $P \subseteq P'$, we have $\mathsf{Rep}(P') \subseteq \mathsf{Rep}(P)$. The base case, when $P' = \varnothing$, is trivial.

Pick a set $P' \subseteq \mathsf{Agt}$, and assume that the result holds for any subset of P'. We show that $\mathsf{Rep}(P') \subseteq \mathsf{Rep}(P)$ by showing that it satisfies Conditions (1) and (2) of Definition 6 (remember that $\mathsf{Rep}(P)$ is the largest set satisfying both conditions):

- any $A \in P$ is also in P', hence $\mathsf{Rep}(P') \cap \Omega_A = \varnothing$ for those players;
- for any $s \in \mathsf{Rep}(P')$, there is a move m_{Agt} such that any $s' \in \mathsf{States}$ satisfies $s' \in \mathsf{Rep}(P' \cap \mathsf{Susp}((s, s'), m_{\mathsf{Agt}}))$, which, by induction hypothesis, entails $s' \in \mathsf{Rep}(P \cap \mathsf{Susp}((s, s'), m_{\mathsf{Agt}}))$. □

Lemma 9. *Let $P \subseteq \mathsf{Agt}$, and $s \in \mathsf{States}$. Then $s \in \mathsf{Rep}(P)$ if and only if there exists an infinite path π in $\mathcal{S}(P)$ starting from s.*

Proof. ($\Rightarrow$) We construct $\pi = (s_p)_{p \geq 0}$ inductively, starting with $s_0 = s$. Then, assuming that $s_p \in \mathsf{Rep}(P)$ for some p, we can define s_{p+1} to be such that (s_p, s_{p+1}) corresponds to a secure move $m_{\mathsf{Agt}} \in \mathsf{Secure}(s_p, P)$. Then $s_{p+1} \in \mathsf{Rep}(P \cap \mathsf{Susp}((s_p, s_{p+1}), m_{\mathsf{Agt}}))$, with $\mathsf{Susp}((s_p, s_{p+1}), m_{\mathsf{Agt}}) = \mathsf{Agt}$ (because (s_p, s_{p+1}) is in $\mathsf{Tab}(s_p, m_{\mathsf{Agt}})$), so that $s_{p+1} \in \mathsf{Rep}(P)$.
($\Leftarrow$) Writing $\pi = (s_p)_{p \geq 0}$, by construction of the transition system $\mathcal{S}(P)$, we have that $(s_0, s_1) \in \mathsf{Secure}(s, P)$, which implies that $s \in \mathsf{Rep}(P)$.

Proposition 10. *Let $P \subseteq \mathsf{Agt}$, and $\pi \in \mathsf{Play}(s)$ be an infinite play with initial state s. Then π is a path in $\mathcal{S}(P)$ if and only if there exists $\sigma_{\mathsf{Agt}} \in \mathsf{Prof}$ such that $\pi \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}})$ and for all $B \in P$ and all $\sigma' \in \mathsf{Strat}^B$ it holds:*

$$\forall \pi \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}}[B \mapsto \sigma']).\ \pi \text{ does not visit } \Omega_B.$$

Proof. The proof of this proposition relies on the following lemma.

Lemma 22. *Let $P \subseteq \mathsf{Agt}$ be a subset of agents, and $(\sigma_A)_{A \in P}$ be a strategy of coalition P. Let $s \in \mathsf{States}$, and $\pi \in \mathsf{Out}^f(s, (\sigma_A)_{A \in P})$ ending in some state s'. For any history π' starting in s', define $\sigma_A^{-\pi}(\pi') = \sigma_A(\pi \cdot \pi')$. Then*

$$\pi \cdot \mathsf{Out}(s', (\sigma_A^{-\pi})_{A \in P}) \subseteq \mathsf{Out}(s, (\sigma_A)_{A \in P}).$$

Proof. Let $\pi' \in \pi \cdot \mathsf{Out}(s', (\sigma_A^{-\pi})_{A \in P})$, and write $\pi = (s_p)_{p \leq |\pi|}$ and $\pi' = (s'_p)_{p \leq |\pi'|}$.

If $p < |\pi|$, then $(s'_p, s'_{p+1}) = (s_p, s_{p+1})$, so that it is in $\mathsf{Tab}(s'_p, (\beta_A)_{A \in \mathsf{Agt}})$ for some $(\beta_A)_{A \in \mathsf{Agt}}$ such that $\beta_A = \sigma_A(\pi_{\leq p})$ for all $A \in P$.

Otherwise, if $p \geq |\pi|$, $(s'_p, s'_{p+1}) \in \mathsf{Tab}(s'_p, (\beta'_A)_{A \in \mathsf{Agt}})$ for some $(\beta'_A)_{A \in \mathsf{Agt}}$ with $\beta'_A = \sigma_A^{-\pi}(\pi^{-1} \cdot \pi'_{\leq p}) = \sigma_A(\pi'_{\leq p})$ (where we write $\pi^{-1} \cdot \pi'_{\leq p}$ for the path ρ s.t. $\pi \cdot \rho = \pi'_{\leq p}$).

We can now prove both directions of Prop. 10:
($\Rightarrow$) We assume that $\pi = (s_p)_{p \geq 0}$ is a play in $\mathcal{S}(P)$, and we define the following two (partial) functions:

- $c\colon \mathsf{States} \times 2^{\mathsf{Agt}} \to \mathsf{Act}^{\mathsf{Agt}}$: for all (s, Q) such that $s \in \mathsf{Rep}(Q)$, we let $c(s, Q) = m_{\mathsf{Agt}}$ for some $m_{\mathsf{Agt}} \in \mathsf{Secure}(s, Q)$.
- $d_\gamma\colon \mathbb{N} \to \mathsf{Act}^{\mathsf{Agt}}$: for all $p \in \mathbb{N}$, we let $d_\gamma(p) = m_{\mathsf{Agt}}$ for some $m_{\mathsf{Agt}} \in \mathsf{Secure}(s_p, P)$ such that $(s_p, s_{p+1}) \in \mathsf{Tab}(s_p, m_{\mathsf{Agt}})$. This is well-defined because π is a play in $\mathcal{S}(P)$.

The strategy profile σ_{Agt} is defined as follows:

- on prefixes of π, we let $\sigma_A(\pi_{\leq p}) = (d_\gamma(p))(A)$ for all $p \in \mathbb{N}$;
- for any $\pi' = (s'_p)_{p \leq |\pi'|}$ that is not a prefix of π, we let

$$\sigma_A(\pi') = c(s'_{|\pi'|}, \mathsf{Susp}(\pi', \sigma_{\mathsf{Agt}}))(A).$$

By construction, we have that $\pi \in \mathsf{Out}(\sigma_{\mathsf{Agt}})$. Let B be a player in P, σ' be a strategy for player B, and $\pi' \in \mathsf{Out}((\sigma_A)_{A \in \mathsf{Agt}}[B \mapsto \sigma'])$. Assuming $\pi' = (s'_p)_{p \geq 0}$, we show by induction on p that

- s'_p belongs to $\mathsf{Rep}(\{B\})$, and
- $A_i \in \mathsf{Susp}((s'_{p-1}, s'_p), (\sigma_A(\pi'_{\leq p}))_{A \in \mathsf{Agt}})$ if $p > 1$.

This clearly holds when $p = 0$. Let us assume that it holds for some $p \geq 0$. Let $m_A = \sigma_A(\pi'_{\leq p})$ for all $A \in \mathsf{Agt}$. Then we have $(s'_p, s'_{p+1}) \in \mathsf{Tab}(s'_p, ((m_A)_{A \in \mathsf{Agt}}[B \mapsto \sigma'(\pi'_{\leq p})])$. By construction, $B \in \mathsf{Susp}((s'_p, s'_{p+1}), m_{\mathsf{Agt}})$.

– If $\pi'_{\leq p}$ is not a prefix of π: the strategies σ_A are given by c, and as $s'_p \in \mathsf{Rep}(\{B\})$, it is the case that

$$s'_{p+1} \in \mathsf{Rep}(\{B\} \cap \mathsf{Susp}((s'_p, s'_{p+1}), m_{\mathsf{Agt}}))$$

hence $s'_{p+1} \in \mathsf{Rep}(\{B\})$.
– If $\pi'_{\leq p}$ is a prefix of π: the strategies σ_A are given by d_γ, and it is the case that m_{Agt} is in $\mathsf{Secure}(s'_p, P)$. Hence

$$\begin{aligned} s'_{p+1} &\in \mathsf{Rep}(P \cap \mathsf{Susp}((s'_p, s'_{p+1}), m_{\mathsf{Agt}})) \\ &\subseteq \mathsf{Rep}(\{B\}). \end{aligned}$$

This proves the inductive step. In particular, π' does not visit Ω_B.

($\Leftarrow$) The proof is by induction on P. If $P = \varnothing$ then $\mathsf{Rep}(P) = \mathsf{States}$, and the result follows. If $P \neq \varnothing$ we assume that the implication holds true for all $P' \subsetneq P$. Let σ_{Agt} be a profile such that $\pi \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}})$ and for all $B \in \mathsf{Agt}$ and all $\sigma' \in \mathsf{Strat}^B_{(\mathcal{G},s)}$, it holds $\mathsf{Out}(s, (\sigma_A)_{A\in\mathsf{Agt}}[B \mapsto \sigma']) \subseteq (\mathsf{States} \setminus \Omega_B)^\omega$.

We define the set:

$$S = \{s' \in \mathsf{States} \mid \exists \pi' \in \mathsf{Hist}(s).\mathsf{last}(\pi') = s' \text{ and } P \subseteq \mathsf{Susp}(\pi', (\sigma_i)_{A_i\in\mathsf{Agt}})\}.$$

Let $s' \in S$, and π' such that $\mathsf{last}(\pi') = s'$ and $P \subseteq \mathsf{Susp}(\pi', \sigma_{\mathsf{Agt}})$. According to Lemma 5, any player $B \in P$ has a strategy $\sigma' \in \mathsf{Strat}^i_{\mathcal{G}}$ such that $\pi' \in \mathsf{Out}^f(s, (\sigma_A)_{A\in\mathsf{Agt}}[B \mapsto \sigma'])$. As a consequence $s \notin \Omega_B$. This shows that for every $B \in \mathsf{Agt}$, it holds $S \cap \Omega_B = \varnothing$.

Let $m_{\mathsf{Agt}} = (\sigma_A(\pi'))_{A\in\mathsf{Agt}}$, and $s'' \in \mathsf{States}$. Then

– if $P \subseteq \mathsf{Susp}((s', s''), m_{\mathsf{Agt}})$ then $s'' \in S$;
– otherwise $P \cap \mathsf{Susp}((s', s''), m_{\mathsf{Agt}}) \subsetneq P$. We write P' for this set. Pick $B \in P'$, and let $\pi'' = \pi' \cdot s''$ (path π' with s'' as an extra state at the end). According to Lemma 22,

$$\pi'' \cdot \mathsf{Out}(s'', (\sigma_A^{\pi''^{-1}})_{A\in\mathsf{Agt}\setminus\{B\}}) \subseteq \mathsf{Out}(s, \sigma_{\mathsf{Agt}} \setminus \{B\}) \subseteq (\mathsf{States} \setminus \Omega_B)^\omega.$$

Using the induction hypothesis for P', we get $s'' \in \mathsf{Rep}(P')$.

This proves that $S \subseteq \mathsf{Rep}(P)$ and that the move $(\sigma_A(\pi'))_{A\in\mathsf{Agt}}$ belongs to $\mathsf{Secure}(\mathsf{last}(\pi'), P)$. □

Proposition 12. *If π is an infinite path in $\mathcal{S}(P)$ from s visiting Ω_B for every $B \notin P$, then there is a pseudo-Nash equilibrium $(\sigma_{\mathsf{Agt}}, \pi)$ where strategies σ_{Agt} consist in playing secure moves in the transition system $\mathcal{S}(P \cap P')$, for some P'.*

Proof. For every $P \subseteq \mathsf{Agt}$, for every $s \in \mathsf{Rep}(P)$, there is a secure move m_{Agt} w.r.t. P such that every $(s, s') \in \mathsf{Tab}(s, m_{\mathsf{Agt}})$ is an edge of $\mathcal{S}(P)$ (Lemma 9). For every $P \subseteq \mathsf{Agt}$, we define memoryless[6] strategies σ^P_{Agt} which select such a secure

[6] A memoryless strategy is a strategy σ such that for all runs π and π', $\mathsf{last}(\pi) = \mathsf{last}(\pi')$ implies $\sigma(\pi) = \sigma(\pi')$.

move from every $s \in \mathsf{States}$. One can easily check that for every $s \in \mathsf{Rep}(P)$, every $\pi \in \mathsf{Out}(s, \sigma^P_{\mathsf{Agt}})$ is a path in $\mathcal{S}(P)$. In particular π does not visit any Ω_A when $A \in P$. The idea is to play those memoryless strategies as soon as one of the players has deviated from the optimal play in the equilibrium. That way, as the player who has deviated will be in set P, her objective will never be reached. It remains to define the strategies for the equilibrium when no player has deviated.

Let $\pi = (s_p)_{p \geq 0}$ be an infinite path in $\mathcal{S}(P)$ that starts in s and visits Ω_B for all B with $\nu(B) = 1$. Strategies σ_{Agt} will aim at playing π, and if a player deviates then strategies will be those mentioned earlier. For every history $\pi' = (s'_p)_{p \leq n}$, we distinguish between several cases:

- if π' is a prefix of π ($s_p = s'_p$ if $p \leq n$), then define $(\sigma_A(\pi'))_{A \in \mathsf{Agt}}$ as a secure move w.r.t. P such that $(s'_n, s_{n+1}) \in \mathsf{Tab}(s'_n, (\sigma_A(\pi'))_{A \in \mathsf{Agt}})$;
- if it is a path in $\mathcal{S}(P)$ which is not a prefix of π, then we define $(\sigma_A(\pi'))_{A \in \mathsf{Agt}}$ as a secure move w.r.t. P from $\mathsf{last}(\pi')$. Note that this choice can be made memoryless and may only depend on $\mathsf{last}(\pi')$;
- otherwise decompose π' as $\pi_1 \cdot \pi_2 \cdots \pi_l$ such that there exist $P_l \subset P_{l-1} \cdots \subset P_2 \subset \mathsf{Agt}$ with:
  - π_1 is a path in $\mathcal{S}(P)$,
  - if $j > 1$, then π_j is a path in $\mathcal{S}(P \cap P_j)$,
  - $\mathsf{Susp}((\mathsf{last}(\pi_1), \mathsf{first}(\pi_2)), (\sigma_A(\pi_1))_{A \in \mathsf{Agt}}) = P_2$ (where $\mathsf{first}(\pi_2)$ denotes the first state of path π_2),
  - if $1 < j < l$, then
  $$P_{j+1} = P_j \cap \mathsf{Susp}\big((\mathsf{last}(\pi_j), \mathsf{first}(\pi_{j+1})),\ (\sigma_A^{P_j}(\mathsf{last}(\pi_j)))_{A \in \mathsf{Agt}}\big),$$

  and we define $\sigma_A(\pi')$ as $\sigma_A^{P_l}(\mathsf{last}(\pi'))$.

We claim that $(\sigma_{\mathsf{Agt}}, \pi)$ is a pseudo-Nash equilibrium. First note that if it is an equilibrium, then it has the claimed payoff. Pick a B-strategy σ', and take $\pi' \in \mathsf{Out}(s, \sigma_{\mathsf{Agt}}[B \mapsto \sigma'])$. We can decompose π' as in the construction above (the last π_l being an infinite path). As only B has changed her strategy, we have that for every P_j from the construction, $B \in P_j$ and in particular none of the π_j (with $2 \leq j \leq l$) visits Ω_B. Hence π' visits Ω_B only if π_1 visits Ω_B. Now, π_1 is a path in $\mathcal{S}(P)$, so if π_1 visits Ω_B, we get that $B \notin P$, and π also visits Ω_B. Hence we get that $\pi' \preceq_B \pi$.

B Proofs of Section 4

Proposition 13. *1. The following problems are* P*-hard with bounded number of players and* NP*-hard in the general case:*
   (a) checking that a payoff ν corresponds to a Nash equilibrium in $\mathfrak{TB}^d$;
   (b) deciding the existence of a pseudo-Nash equilibrium in $\mathfrak{TB}^{nd}$;
   (c) deciding the existence of a Nash equilibrium in $\mathfrak{C}^d$.

2. Solving the constrained-existence problem in $\mathfrak{C}^{nd}$ is in P *for a bounded number of players, and in* NP *in the general case.*

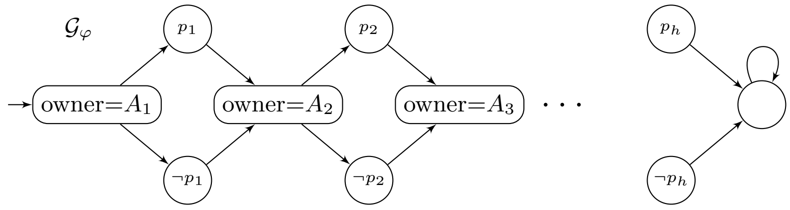

Fig. 6. Game for the reduction of 3SAT

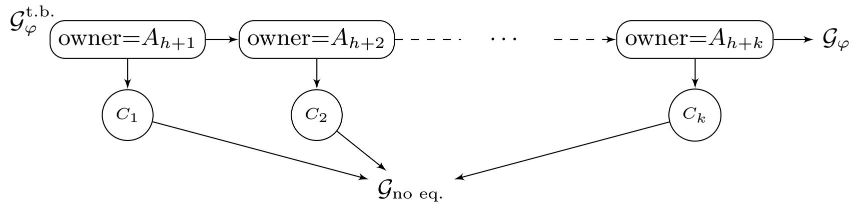

Fig. 7. Reduction of 3SAT to turn-based non-deterministic games

Proof. (1a): By a reduction of the CIRCUIT-VALUE problem (or, equivalently, AGAP (Alternating Graph Accessibility Problem)): a circuit (which we assume w.l.o.g. has only and- and or-gates) is transformed into a two-player turn-based game, where one player controls the and-gates and the other player controls the or-gates. Positive leaves of the circuit are goals of the or-player, and negative leaves are goals of the and-player. Then obviously, the circuit evaluates to true iff the or-player has a winning strategy, which in turn is equivalent to the fact that $(\mathsf{or} = 1, \mathsf{and} = 0)$ is an equilibrium.

We now prove NP-hardness when the number of players is not fixed: we encode an instance of 3SAT as follows. We assume $\mathsf{AP} = \{p_1, p_2, \ldots, p_h\}$:

$$\varphi = \bigwedge_{i=1}^{k} c_i \qquad \text{where } c_i = \ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3}$$

where $\ell_{i,j} \in \{p, \neg p \mid p \in \mathsf{AP}\}$. We build the turn-based deterministic game $\mathcal{G}_\varphi$ with $h + k$ players $\mathsf{Agt} = \{L_1, L_2, \ldots, L_h, C_1, \ldots, C_k\}$ as follows: each player L_i (for i between 1 and h) in turn chooses to visit either location p_i or $\neg p_i$. Location p_i is winning for the clause players C_m iff p_i is one of the literals in c_m, and similarly, location $\neg p_i$ is winning for C_m iff $\neg p_i$ is one of the literals of c_m. This is depicted on Fig. 6, with the reachability objectives defined as:

$$\begin{cases} \Omega_{L_i} = \{p_i, \neg p_i\} & \text{for } 1 \le i \le h \\ \Omega_{C_j} = \{\ell_{i,1}, \ell_{i,2}, \ell_{i,3}\} & \text{for } 1 \le j \le k \end{cases}$$

Now, obviously, this game has a Nash equilibrium with payoff 1 for all players iff φ is satisfiable.

Remark 23. This reduction can be used to prove FNP-hardness of the function problem of computing a witnessing strategy profile, as such a strategy profile immediately gives a valuation which makes φ hold true.

(1b): The first step for this proof is to build a non-deterministic turn-based game $\mathcal{G}_{\text{no eq.}}$ with no pseudo-Nash equilibria. Such a game is depicted in the example of Section 2.1.

Now, the reduction of 3SAT builds upon the previous reduction is as follows: each "clause" player C_j (for j between 1 and k) in turn is first given the opportunity to directly satisfy her goal and go to the game with no equilibria, or to stay in the game. This is depicted on Fig. 7, with the reachability objectives defined as:

$$\begin{cases} \Omega_{L_i} = \{p_i, \neg p_i\} & \text{for } 1 \leq i \leq h \\ \Omega_{C_j} = \{\ell_{i,1}, \ell_{i,2}, \ell_{i,3}, c_i\} & \text{for } h+1 \leq i \leq h+k \end{cases}$$

Now, if φ is satisfiable, then the strategy profile where players C_1 to C_k all play towards $\mathcal{G}_\varphi$, and players L_1 to L_h play according to a witnessing valuation of φ, is an equilibrium where the payoff is 1 for all the players. On the other hand, any equilibrium in $\mathcal{G}_\varphi^{\text{t.b.}}$ should avoid $\mathcal{G}_{\text{no eq.}}$ That it is an equilibrium means that players C_1 to C_k all reach their goals, since they have the opportunity to do it in the first steps of the game. Hence the existence of a valuation for p_1 to p_h satisfying all the clauses, so that φ is satisfiable.

(1c): The technique is similar to the previous case: it suffices to come with a concurrent game having no equilibria (to be played as $\mathcal{G}_{\text{no eq.}}$). Such a game is depicted on Fig. 2.

(2): We begin with our NP algorithm for the general case. It consists in non-deterministically selecting a payoff ν, and checking that Theorem 11 holds for this payoff. This is achieved by building the transition system $\mathcal{S}(P)$ (where $P = \{A \in \mathsf{Agt} \mid \nu(A) = 0\}$) and checking that there is a path visiting Ω_A for each player having $\nu(A) = 1$.

Our algorithm is as follows: it first guesses the path in $\mathcal{S}(P)$, *i.e.*, a sequence of states of States for which it will have to prove that it corresponds to a path in $\mathcal{S}(P)$ and it visits Ω_A when $\nu(A) = 1$.

First, obviously, if such a path exists, then its size can be made polynomial in the number of states and in the number of players. Checking that it visits Ω_A if needed is also straightforwardly achieved in polynomial time.

The last, and most difficult step is to prove that the transitions along this path are transitions of $\mathcal{S}(P)$. This is achieved by computing the repellor of $P \cap P'$ for several subsets P' of Agt. We begin with presenting a deterministic (but possibly exponential) algorithm, and then explain how to make it run in non-deterministic polynomial time.

The procedure consists in filling a table $R[s, P]$ where s ranges over States and P ranges over 2^{Agt}, with the intended meaning that $R[s, P] = 1$ if $s \in \mathsf{Rep}(P)$, and

$R[s,P]=0$ otherwise. This is achieved inductively: $R[s,\varnothing]=1$ (by definition). Now, given P and assuming that $R[s,P']$ has been computed for any strict subset P' of P, we compute $R[s,P]$ using the usual procedure for computing greatest fixpoints: start with letting $R[s,P]=1$ for all $s \notin \bigcup_{A\in P} \Omega_A$, and iteratively set $R[s,P]$ to 0 for states that do not satisfy Condition (2) of Def. 6. Checking that a state does not satisfy Condition (2) is achieved by enumerating the set of moves $(m_A)_{A\in\mathsf{Agt}}$ (by scanning the transition table), and for each state computing the set of suspects and checking that the state belongs to the corresponding repellor set.

This procedure computes the repellor set for P in at most $|\mathsf{States}|$ steps, each step being quadratic in the size of the transition table and in the number of states. Moreover, during this computation, we can also get the set of secure moves from each state, thus building the transition system $\mathcal{S}(P)$.

However, since there may be exponentially many subsets P' of P, it might take exponential time. We make it run in polynomial time using non-determinism, by guessing the subsets of P which are really needed during the computation of $\mathsf{Rep}(P)$. We begin with defining this set, inductively. This uses the following sets, for each $s \in \mathsf{States}$:

$$\mathfrak{S}_s = \{\mathsf{Susp}((s,s'),(m_A)_{A\in\mathsf{Agt}}) \mid \\ s' \in \mathsf{States},\ (m_A)_{A\in\mathsf{Agt}} \in \prod_{A\in\mathsf{Agt}} \mathsf{Mov}(s,A) \\ \text{and } (s,s') \notin \mathsf{Tab}(s,(m_A)_{A\in\mathsf{Agt}})\}$$

First remark that those sets are *not too big*: let $U_s = \bigcup_{P'\in\mathfrak{S}_s} P'$; being a suspect for some move $(m_A)_{A\in\mathsf{Agt}}$, any player B in U_s must have at least a second action m', besides m_B, for which $(s,s') \in \mathsf{Tab}(s,(m_A)_{A\in\mathsf{Agt}}[B\mapsto m'])$. Thus the transition table from state s has size at least $2^{|U_s|}$. On the other hand, $\mathfrak{S}_s$ contains subsets of players from U_s, and thus has size at most $2^{|U_s|}$, hence $|\mathfrak{S}_s| \le |\mathsf{Tab}(s)|$. For each $s \in \mathsf{States}$, we let n_s be the size of the largest set of suspects in $\mathfrak{S}_s$. Using the same argument as above, all players in any set of $\mathfrak{S}_s$ has at least two moves from s, so that $2^{n_s} \le |\mathsf{Tab}(s)|$.

Now, the computation of $R[s,P]$ for all s involves the following sets of players:

- $\mathfrak{P}_0 = \{P\}$, the initial set,
- for each $P_i \in \mathfrak{P}_i$, computing $R[s',P_i]$ (possibly for all $s' \in \mathsf{States}$) involves $\{P_i \cap S \mid S \in \mathfrak{S}_{s'}\} \smallsetminus \{P_i\}$. We let

$$\mathfrak{P}_{i+1} = \bigcup_{P_i\in\mathfrak{P}_i, s'\in\mathsf{States}} \{P_i \cap S \mid S \in \mathfrak{S}_{s'}\} \smallsetminus \{P_i\}.$$

  Notice that any set in $\mathfrak{P}_i$ has size at most $|P|-i$, so that $\mathfrak{P}_{|P|+1}$ is empty (and $\mathfrak{P}_{|P|}$ contains at most the empty set).

In order to compute $R[s,P]$, it is sufficient to compute $R[s',P']$ for all P' in the union $\bigcup_{i\le|P|} \mathfrak{P}_i$. We now show that this union contains polynomially many sets.

By definition of $\mathfrak{P}_i$, any $P' \in \mathfrak{P}_i$ is an intersection of P and of one or several sets in some of the $\mathfrak{S}_{s''}$:

$$\exists (J_s)_{s\in\mathsf{States}}.\ (\forall s \in \mathsf{States}.\ J_s \subseteq \mathfrak{S}_s) \text{ and } \qquad P' = P \cap \bigcap_{\substack{s\in\mathsf{States}\\ J_s\neq\varnothing}} \bigcap_{P''\in J_s} P''$$

For $P' \neq P$, let s_0 be the[7] state for which $n_{s_0} = \min\{n_s \mid J_s \neq \varnothing\}$. Then $|P'| \leq n_{s_0}$, and P' is a subset of some set in $\mathfrak{S}_{s_0}$. Hence, for this s_0, there are at most $2^{n_{s_0}} \times |\mathfrak{S}_{s_0}|$ different possible P' occurring in some $\mathfrak{P}_i$. As a consequence, the total number of sets in $\bigcup_{i\leq|P|} \mathfrak{P}_i$ is at most

$$\sum_{s\in\mathsf{States}} 2^{n_s} \times |\mathfrak{S}_s| \leq |\mathsf{Tab}|^2.$$

In the end, our non-deterministic algorithm runs in time $O(|\mathsf{Tab}|^4 \cdot |\mathsf{States}|^2)$, while its deterministic version was in time $O(2^{|\mathsf{Agt}|} \cdot |\mathsf{Tab}|^2 \cdot |\mathsf{States}|^2)$. Notice that this algorithm only checks the existence of a (pseudo-)Nash equilibrium having payoff ν (verification problem). For the constrained-existence problem, we first had to non-deterministically choose ν at the beginning.

In the case where the number of players is bounded, the number of possible payoffs is also bounded, and we can list them all. For each of them, we can apply the deterministic version of the algorithm above, which is polynomial time in this case where the number of players is bounded.

Remark 24. In the class $\mathfrak{T}\mathfrak{B}^{\mathrm{nd}}$, the argument for having a polynomial algorithm is simpler, once the payoff ν has been guessed. Indeed if $s \in \mathsf{States}$ and $\mathsf{Owner}(s) = B$, then for every edge $e = (s, s')$, for every move $(m_A)_{A\in\mathsf{Agt}}$, then $\mathsf{Susp}(e, (m_A)_{A\in\mathsf{Agt}})$ is either Agt in case $e \in \mathsf{Tab}(s, (m_A)_{A\in\mathsf{Agt}})$, or $\{B\}$ otherwise. Hence the filling of $R[\cdot, P]$ requires only the pre-computation of all lines $R[\cdot, \{B\}]$, and can thus be done in time $O(|\mathsf{Agt}| \cdot |\mathsf{Tab}|^2 \cdot |\mathsf{States}|^2)$. □

C Definitions and proofs of Section 5

C.1 Semantics of timed games

With a timed game $\mathcal{G} = \langle \mathsf{Loc}, X, \mathsf{Inv}, \mathsf{Trans}, \mathsf{Agt}, \mathsf{Owner}, \preccurlyeq \rangle$, we associate the infinite concurrent game $\mathcal{G}' = \langle \mathsf{States}, \mathsf{Edg}, \mathsf{Agt}, \mathsf{Act}, \mathsf{Mov}, \mathsf{Tab}, \preccurlyeq \rangle$ such that

- the set of states is the set of configurations of the timed game: $\mathsf{States} = \{(\ell, v) \mid \ell \in \mathsf{Loc}, v\colon X \to \mathbb{R}_+ \text{ such that } v \models \mathsf{Inv}(\ell)\}$;
- $s_0 = (\ell_0, \mathbf{0})$ is the initial state;

[7] In order to have s_0 uniquely defined, we assume an ordering of the states in States, and pick the smallest set in case several would match.

– transitions give rise to set of edges Edg as follows: for each $d \in \mathbb{R}_+$ and each $\delta = (\ell, g, z, \ell')$ in Trans, for each $(\ell, v) \in \mathsf{States}$ such that $v + d \models \mathsf{Inv}(\ell) \wedge g$, there is an edge $((\ell, v), (\ell', [z \leftarrow 0](v + d)))$;
– the set of actions is $\mathsf{Act} = \{(d, \delta) \mid d \in \mathbb{R}_+, \delta \in \mathsf{Trans}\} \cup \{\bot\}$;
– an action (d, δ) is allowed to player A in state (ℓ, v) iff the following three conditions hold:
  - $(\ell, v + d) \in \mathsf{States}$;
  - $\delta = (\ell, g, z, \ell')$ is such that $\mathsf{Owner}(\delta) = A$;
  - $v + d \models g$ and $[z \leftarrow 0](v + d) \models \mathsf{Inv}(\ell')$.

  Then $\mathsf{Mov}((\ell, v), A)$ is the set of actions allowed to player A when this set is non empty, and it is $\{\bot\}$ otherwise;
– finally, given a state (ℓ, v) and a set of moves $(m_A)_{A\in\mathsf{Agt}}$ allowed from this state, $\mathsf{Tab}((\ell, v), (m_A)_{A\in\mathsf{Agt}})$ is the set

$$\Big\{((\ell, v), (\ell', v')) \;\Big|\; \exists A.\ d_A = \min\{d_B \mid B \in \mathsf{Agt} \text{ s.t. } m_B = (d_B, \delta_B)\} \\ \text{and } \delta_A = (\ell, g_A, z_A, \ell') \text{ and } v' = [z_A \leftarrow 0](v + d_A)\Big\}.$$

C.2 Proofs of Section 5

Proposition 16. *Let $\mathcal{G}$ be a timed game, and $\mathcal{R}$ its associated region game. Then there is a pseudo-Nash equilibrium in $\mathcal{G}$ from $(s, \mathbf{0})$ iff there is a pseudo-Nash equilibria in $\mathcal{R}$ from $(s, [\mathbf{0}])$, where $[\mathbf{0}]$ is the region associated to $\mathbf{0}$. Furthermore, this equivalence is constructive.*

Proof. The proof of Prop. 16 is in three steps: we first define a kind of generic *simulation* relation between games, which gives information on their respective repellors. We then show that a timed game and its associated region game simulate one another, which entail that they have the same sets of repellors. Theorem 11 will conclude the proof.

Lemma 17. *Consider two games $\mathcal{G}$ and $\mathcal{G}'$ involving the same set of agents, with preference relations defined in terms of reachability conditions $(\Omega_A)_{A\in\mathsf{Agt}}$ and $(\Omega'_A)_{A\in\mathsf{Agt}}$, respectively. Assume that there exists a binary relation $\lhd$ between states of $\mathcal{G}$ and states of $\mathcal{G}'$ such that, if $s \lhd s'$, then:*

1. *if $s' \in \Omega'_A$ then $s \in \Omega_A$ for any $A \in \mathsf{Agt}$;*
2. *for all move m_{Agt} in $\mathcal{G}$, there exists a move m'_{Agt} in $\mathcal{G}'$ such that:*
   - *for any t' in $\mathcal{G}'$, there is $t \lhd t'$ in $\mathcal{G}$ s.t. $\mathsf{Susp}((s', t'), m'_{\mathsf{Agt}}) \subseteq \mathsf{Susp}((s, t), m_{\mathsf{Agt}})$;*
   - *for any (s, t) in $\mathsf{Tab}(s, m_{\mathsf{Agt}})$, there is a (s', t') in $\mathsf{Tab}(s', m'_{\mathsf{Agt}})$ s.t. $t \lhd t'$.*

Then for any $P \subseteq \mathsf{Agt}$ and for any s and s' such that $s \lhd s'$, it holds:

1. *if s is in $\mathsf{Rep}_{\mathcal{G}}(P)$, then s' is in $\mathsf{Rep}_{\mathcal{G}'}(P)$;*
2. *for any $(s, t) \in \mathsf{Edg}_{\mathsf{Rep}}$, there exists (s', t') in $\mathsf{Edg}'_{\mathsf{Rep}}$ s.t. $t \lhd t'$, where $\mathsf{Edg}_{\mathsf{Rep}}$ and $\mathsf{Edg}'_{\mathsf{Rep}}$ are the set of edges in the transition systems $\mathcal{S}_{\mathcal{G}}(P)$ and $\mathcal{S}_{\mathcal{G}'}(P)$, respectively.*

Proof. The proof is by induction on P. The result is trivial when $P = \emptyset$. Now, we pick some P and assume that the implication holds true for any $P' \subsetneq P$. Let

$$R = \{s' \in \mathsf{States}' \mid \exists s \in \mathsf{Rep}(P).\ s \lhd s'\};$$

pick $s' \in R$, and a corresponding $s \in \mathsf{Rep}(P)$ s.t. $s \lhd s'$. In particular, for all $A \in P$, we have $s \notin \Omega_A$, hence $s' \notin \Omega'_A$. It follows $R \cap \Omega'_A = \varnothing$ for all $A \in P$.

Also there exists $(m_A)_{A\in\mathsf{Agt}}$ s.t. for every $t \in \mathsf{States}$, we have $t \in \mathsf{Rep}(P \cap \mathsf{Susp}((s,t),(m_A)_{A\in\mathsf{Agt}}))$. Since $s \lhd s'$, there exists $(m'_A)_{A\in\mathsf{Agt}}$ s.t. for all t', there exists $t \lhd t'$ s.t. $\mathsf{Susp}((s',t'),(m'_A)_{A\in\mathsf{Agt}}) \subseteq \mathsf{Susp}((s,t),(m_A)_{A\in\mathsf{Agt}})$. Then

- if $P \subseteq \mathsf{Susp}((s,t),(m_A)_{A\in\mathsf{Agt}})$ then $t \in \mathsf{Rep}(P)$, and $t' \in R$;
- otherwise, $\mathsf{Susp}((s,t),(m_A)_{A\in\mathsf{Agt}}) \cap P \subsetneq P$; as a consequence, we also have $\mathsf{Susp}((s',t'),(m'_A)_{A\in\mathsf{Agt}}) \cap P \subsetneq P)$, and by induction hypothesis, we get $t' \in \mathsf{Rep}_{\mathcal{G}'}(P \cap \mathsf{Susp}((s,t),(m_A)_{A\in\mathsf{Agt}}))$. By Lemma 7, we have $t' \in \mathsf{Rep}_{\mathcal{G}'}(P \cap \mathsf{Susp}((s',t'),(m'_A)_{A\in\mathsf{Agt}}))$.

This proves that $R \subseteq \mathsf{Rep}_{\mathcal{G}'}(P)$, since it fulfills both conditions of Def. 6. The first property follows.

Let $s \lhd s'$, and $(s,t) \in \mathsf{Edg}_{\mathsf{Rep}}$: there exists $(m_A)_{A\in\mathsf{Agt}} \in \mathsf{Secure}(s,P)$ such that $(s,t) \in \mathsf{Tab}(s,(m_A)_{A\in\mathsf{Agt}})$. Since $(m_A)_{A\in\mathsf{Agt}}$ is secure, any state $u \in \mathsf{States}$ belongs to the repellor set $\mathsf{Rep}(P \cap \mathsf{Susp}((s,u),(m_A)_{A\in\mathsf{Agt}}))$.

Now, from $s \lhd s'$, there exists $(m'_A)_{A\in\mathsf{Agt}}$ such that for all $u' \in \mathsf{States}'$, there exists $u \lhd u'$ s.t.

$$\mathsf{Susp}((s',u'),(m'_A)_{A\in\mathsf{Agt}}) \subseteq \mathsf{Susp}((s,u),(m_A)_{A\in\mathsf{Agt}}).$$

From the first property of the induction, we get that u' belongs to $\mathsf{Rep}(P \cap \mathsf{Susp}((s,u),(m_A)_{A\in\mathsf{Agt}}))$, hence $u' \in \mathsf{Rep}(P \cap \mathsf{Susp}((s',u'),(m'_A)_{A\in\mathsf{Agt}}))$. This shows that $(m'_A)_{A\in\mathsf{Agt}}$ is secure. For this $(m'_A)_{A\in\mathsf{Agt}}$, we also have that $(s',t') \in \mathsf{Tab}(s',(m'_A)_{A\in\mathsf{Agt}})$ for some t' with $t \lhd t'$. This conclude the induction. ☐

We now pick three partial functions $f_1, f_2, f_3 \colon \mathbb{R}_+^X \times \mathfrak{R} \to \mathbb{R}_+$ such that for every valuation v and every region r, if there is some $t \in \mathbb{R}_+$ such that $v + t \in r$, then all three functions are defined at (v,r), and we require that $v + f_i(v,r) \in r$ for all $i \in \{1;2;3\}$, and that $f_1(v,r) < f_2(v,r) < f_3(v,r)$ if r is open, and $f_1(v,r) = f_2(v,r) = f_3(v,r)$ otherwise. If no such t exists, all three functions are undefined at (v,r).

▶ ***From timed game $\mathcal{G}$ to region game $\mathcal{R}$.*** We show that there is a relation $\lhd$ such that $(\ell,v) \lhd (\ell,r)$ if r is the region containing v. By definition of the preference relations in $\mathcal{R}$, the first hypothesis of Lemma 17 holds.

Now, for any state (ℓ,v) and any move vector $(m_A)_{A\in\mathsf{Agt}}$ in $\mathcal{G}$, we define the move vector $(\lambda_A)_{A\in\mathsf{Agt}} = \lambda((\ell,v),(m_A)_{A\in\mathsf{Agt}})$ in $\mathcal{R}$ as follows: let $d^1 = \min\{d_A \mid A \in \mathsf{Agt} \text{ s.t. } m_A = (d_A,\delta_A)\}$ and $d^2 = \min\{d_A \mid A \in \mathsf{Agt} \text{ s.t. } m_A = (d_A,\delta_A) \text{ with } d_A > d^1\}$. Then for all $A \in \mathsf{Agt}$,

- if $m_A = (d^1,\delta)$, then we set $\lambda_A = (r,1,\delta)$, where r is the region corresponding to valuation $v + d^1$;

– if $m_A = (d^2, \delta)$, then we set $\lambda_A = (r, p, \delta)$, where r is the region corresponding to valuation $v + d^1$, and $p = 2$ if r is time-elapsing and $p = 1$ otherwise.
– if $m_A = (d_A, \delta)$, with $d_A > d^2$ then set $\lambda_A = (r, p, \delta)$, where r is the region corresponding to valuation $v + d_A$, and $p = 3$ if r is time-elapsing and $p = 1$ otherwise;
– if $m_A = \perp$, then set $\lambda_A = \perp$.

Clearly, if m_A is allowed to A in (ℓ, v), then λ_A is allowed to A in (ℓ, r) where r is the region containing v, since it corresponds to the same transition played in the correct region.

The following two lemmas will imply one direction of Proposition 16 (*via* Lemma 17 and Theorem 11).

Lemma 25. *For any* $((\ell, v), (\ell', v')) \in \mathsf{Tab}((\ell, v), (m_A)_{A\in\mathsf{Agt}})$*, it holds* $((\ell, r), (\ell', r')) \in \mathsf{Tab}_{\mathcal{R}}((\ell, r), (\lambda_A)_{A\in\mathsf{Agt}})$*, where* r *and* r' *are the regions containing* v *and* v' *respectively.*

Proof. As $((\ell, v), (\ell', v')) \in \mathsf{Tab}((\ell, v), (m_A)_{A\in\mathsf{Agt}})$, there is some A such that $m_A = (d_A, \delta_A)$ with $d_A = \min\{d_B \mid B \in \mathsf{Agt} \text{ s.t. } m_B = (d_B, \delta_B)\}$ and $\delta_A = (\ell, g, z, \ell')$ s.t. $v' = [z \leftarrow 0](v + d)$. In this case, we have $\lambda_A = (r_A, 1, \delta_A)$ with $v + d_A \in r_A$ and $r_A = \min\{r_B \mid B \in \mathsf{Agt} \text{ and } \lambda_B = (r_B, p_B, \delta_B)\}$. Therefore $((l, r), (l', r')) \in \mathsf{Tab}_{\mathcal{R}}((l, r), (\lambda_B)_{B\in\mathsf{Agt}})$ with $r' = [z \leftarrow 0]r_i$, so that r' contains $[z \leftarrow 0](v + d)$.

Lemma 26. *For any region* r'*, there is a valuation* $v' \in r'$ *such that:*

$$\mathsf{Susp}_{\mathcal{R}}(((\ell, r), (\ell', r')), (\lambda_i)_{A_i\in\mathsf{Agt}}) \subseteq \mathsf{Susp}_{\mathcal{G}}(((\ell, v), (\ell', v')), (m_i)_{A_i\in\mathsf{Agt}}).$$

Proof. Let B be a suspect in the first set, and $\lambda'_B = (r'_B, p'_B, \delta'_B)$ be a move such that

$$((\ell, r), (\ell', r')) \in \mathsf{Tab}_{\mathcal{R}}((\ell, r), (\lambda_A)_{A\in\mathsf{Agt}}[B \mapsto \lambda'_B]).$$

We want to prove that there exists $v' \in r'$ and m'_B such that $((\ell, v), (\ell', v')) \in \mathsf{Tab}((\ell, v), (m_A)_{A\in\mathsf{Agt}}[B \mapsto m'_B])$. We first define m'_B:

– if $r'_B < \min\{r_A \mid A \neq B\}$ then $m'_B = (f_1(v, r'_B), \delta'_B)$;
– if $r'_B = \min\{r_A \mid A \neq B\}$ then:
  - if $p'_B \leq \min\{p_A \mid A \neq B \text{ and } r_A = r'_B\}$ then $m'_B = (d^{p'_B}, \delta'_B)$;
  - otherwise $m'_B = (d, \delta'_B)$ such that $v + d \in r'_B$ and $d > \min\{d_A \mid A \neq B \text{ and } r_A = r'_B\}$;
– otherwise $m'_B = (f_{p'_B}(v, r'_B), \delta'_B)$.

Then $m'_i \in \mathsf{Mov}((\ell, v), A_i)$ since $v + d'_i \in r_i$ in all cases.

Moreover, in all four cases, writing $m'_B = (d'_B, \delta'_B)$, it can be checked that:

$$d'_B = \min\Big(\{d_A \mid A \in \mathsf{Agt} \setminus \{B\} \text{ s.t. } m_A = (d_A, \delta_A)\} \cup \{d'_B\}\Big)$$
$$\Leftrightarrow$$
$$\left(\begin{array}{c} r'_B = \min\Big(\{r_A \mid A \in \mathsf{Agt} \setminus \{B\} \text{ s.t. } \lambda_A = (r_A, p_A, \delta_A)\} \cup \{r'_B\}\Big) \\ \text{and} \\ p'_B = \min\Big(\{p_A \mid A \in \mathsf{Agt} \setminus \{B\} \text{ s.t. } \lambda_A = (r'B, p_A, \delta_A)\} \cup \{p'_B\}\Big) \end{array}\right)$$

It remains to prove that $((\ell, v), (\ell', v'))$ belongs to $\mathsf{Tab}((\ell, v), (m_A)_{A\in\mathsf{Agt}}[B \mapsto m'_B])$ for some $v' \in r'$. Since $((\ell, r), (\ell', r')) \in \mathsf{Tab}_{\mathcal{R}}((\ell, r), (\lambda_A)_{A\in\mathsf{Agt}}[B \mapsto \lambda'_B])$, we have to prove that any player who plays the minimal region and index in $(\lambda_A)_{A\in\mathsf{Agt}}[B \mapsto \lambda'_B]$ also plays the shortest delay in $(m_A)_{A\in\mathsf{Agt}}[B \mapsto m'_B]$. From the equivalence above, this is true for the suspect. Now, assume that C, not the suspect, plays the smallest region and index in $(\lambda_A)_{A\in\mathsf{Agt}}[B \mapsto \lambda'_B]$. Then p_C is either 1 or 2, by construction of $(\lambda_A)_{A\in\mathsf{Agt}}$, and:

- if $p_C = 1$, then $d_C = d^1$. Moreover, r'_B (the region proposed by the suspect) is either r_C or a later region. In both cases, d'_B (the delay proposed by the suspect) is larger than or equal to d^1, and the result follows.
- otherwise, $p_C = 2$, and $d_C = d^2$. But again, r'_B is either r_C or a later region, and in both cases, d'_B is larger than or equal to d^2. ☐

▶ ***From region game $\mathcal{R}$ to timed game $\mathcal{G}$.*** We consider an action profile $(\alpha_A)_{A\in\mathsf{Agt}}$ in $\mathcal{R}$. We define the action profile $(\mu_A)_{A\in\mathsf{Agt}} = (\mu((\ell, v), (\alpha_A)))_{A\in\mathsf{Agt}}$ in $\mathcal{G}$ as follows:

- if $\alpha_A = \bot$, then $\mu((\ell, v), (\alpha_A)) = \bot$;
- if $\alpha_A = (r_A, p_A, \delta_A)$, then we let $\mu((\ell, v), (\alpha_A)) = (f_{p_A}(v, r_A), \delta_A)$.

Again, if α_A is allowed to Player A in (ℓ, r), then μ_A is also allowed to A in (ℓ, v), since it corresponds to playing the same transition in the same region.

Lemma 27. *For any transition $((\ell, r), (\ell', r')) \in \mathsf{Tab}_{\mathcal{R}}((\ell, r), (\alpha_A)_{A\in\mathsf{Agt}})$, there exists a transition $((\ell, v), (\ell', v')) \in \mathsf{Tab}((\ell, v), (\mu_A)_{A\in\mathsf{Agt}})$ where v and v' are contained in the regions r and r', respectively.*

Proof. Since $((\ell, r), (\ell', r')) \in \mathsf{Tab}_{\mathcal{R}}((\ell, r), (\alpha_A)_{A\in\mathsf{Agt}})$, there is a player B such that $r_B = \min\{r_A \mid A \in \mathsf{Agt} \text{ s.t. } \alpha_A = (r_A, p_A, \delta_A)\}$, $p_B = \min\{p_A \mid A \in \mathsf{Agt} \text{ s.t. } \alpha_A = (r_B, p_A, \delta_A)\}$, and $\delta_A = (\ell, g, z, \ell')$ and $r' = [z \leftarrow 0]r_A$. Therefore $f_{p_B}(v, r_B) = \min\{f_{p_A}(v, r_A) \mid A \in \mathsf{Agt} \text{ s.t. } \alpha_A = (r_A, p_A, \delta_A)\}$ and $v' = [z \leftarrow 0](v + f_{a_B}(v, r_B))$ is such that $(l', v') \in \mathsf{Tab}((l, v), (\mu_A)_{A\in\mathsf{Agt}})$ with $v' \in r'$.

Lemma 28. *For any v', writing r' for the region containing v', we have*

$$\mathsf{Susp}_{\mathcal{G}}((((\ell, v), (\ell', v')), (\mu_A)_{A\in\mathsf{Agt}}) \subseteq \mathsf{Susp}_{\mathcal{R}}((((\ell, r), (\ell', r')), (\alpha_A)_{A\in\mathsf{Agt}})$$

Proof. Let B be a suspect in the first set, and $\mu'_B = (d'_B, \delta'_B)$ be such that $((\ell, v), (\ell', v')) \in \mathsf{Tab}((\ell, v), ((\mu_A)_{A\in\mathsf{Agt}}[B \mapsto \mu'_B]))$. We show that $((l, r), (l', r')) \in \mathsf{Tab}_{\mathcal{R}}((l, r), ((\alpha_A)_{A\in\mathsf{Agt}}[B \mapsto \alpha'_B]))$ with $\alpha'_B = (r'_B, p'_B, \delta'_B)$, where r'_B is the region associated to $v + d'_B$, and

$$p'_B = \begin{cases} 1 & \text{if } r'_B \text{ is not time-elapsing or } d'_B \leq d^B_{\min} \\ 3 & \text{otherwise.} \end{cases}$$

where $d^B_{\min} = \min\{d_A \mid A \in \mathsf{Agt} \smallsetminus \{B\} \text{ s.t. } \mu_A = (d_A, \delta_A)\}$. First notice that this move is allowed for Player B in (ℓ, r). Moreover, we have to prove that the player(s) playing the minimal delay in $(\mu_A)_{A\in\mathsf{Agt}}[B \mapsto \mu'_B]$ also play(s) the minimal region and index in $(\alpha_A)_{A\in\mathsf{Agt}}[B \mapsto \alpha'_B]$.

– in the case where $d'_B \leq d^{\min}$, then if Player B plays in a region not containing $v + d^1$, then $d'_B < d^1$ and she is the only player proposing minimal delay in $\mathcal{G}$ and minimal region and index in $\mathcal{R}$; if she plays in a region containing d^1, then again any player proposing the minimal delay in $(\mu_A)_{A\in\mathsf{Agt}}[B \mapsto \mu'_B]$ also proposes the minimal region and index in $(\alpha_A)_{A\in\mathsf{Agt}}[B \mapsto \alpha'_B]$.
– otherwise, Player A does not propose the minimal delay in $(\mu_A)_{A\in\mathsf{Agt}}[B \mapsto \mu'_B]$ (but it could be the case that she proposes minimal region and index in $(\alpha_A)_{A\in\mathsf{Agt}}[B \mapsto \alpha'_B]$, which does not contradict our claim). Any player proposing minimal delay in $(\mu_A)_{A\in\mathsf{Agt}}[B \mapsto \mu'_B]$ then also proposes minimal region and index in $(\alpha_A)_{A\in\mathsf{Agt}}[B \mapsto \alpha'_B]$. ☐

Applying Lemma 17, for any $P \subseteq \mathsf{Agt}$, there is a path ρ in $\mathcal{S}_{\mathcal{G}}(P)$ (associated with $\mathcal{G}$) if and only if there is a path ρ' in $\mathcal{S}_{\mathcal{G}'}(P)$ which visits exactly the same regions visited by ρ. In particular, if ρ visits every Ω_A when $A \notin P$, then ρ' visits every Ω'_A when $A \notin P$. Therefore there is an equilibrium in the game $\mathcal{G}$ with valuation $\nu = \mathbb{1}_P$ if and only if there is an equilibrium in the game $\mathcal{R}$ with the same valuation ν.

Theorem 18. *The constrained existence problem (and thus the existence- and verification problems) in timed game can be solved in* $\mathsf{EXPTIME}$*.*

Proof. We can apply the algorithms for finite, non-deterministic concurrent games to game $\mathcal{R}$ we developed in Section 4. Since the transformation of $\mathcal{G}$ into $\mathcal{R}$ involves an exponential blowup, the non-deterministic procedure yields an NEXPTIME algorithm. On the other hand, our deterministic procedure applied to $\mathcal{R}$ would still e in exponential time: we have

$$\begin{aligned} |\mathsf{States}| &= |\mathsf{Loc}| \cdot |\mathfrak{R}| \\ |\mathsf{Edg}'| &\leq |\mathsf{Edg}| \cdot |\mathfrak{R}| \\ |\mathsf{Act}| &\leq 3 \cdot |\mathfrak{R}| \cdot |\mathsf{Trans}| \\ |\mathsf{Mov}| &\leq |\mathsf{States}| \cdot |\mathsf{Agt}| \cdot |\mathsf{Act}| \\ |\mathsf{Tab}| &\leq |\mathsf{States}| \cdot |\mathsf{Act}|^{|\mathsf{Agt}|} \cdot |\mathsf{Edg}| \end{aligned}$$

The deterministic algorithm then runs in time

$$O(2^{|\mathsf{Agt}|} \cdot |\mathsf{Loc}|^4 \cdot |\mathfrak{R}|^4 \cdot (3 \cdot |\mathfrak{R}| \cdot |\mathsf{Trans}|)^{2\cdot|\mathsf{Agt}|} \cdot |\mathsf{Edg}|^2)$$

hence in (deterministic) exponential time.

Proposition 20. *The constrained-existence and verification problems for deterministic turn-based timed games with at least two clocks and two players is* $\mathsf{EXPTIME}$*-hard. The existence problem is* $\mathsf{EXPTIME}$*-hard for concurrent timed games (with at least two clocks and two players).*

Proof. First we introduce *countdown games* [10] A *countdown game* $\mathcal{C}$ is a weighted graph (S, T), where S is the finite set of states and $T \subseteq S \times (\mathbb{N} \setminus \{0\}) \times S$ is the transition relation.

The game is played as follows : starting from a state s, the first player chooses a positive integer d such that there exists a transition going from s labeled with d; the second player then chooses a the next state s' such that $(s, d, s') \in T$, and the game continues from that state. Countdown games are zero-sum: the goal of one of the player (player 1 in the sequel) is make the sum of the integers equal a given target value c, while the other player tries to prevent this.

The semantics of a countdown game can easily be expressed as an exponential-size[8] turn-based game, hence the exponential-time algorithm. It is proved in [10] that deciding the winner in such a game is EXPTIME-complete.

Let $\mathcal{C}$ be a countdown game, s be a state of $\mathcal{C}$, and $c \in \mathbb{N}$ be a target value. We build a 2-player timed game $\mathcal{T}$ as follow.

- $\mathsf{Loc} = S \cup \{(s, d) \in S \times \mathbb{N} \mid \exists (s, d, s') \in T\} \cup \{w_1\} \cup \{w_2\}$;
- $X = \{x, y\}$
- $\mathsf{Trans} = \mathsf{Trans}_1 \cup \mathsf{Trans}_2$, where :

$$\begin{aligned}
\mathsf{Trans}_1 &= \{(s, (x = d), \{x\}, (s, d)) \mid \exists (s, d, s') \in T\} \\
\mathsf{Trans}_2 &= \{((s, d), (x = 0 \ \wedge \ y \leq c - d'), \varnothing, s') \mid \\
&\qquad\qquad\qquad d' = \min\{d'' \mid (s', d'', s'') \in T\}\} \\
&\quad \cup \{((s, d), (x = 0 \ \wedge \ y = c), \varnothing, w_1)\} \\
&\quad \cup \{((s, d), (x = 0 \ \wedge \ y > c - d_0 \ \wedge \ y \neq c), \varnothing, w_2) \mid \\
&\qquad\qquad\qquad d_0 = \max_{(s,d,s') \in T} \min_{(s',d',s'') \in T} d'\}
\end{aligned}$$

- $\mathsf{Owner}(t) = A_1$ if $t \in \mathsf{Trans}_1$, and $\mathsf{Owner}(t) = A_2$ if $t \in \mathsf{Trans}_2$.
- $\Omega'_1 = \{w_1\}$ and $\Omega'_2 = \{w_2\}$.

Lemma 29. *There is a (pseudo-)Nash equilibrium winning for player A_1 in $\mathcal{T}$ from location $\ell_0 = s$ if and only if player A_1 has a winning strategy in $\mathcal{C}$ with initial configuration (s, c).*

Proof. We define a binary relation $\sim$ between configurations of $\mathcal{C}$ and $\mathcal{T}$ as follows:

- $(s, 0) \sim w_1$
- $(s, c') \sim w_2$ if $c' > 0$ and $\forall (s, d, s') \in T.\ c' - d < 0$
- $(s, c') \sim (s, v)$ when $c' > 0$, $v(x) = 0$ and $v(y) = c - c'$
- $(s, c', d) \sim ((s, d), v)$ when $c' > 0$, $v(x) = 0$ and $v(y) = c - (c' - d)$

We show that $\sim$ satisfies the conditions of Lemma 17. The first condition is ensured by the first two points :

$$(s, c') \sim (s, v) \quad \Rightarrow \quad (s, c') \in \Omega_i \Leftrightarrow (s, v) \in \Omega'_i$$

The second condition can be simplified as follows for turn-based deterministic games :

[8] In fact, pseudo-polynomial: the exponential blowup only comes from the use of binary encoding of integers in the input.

– for all $(m_i)_{A_i \in \mathsf{Agt}}$, there exists $(l_i)_{A_i \in \mathsf{Agt}}$ s.t.

$$\mathsf{Tab}(s, (m_i)_{A_i \in \mathsf{Agt}}) \sim \mathsf{Tab}(t, (l_i)_{A_i \in \mathsf{Agt}});$$

– for all $(l_i)_{A_i \in \mathsf{Agt}}$, there exists $(m_i)_{A_i \in \mathsf{Agt}}$ s.t.

$$\mathsf{Tab}(s, (m_i)_{A_i \in \mathsf{Agt}}) \sim \mathsf{Tab}(t, (l_i)_{A_i \in \mathsf{Agt}}).$$

Actually, only $m_{\mathsf{Owner}(s)}$ and $l_{\mathsf{Owner}(s)}$ are relevant here.

Let (s, c') with $c \geq c' > 0$ be a configuration in the countdown game, (s, v) such that $(s, c') \sim (s, v)$, and $m_1 = (s, c') \to (s, c', d)$ be a possible action for player A_1; we define l_1 to be $(s, (x = d), (x \leftarrow 0), (s, d))$. Then $\mathsf{Tab}((s, c'), m_1) = \{(s, c', d)\}$, and $\mathsf{Tab}((s, v), l_1) = \{(s, d), v'\}$ with $v'(x) = 0$ and $v'(y) = c - c' + d$. Hence $(s, c', d) \sim ((s, d), v')$.

In the same way, for an action $l_1 = (s, (x = d), (x \leftarrow 0), (s, d))$ we can define $m_1 = (s, c') \to (s, c', d)$ and the same relation holds :

$$\mathsf{Tab}((s, c'), m_1) \sim \mathsf{Tab}((s, v), l_1)$$

Let (s, c', d) be a configuration in the countdown game, $((s, d), v)$ such that $(s, c', d) \sim ((s, d), v)$,

- if $c' - d = 0$ then $l_2 = ((s, d), (y = c), \varnothing, w_1)$ is the only possible action for player A_2 in the timed game. For any action $m_2 = (s, c', d) \to (s', c' - d)$ we will have $(s', c' - d) \sim w_1$
- if $m_2 = (s, c', d) \to (s', c' - d)$ with $(s', c' - d) \in \Omega_2$, then $\forall (s', d', s'')$. $c' - d - d'' < 0$, which implies that $c' - d < \min_{s', d', s''} d''$, and therefore $c' - d < d_0$, where $d_0 = \max_{(s,d,s') \in T} \min_{(s', d', s'') \in T} d'$. Finally $v(y) = c - (c' - d) > c - d_0$, hence $l_2 = ((s, d), (x = 0 \ \wedge \ y > c - d'), \varnothing, w_2)$ is a possible action and we have $(s', c' - d) \sim w_2$.
- if $l_2 = ((s, d), (x = 0 \ \wedge \ y > c - d'), \varnothing, w_2)$, we define $m_2 = (s, c', d) \to (s', c' - d)$ where s' is a state maximising $\min_{(s', d'', s'') \in T} d''$. The configuration $(s', c' - d)$ is winning for player A_2. $(s', c' - d) \sim w_2$
- if $m_2 = (s, c', d) \to (s', c' - d)$ with $(s', c' - d) \notin \Omega_2$, we define $((s, d), (x = 0 \ \wedge \ y \leq c - d'), \varnothing, s')$. $(s', c' - d) \sim (s', v)$
- if $l_2 = ((s, d), (x = 0 \wedge y \leq c - d'), \varnothing, s')$ where $d' = \min\{d'' \mid (s', d'', s'') \in T\}$, we choose $m_2 = (s, c', d) \to (s', c' - d)$, $(s', c' - d)$ is not a winning configuration for any player. $(s', c' - d) \sim (s', v)$

Applying Lemma 17, we obtain the equivalence between equilibria in both games. Now, clearly, player A_1 has a winning strategy in $\mathcal{C}$ if, and only if, there is a Nash equilibrium where she wins.

It immediately follows that the "constrained existence" problem is EXPTIME-hard for 2-clock 2-player turn-based games. Since the game is zero-sum, the result also follows for the "verification" problem.

Now, as mentioned above, the "existence" problem for turn-based timed games is trivial. However, using the above reduction and a timed game with no equilibria (see Fig. 5) as we did in the proof of Prop. 13), we easily get EXPTIME-hardness of the "existence" problem for non-turn-based and non-deterministic timed games.