Courcelle's Theorem Made Dynamic*

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Abstract

Dynamic complexity is concerned with updating the output of a problem when the input is slightly changed. We study the dynamic complexity of model checking a fixed monadic second-order formula over evolving subgraphs of a fixed maximal graph having bounded tree-width; here the subgraph evolves by losing or gaining edges (from the maximal graph). We show that this problem is in DynFO (with LOGSPACE precomputation), via a reduction to a Dyck reachability problem on an acyclic automaton.

1 Introduction

Monadic second-order logic, tree-width of graphs, and Courcelle's theorem. Monadic second-order logic (MSO) is a powerful formalism for expressing and checking properties of graphs. It allows first-order quantification (over states of the graph) as well as *monadic* second-order quantification (over sets of states), and can thus express properties such as connectivity or 3-colorability of finite graphs. While the satisfiability problem of this logic is in general undecidable, model checking (i.e., deciding if a formula holds true in a given graph) is PSPACE-complete [25]; when the MSO formula is fixed, the problem is hard for every level of the polynomial hierarchy over finite graphs [24], and can be performed in PTIME over finite trees.

The tree-width of graphs has been defined by Robertson and Seymour [21] as a measure of the *complexity* of graphs—intuitively, of how close a graph is to a tree. Many classes of graphs have been shown to have *bounded* tree-width [3]. Over such graphs, several (NP-)hard problems can be solved in polynomial time [3]. Model checking a fixed MSO formula is such an example, as proven by Courcelle in 1990 [5].

Dynamic problems and dynamic complexity. In this paper, we focus on the *dynamic complexity* of MSO model checking over finite graphs. Dynamic-complexity theory aims at developing algorithms that are capable of efficiently updating the output of a problem after a slight change in its input [10, 19]. Such algorithms would keep track of auxiliary information about the current instance, and update it efficiently when the instance is modified. Consider the problem of reachability in directed graphs, and equip such graphs with two operations, for respectively inserting and deleting edges (one at a time). It has recently been proven that this problem is in the class DynFO [8], which was a long-standing open problem. Roughly speaking, a problem is in DynFO when it admits an algorithm updating the solution and some auxiliary information through FO formulas (or, equivalently by AC^0 circuits satisfying some uniformity requirements) after a small change in the input.

Our contributions. We study the MSO model-checking problem from a dynamic perspective, considering the following basic operations on graphs: insertion and deletion of an edge.

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We assume that we are given a maximal graph, which embeds all constructed game graphs along the dynamic process: this maximal graph represents the set of all possible connections in the subgraphs we will consider. We first realize that, since the MSO model-checking problem over arbitrary finite graphs is NP-hard (see [16, 24]), the MSO model-checking problem over arbitrary maximal graphs is unlikely to be solvable in DynFO, even allowing PTIME precomputation (unless Dyn(PTIME,FO)=PTIME =NP).

We therefore make a standard restriction and assume that the maximal graph has bounded tree-width. Under this hypothesis, we show that the MSO model checking over such graphs can be solved in DynFO with LOGSPACE precomputation. To obtain this result, we rely on (and extend) a DynFO algorithm for finding a Dyck path in an acyclic automaton [26], and build a transformation of our model checking problem into such a Dyck reachability problem. The latter transformation is performed by using Courcelle's theorem, and by realizing that the runs of bottom-up, deterministic tree automata can be computed step-by-step. These simple steps can be stored in an auxiliary graph, in which only few edges depend on the real edges that exist in the original game; the correctness of the construction then goes through the search for paths labeled with Dyck words.

Related works. MSO has been extensively studied over various classes of structures in the last 40 years, both regarding its expressiveness and regarding the algorithmic properties of its decision problems (see [6] and references therein). Similarly, numerous measures of the *complexity* of graphs, such as tree-width [21], clique-width [7] or entanglement [2], have been defined and studied; they provide large classes of graphs in which different kinds of hard problems become tractable (see [20] and references therein).

On the other hand, dynamic complexity is much less developed: while the main dynamic complexity classes were defined and studied 20 years ago [10, 19], only few problems have been considered from that point of view [26, 8, 27]. As cited above, directed-graph reachability has recently been proven in DynFO, which was an important open problem in the area.

Finally, let us mention that the results reported in this paper were originally presented in the setting of parity games played on a graph having bounded tree-width [4].

2 Definitions

2.1 Monadic second-order logic over graphs

A graph is a pair $G = \langle V, E\rangle$ where V is a finite set of vertices, and $E \subseteq V \times V$ is a finite set of edges. The size of G is the cardinality of V. Formulas of the monadic second-order logic (denoted MSO) are built using first-order and (monadic) second-order variables, used respectively to quantify over vertices and sets of vertices; formulas may also use equality, and the edge relation E of the graph. As an example, *connectivity* of a graph can be expressed as

$$\forall S.\ [(\forall x.\ x \in S) \vee (\forall x.\ x \notin S) \vee (\exists x, y.\ (x \in S \wedge y \notin S \wedge E(x, y))] .$$

The standard static question regarding MSO over graphs is to decide whether a given formula is satisfiable, or to check whether it is satisfied in a given graph model. These problems have been extensively considered in the literature; in particular, satisfiability is undecidable [23], while model checking is PSPACE-complete [25]. We refer to [17] for a survey.

2.2 Tree decomposition

The notion of *tree decomposition* [21, 22] was introduced by Robertson and Seymour. It gives rise to classes of graphs on which many problems that are NP-hard in general become tractable.

▶ **Definition 1.** An ordered tree decomposition of G is a pair $\mathcal{D} = \langle \mathcal{T}, \mathbf{T} \rangle$, where $\mathcal{T} = \langle \mathcal{N}, \mathcal{E} \rangle$ is an ordered tree, and $\mathbf{T} \colon \mathcal{N} \to 2^V$ is a function such that:

- for each edge $e \in E$, there exists a node $n \in \mathcal{N}$ such that $e \in \mathbf{T}(n)^2$;
- for each vertex $v \in V$, the set $\mathcal{N}_v = \{n \in \mathcal{N} \mid v \in \mathbf{T}(n)\}$ is non-empty, and the restriction of $\mathcal{T}$ to $\mathcal{N}_v$ is connected.

The *width* of $\mathcal{D}$ is defined as the integer $\max\{|\mathbf{T}(n)| \mid n \in \mathcal{N}\} - 1$, and the *tree-width* of G is the least width of all tree decompositions of G.

2.3 Tree automaton

The notion of (deterministic, bottom-up) *tree automaton* is a powerful tool for expressing and checking properties of finite trees.

▶ **Definition 2.** A *tree automaton* is a tuple $\mathcal{A} = \langle Q, \Sigma, \iota, Q_{\mathsf{end}}, \delta \rangle$ where Q is a finite set of states, Σ is a finite input alphabet, $\iota \in Q$ is the initial state, $Q_{\mathsf{end}} \subseteq Q$ is the set of accepting states and $\delta \colon Q^2 \times \Sigma \to Q$ is the transition function.

Let $\mathcal{T} = \langle \mathcal{N}, \mathcal{E} \rangle$ be a binary ordered labeled tree, with label set Σ. The *run* of $\mathcal{A}$ over $\mathcal{T}$ is the function $\rho \colon \mathcal{N} \to Q$ such that:

- for every leaf n of $\mathcal{T}$ with label λ, we have $\rho(n) = \delta(\iota, \iota, \lambda)$;
- for every internal node n of $\mathcal{T}$ with label λ and with children m_1 and m_2, we have $\rho(n) = \delta_2(\rho(m_1), \rho(m_2), \lambda)$.

If, furthermore, the run ρ maps the root of $\mathcal{T}$ to an accepting state $q \in Q_{\mathsf{end}}$, then we say that ρ is *accepting*, and that the automaton $\mathcal{A}$ *accepts* the tree $\mathcal{T}$.

2.4 Dynamic complexity theory

In this paper, we adapt Courcelle's theorem to a dynamic-complexity framework. We briefly introduce the formalisms of descriptive- and dynamic complexity here, and refer to [19, 15, 13, 26] for more details.

Descriptive complexity aims at characterizing positive instances of a problem using logical formulas: the input is then given as a logical structure described by a set of k-ary predicates (the *vocabulary*) over its universe. For example, a directed graph can be represented as a binary predicate representing its edges, with the set of vertices (usually identified with $\{1, \ldots, n\}$ for some n) as the universe. Whether each vertex has at most one outgoing edge is expressed by the first-order formula $\forall x, y, z.(E(x, y) \wedge E(x, z)) \Rightarrow (y = z)$. The complexity class FO contains all problems that can be characterized by such first-order formulas. This class corresponds to the circuit-complexity class AC^0 (under adequate uniformity assumptions) [1].

Dynamic complexity aims at developing algorithms that can efficiently update the output of a problem (e.g. reachability of a given vertex in a graph) when the input is slightly changed. In this setting, algorithms may take advantage of previous computations in order to very quickly recompute the solution for the modified input.

Formally, following [26], a decision problem S is a subset of the set of τ-structures $\mathsf{Struct}(\tau)$ built on a vocabulary τ. In order to turn S into a dynamic problem DynS, we need to define a finite set of initial inputs and a finite set of allowed updates. For instance, we might use an arbitrary graph as initial input, then use a 2-ary operator $\mathsf{ins}(x, y)$ that would insert an edge between vertices x and y.

Hence, we associate the decision problem S with a set $\mathsf{Updates}$ of update functions $\mathsf{up}\colon \mathsf{Struct}(\tau) \to \mathsf{Struct}(\tau)$. We identify every non-empty word in $\mathsf{Struct}(\tau) \cdot \mathsf{Updates}^*$ with the τ-structure obtained by applying a sequence of update operations to an initial structure. Denoting by $\mathsf{Struct}_n(\tau)$ and by $\mathsf{Updates}_n$ the set of τ-structures and of updates restricted to a universe of size n, we define the dynamic language DynS_n as the set of those words in $\mathsf{Struct}_n(\tau) \cdot \mathsf{Updates}_n^*$ that correspond to structures of S. The dynamic language DynS is then defined as the union (over all n) of all such languages.

Given two complexity classes $\mathcal{C}$ and $\mathcal{C}'$, a dynamic problem DynS with set of updates $\mathsf{Updates}$ belongs to the class $\mathsf{Dyn}(\mathcal{C}, \mathcal{C}')$ if, and only if, there exists an auxiliary vocabulary τ^{aux}, a $\mathcal{C}$-computable *initialisation* function $f^{\mathsf{init}}\colon \mathsf{Struct}(\tau) \to \mathsf{Struct}(\tau^{\mathsf{aux}})$, a $\mathcal{C}'$-computable *update* function $f^{\mathsf{up}}\colon \mathsf{Struct}(\tau^{\mathsf{aux}}) \times \mathsf{Updates} \to \mathsf{Struct}(\tau^{\mathsf{aux}})$, and a $\mathcal{C}'$-computable *decision* function $f^{\mathsf{dec}}\colon \mathsf{Struct}(\tau^{\mathsf{aux}}) \to \{0, 1\}$ such that:

- for every structure $A \in \mathsf{Struct}(\tau)$ and every upate $\mathsf{up} \in \mathsf{Updates}$, we have $f^{\mathsf{init}}(\mathsf{up}(A)) = f^{\mathsf{up}}(f^{\mathsf{init}}(A), \mathsf{up})$;
- for every structure $A \in \mathsf{Struct}(\tau)$, we have $A \in \mathsf{S} \Leftrightarrow f^{\mathsf{dec}}(f^{\mathsf{init}}(A)) = 1$.

If, furthermore, f^{init} maps the empty structure of $\mathsf{Struct}(\tau)$ to the empty structure of $\mathsf{Struct}(\tau^{\mathsf{aux}})$, then we say that DynS belongs to the class $\mathsf{Dyn}\mathcal{C}'$.

Informally, DynS belongs to $\mathsf{Dyn}(\mathcal{C}, \mathcal{C}')$ if, by maintaining an auxiliary structure (which may have an initial cost in $\mathcal{C}$), an algorithm can tackle every update on the input structure with a cost in $\mathcal{C}'$. If the initial cost is reduced to zero when the initial input is the empty structure, then DynS belongs to $\mathsf{Dyn}\mathcal{C}'$.

In this paper, we consider the case where $\mathcal{C} = \mathsf{LOGSPACE}$ and $\mathcal{C}' = \mathsf{FO}$, meaning that precomputations will be carried out in $\mathsf{LOGSPACE}$ and that first-order formulas will be used to describe how predicates are updated along transitions.

2.5 Main result

We are now in a position to formally define our problem and state our main result. We fix an MSO formula φ. We follow the approach of [9], and represent graphs as tuples $\langle V, E\rangle$. Given a universe V, our initial structure consists of a tuple $\langle V, E_\star, E\rangle$, where $E_\star$ is a maximal set of edges and $E \subseteq E_\star$ is an initial set of edges.

We focus below on the operations of insertion and deletion of edges that belong to $E_\star$. More precisely, we let $\mathsf{Updates}_{E_\star} = \{\mathsf{ins}(e), \mathsf{del}(e) \mid e \in E_\star\}$. The effect of a sequence of update operations, represented as a word $w \in \mathsf{Updates}_{E_\star}^*$, over a set $E \subseteq E_\star$ of edges, is denoted by $w(E)$, and is defined inductively as:

E	if $w = \epsilon$	$E \cup \{e\}$	if $w = \mathsf{ins}(e)$
$w'(a(E))$	if $w = w' \cdot a$	$E \setminus \{e\}$	if $w = \mathsf{del}(e)$

For $w \in \mathsf{Updates}_{E_\star}^*$ and $E \subseteq E_\star$, we write $G_{w(E)}$ for the graph with vertex set V and edge set $w(E)$. It is to be noted that $G_{w(E)}$ is a subgraph of $\langle V, E_\star\rangle$. Finally, we let

$$\mathsf{DynSat}_\varphi = \{\langle V, E_\star, E\rangle \cdot w \mid w \in \mathsf{Updates}_{E_\star}^* \text{ and } G_{w(E)} \vDash \varphi\}.$$

As mentioned in the introduction, the above problem is unlikely to be solvable in $\mathsf{Dyn}(\mathsf{PTIME},\mathsf{FO})$. We therefore adopt the idea of bounding the tree-width of the maximal graph $\langle V, E_\star\rangle$. We fix a positive integer κ and restrict the set of admissible initial inputs: the graph $\langle V, E_\star\rangle$ should be of tree-width at most κ. We thus refine our problem as follows:

$$\mathsf{DynSat}_{\kappa,\varphi} = \{\langle V, E_\star, E\rangle \cdot w \mid \langle V, E_\star\rangle \text{ has tree-width at most } \kappa\} \cap \mathsf{DynSat}_\varphi.$$

Our main contribution is a dynamic algorithm for deciding $\mathsf{DynSat}_{\kappa,\varphi}$.

▶ **Theorem 3.** *Fix a positive integer κ and an* MSO *formula φ. The problem $\mathsf{DynSat}_{\kappa,\varphi}$ can be solved in* Dyn(LOGSPACE, FO).

We give a short overview of the proof here. Our algorithm consists in transforming our MSO model checking problem into an equivalent Dyck reachability problem over a labeled acyclic graph. The latter problem is known to be in DynFO [26], although we had to adapt the algorithm to our setting. Our approach for building this acyclic graph follows from an automata-based construction used for proving Courcelle's theorem: along some linearization of a tree decomposition of the maximal graph, we can inductively compute local information about the possible computations of a bottom-up tree automaton. These computations can be represented as finding a path in an acyclic graph. However, we have to resort to *Dyck paths* in order to make our acyclic graph efficiently updatable when the input graph is modified.

3 Courcelle's theorem

Courcelle's theorem is not based on working directly with tree decompositions of graphs, but on labeled ordered trees whose labels are chosen from a finite alphabet, and that represent such tree decompositions. We begin with defining such trees.

▶ **Definition 4.** Let $G = \langle V, E\rangle$ be a graph, and let $\mathcal{D} = \langle \mathcal{T}, \mathbf{T}\rangle$ be a binary ordered tree decomposition of G of width κ. Let $\Sigma = 2^{\{0,\ldots,\kappa\}} \times 2^{\{0,\ldots,\kappa\}} \times 2^{\{0,\ldots,\kappa\}^2}$ be a (finite) set of labels. We call *proper $\mathcal{D}$-coloring* of G a function $\chi\colon V \to \{0,\ldots,\kappa\}$ such that, for all nodes n of $\mathcal{T}$, the restriction of χ to $\mathbf{T}(n)$ is injective. We then call $(\chi,\mathcal{D})$*-succinct tree decomposition* of G (we may omit to mention χ and $\mathcal{D}$ if it is clear from the context) the rooted tree obtained by labeling every node n of $\mathcal{T}$ with a label $\lambda(n) = \langle \chi(A), \chi(B), \chi(C)\rangle \in \Sigma$ as follows:

- we set $A = \mathbf{T}(n) \cap \mathbf{T}(m)$ if n has a parent m in $\mathcal{T}$, and $A = \emptyset$ if n is the root of $\mathcal{T}$;
- we set $B = \mathbf{T}(n) \setminus A$;
- we set $C = \{\langle v, w\rangle \in E \mid \langle v, w\rangle \in \mathbf{T}(n)^2 \setminus A^2\}$;
- if X is a set of vertices, then $\chi(X) = \{\chi(v) \mid v \in X\}$, and if X is a set of edges, then $\chi(X) = \{\langle \chi(v), \chi(w)\rangle \mid \langle v, w\rangle \in X\}$.

These constructions are illustrated in Figure 1, which displays a graph G, a tree decomposition $\mathcal{D} = \langle \mathcal{T}, \mathbf{T}\rangle$ of G, a proper $\mathcal{D}$-coloring of G, and its associated succinct tree decomposition.

Observe that, given a tree decomposition $\mathcal{D}$, there always exist $\mathcal{D}$-colorings χ of G and an associated $(\chi,\mathcal{D})$-succinct tree decomposition. They are typically computed from $\mathcal{D}$ in a top-down fashion. Furthermore, note that a graph G may have several tree decompositions $\mathcal{D}$ of width κ and, for each of them, several proper $\mathcal{D}$-colorings. Hence, G may have several succint tree decompositions. Yet, from all of them we are able to reconstruct G (up to graph isomorphism), and therefore to check whether G satisfies the formula φ. A more precise and powerful version of this statement is the theorem stated below, which is a variant of the versions of Courcelle's theorem of [12, Section 11.4] and [17, Section 3.3].

▶ **Theorem 5.** *Fix a positive integer κ and an* MSO *formula φ. There exists a (bottom-up, deterministic) tree automaton $\mathcal{A}_{\varphi,\kappa}$ such that, for all graphs $G = \langle V, E\rangle$ and all succinct tree decompositions $\mathcal{T}$ of G of width κ, G satisfies φ if, and only if, $\mathcal{A}_{\varphi,\kappa}$ accepts $\mathcal{T}$.*

Making Theorem 5 useful further requires being able to compute succinct tree decompositions efficiently. This is possible thanks to the following result, which is proven in [11]:

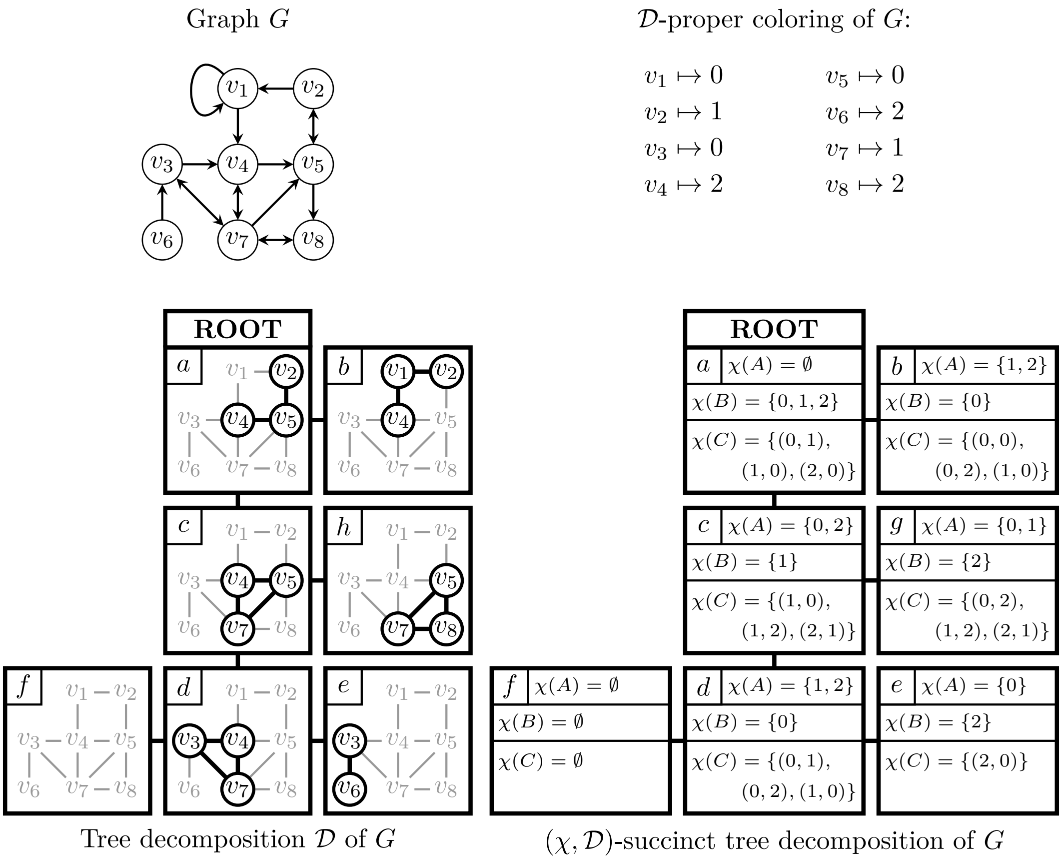


Figure 1 Graph, tree decomposition, proper coloring and succinct tree decomposition

▶ **Lemma 6.** *Let G be a graph of size N and tree-width κ. We can construct in space $\mathcal{O}(c(\kappa)\log_2 N)$ an ordered binary tree decomposition of G of size at most $2N$, width at most $4\kappa+3$, and height at most $c(\kappa)\cdot(\log_2(N)+1)$, where $c(\kappa)$ only depends on κ.*

4 Towards a dynamic algorithm

In this section, we focus on making Courcelle's theorem dynamic. We fix an input (bottom-up, deterministic) tree automaton $\mathcal{A}$ and an input graph G, and we assume that we have a succinct tree decomposition $\mathcal{T}$ of G. We transform the language-theoretic problem of checking whether $\mathcal{A}$ accepts $\mathcal{T}$ into a Dyck reachability problem. More precisely, we build a graph Γ_G and establish a correspondence between some Dyck paths in this graph and the (accepting) runs of $\mathcal{A}$ on $\mathcal{T}$.

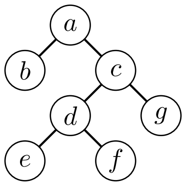
4.1 State progression

A first step towards our goal consists in performing a sequence of transformations on $\mathcal{T}$.

▶ **Definition 7.** Let $\mathcal{T}$ be an ordered tree with N nodes. The *post order* on $\mathcal{T}$ is defined as the linear order $\prec$ such that:

- if m is a strict ancestor of n, then $n \prec m$;

Tree $\mathcal{T}$:



Post indices:

$\mathsf{post}(a) = 7$ $\mathsf{post}(e) = 2$
$\mathsf{post}(b) = 1$ $\mathsf{post}(f) = 3$
$\mathsf{post}(c) = 6$ $\mathsf{post}(g) = 5$
$\mathsf{post}(d) = 4$

Bottom-up progression:

$\mathcal{S}_0 = \emptyset$ $\mathcal{S}_4 = \{b, d\}$
$\mathcal{S}_1 = \{b\}$ $\mathcal{S}_5 = \{b, d, g\}$
$\mathcal{S}_2 = \{b, e\}$ $\mathcal{S}_6 = \{b, c\}$
$\mathcal{S}_3 = \{b, e, f\}$ $\mathcal{S}_7 = \{a\}$

Figure 2 Ordered tree, depth-first traversal and bottom-up progression

- if an internal node n has a left child m_1 and a right child m_2, then m_1 and its descendants are all smaller than m_2 and its descendants (for the order $\prec$).

There exists a unique labeling $\mathsf{post}\colon T \to \{1, \ldots, N\}$, which we call *post index*, such that $n \preccurlyeq m \Leftrightarrow \mathsf{post}(n) \leqslant \mathsf{post}(m)$. We also commonly denote by n_i the unique node of $\mathcal{T}$ such that $i = \mathsf{post}(n_i)$.

We further call *bottom-up progression* of $\mathcal{T}$ the sequence $\mathcal{S}_0, \ldots, \mathcal{S}_N$ of subsets of vertices of $\mathcal{T}$ defined by $\mathcal{S}_i = \{n \mid \mathsf{post}(n) \leqslant i \text{ and } \mathsf{post}(m) > i \text{ for all strict ancestors } m \text{ of } n\}$.

Observe that, by construction, we always have $\mathcal{S}_0 = \emptyset$.

Figure 2 presents a binary ordered tree $\mathcal{T}$, post indices labeling, and bottom-up progression. Observe that $\mathcal{T}$ is the (succinct) tree decomposition presented in Figure 1.

When $\mathcal{T}$ is binary and has height h, then the bottom-up progression enjoys some conciseness and smoothness properties, which we state below.

▶ **Lemma 8.** *Let $\mathcal{T}$ be a binary tree with N nodes, of height h, and let $(\mathcal{S}_i)_{0 \leqslant i \leqslant N}$ be the bottom-up progression of $\mathcal{T}$. For all $i \geqslant 1$, it holds that:*

- *the set $\mathcal{S}_i$ is of cardinality $2h$ or less;*
- *if the node n_i is a leaf, then $\mathcal{S}_i = \mathcal{S}_{i-1} \cup \{n_i\}$;*
- *if the node n_i is an internal node, with children m_1 and m_2, then both m_1 and m_2 belong to $\mathcal{S}_{i-1}$, and $\mathcal{S}_i = \mathcal{S}_{i-1} \setminus \{m_1, m_2\} \cup \{n_i\}$.*

Proof. First, since $i < \mathsf{post}(m)$ for all strict ancestors m of n_i, it comes at once that $\mathcal{S}_i \setminus \mathcal{S}_{i-1} = \{n_i\}$. Furthermore, a node n belongs to $\mathcal{S}_{i-1} \setminus \mathcal{S}_i$ if, and only if, $\mathsf{post}(n) < i$, n_i is an ancestor of n, and $\mathsf{post}(m) \geqslant i$ for all strict ancestors m of n. Since we have $\mathsf{post}(x) < i < \mathsf{post}(y)$ for all strict descendants x and all strict ancestors y of n_i, it follows that $\mathcal{S}_{i-1} \setminus \mathcal{S}_i$ consists of the children of n_i only (if they exist).

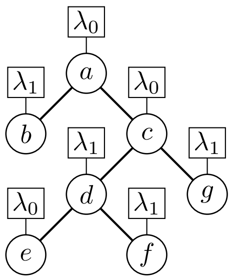
Finally, assume that there exist two nodes $x \prec x'$ in $\mathcal{S}_i$ that are at the same height, and let y be the parent of x. By definition, we know that $\mathsf{post}(x) < \mathsf{post}(x') \leqslant i < \mathsf{post}(y)$, hence x' belongs to the right subtree of y, i.e., x' is the right sibling of x. It follows that $\mathcal{S}_i$ contains at most two nodes at each level, whence $|\mathcal{S}_i| \leqslant 2h$. ◀

The above notion of bottom-up progression also leads to the notion of *state progression*.

▶ **Definition 9.** Let $\mathcal{T}$ be a labeled binary ordered tree with N nodes, let $(\mathcal{S}_i)_{0 \leqslant i \leqslant N}$ be its bottom-up progression, and let $\mathcal{A}$ be a (deterministic, bottom-up) automaton. Let ρ be the (unique) run of $\mathcal{A}$ over $\mathcal{T}$. We call *state progression* of $\mathcal{A}$ on $\mathcal{T}$ the sequence $(\rho\restriction_{\mathcal{S}_i})_{0 \leqslant i \leqslant N}$, where $\rho\restriction_{\mathcal{S}_i}$ denotes the restriction of ρ to the domain $\mathcal{S}_i$.

Figure 2 presented a tree $\mathcal{T}$ and a bottom-up progression of $\mathcal{T}$. Figure 3 presents a tree automaton $\mathcal{A}$, a labeling of $\mathcal{T}$, the (rejecting) run ρ of $\mathcal{A}$ on $\mathcal{T}$ and the associated state progression. We omit to represent the restriction $\rho\restriction_{\mathcal{S}_0}$ since $\mathcal{S}_0$ is empty.

Labeled ordered tree $\mathcal{T}$:



Automaton $\mathcal{A}$:

$Q = \{q_0, q_1\}$
$\Sigma = \{\lambda_0, \lambda_1\}$
$\iota = q_0$
$Q_{\mathsf{end}} = \{q_0\}$
$\delta : (q_i, q_j, \lambda_k) \mapsto q_{\max\{i \cdot j, k\}}$

Run ρ:

$a \mapsto q_1$ $\quad d \mapsto q_1$ $\quad f \mapsto q_1$
$b \mapsto q_1$ $\quad e \mapsto q_0$ $\quad g \mapsto q_1$
$c \mapsto q_1$

State progression:

$\rho\restriction_{\mathcal{S}_1}: b \mapsto q_1$
$\rho\restriction_{\mathcal{S}_2}: b \mapsto q_1,\ e \mapsto q_0$
$\rho\restriction_{\mathcal{S}_3}: b \mapsto q_1,\ e \mapsto q_0,\ f \mapsto q_1$
$\rho\restriction_{\mathcal{S}_4}: b \mapsto q_1,\ d \mapsto q_1$
$\rho\restriction_{\mathcal{S}_5}: b \mapsto q_1,\ d \mapsto q_1,\ g \mapsto q_1$
$\rho\restriction_{\mathcal{S}_6}: b \mapsto q_1,\ c \mapsto q_1$
$\rho\restriction_{\mathcal{S}_7}: a \mapsto q_1$

Figure 3 Labeled tree, tree automaton, run and associated state progression

For all $i \geqslant 1$, recall that Lemma 8 states that there exists a unique node $n_i \in \mathcal{S}_i \setminus \mathcal{S}_{i-1}$, and that either n_i is a leaf or both of its children belong to $\mathcal{S}_{i-1}$. The functions $\rho\restriction_{\mathcal{S}_i}$ can therefore be computed in a step-wise manner once the automaton $\mathcal{A} = \langle Q, \Sigma, \iota, Q_{\mathsf{end}}, \delta\rangle$ is fixed. More precisely, and denoting by λ the labeling function of $\mathcal{T}$, we have:

- if $i = 0$, then $\mathcal{S}_0 = \emptyset$, hence $\rho\restriction_{\mathcal{S}_0}$ is the empty-domain function;
- if $1 \leqslant i$ and n_i is a leaf, then $\rho\restriction_{\mathcal{S}_i} = \Pi_i(\rho\restriction_{\mathcal{S}_{i-1}}, \lambda(n_i))$, where

$$\Pi_i(\varphi, \gamma) \colon n \in \mathcal{S}_i \mapsto \begin{cases} \varphi(n) & \text{if } n \in \mathcal{S}_{i-1}; \\ \delta(\iota, \iota, \gamma) & \text{if } n = n_i; \end{cases}$$

- if $1 \leqslant i$ and n_i is an internal node with children m_1 and m_2, then $\rho\restriction_{\mathcal{S}_i} = \Pi_i(\rho\restriction_{\mathcal{S}_{i-1}}, \lambda(n_i))$, where

$$\Pi_i(\varphi, \gamma) \colon n \in \mathcal{S}_i \mapsto \begin{cases} \varphi(n) & \text{if } n \in \mathcal{S}_{i-1}; \\ \delta(\varphi(m_1), \varphi(m_2), \gamma) & \text{if } n = n_i. \end{cases}$$

We will rely on this step-wise computation in the following section.

4.2 Reduction to the Dyck reachability problem

In this section, we present the reduction of $\mathsf{DynSat}_{\kappa,\varphi}$ to a Dyck reachability problem on an acyclic labeled graph. Our reduction is such that any update (of the edges) in the input graph corresponds to a simple update of the acyclic graph. As we explain in Section 5, this reduction proves Theorem 3.

4.2.1 The Dyck reachability problem in acyclic graphs

Before presenting our reduction, we first define Dyck reachability problems, then recall briefly some results about their dynamic complexity in the case of acyclic graphs: in such graphs, context-free graph queries, and therefore Dyck reachability problems, belong to the dynamic complexity class DynFO [18, 26].

▶ **Definition 10.** Let $G = \langle V, E, L\rangle$ be a labeled graph, with set of labels L, and with edge set $E \subseteq V^2 \times L$. Let v_1 and v_2 be two marked vertices of G. We assume that L can be partitioned as $L = L^+ \uplus L^- \uplus \{\bullet\}$, where L^+ and L^- are in bijection with each other, and $\bullet$ is a fresh "neutral" label symbol, and that a bijection $\lambda^+ \mapsto \lambda^-$ from L^+ to L^- is given.

The labeling on edges induces in a direct way a labeling on paths in G. The *Dyck reachability problem* asks whether there exists a path π (in the graph G) from v_1 to v_2, such that π is labeled with a string in the language $\mathbf{D}$ of *Dyck words* built over the grammar: $S \to \varepsilon \mid S \cdot \bullet \cdot S \mid S \cdot \lambda^+ \cdot S \cdot \lambda^- \cdot S$, where λ^+ ranges over the set L^+.

While [26] assumed a constant-size label set (which is not the case here), the result of [26] can be generalized, as stated below (see [4] for a proof).

▶ **Lemma 11.** *The Dyck reachability problem in acyclic graphs is solvable in* DynFO *(under the assumption that updates consist in adding or deleting individual labeled edges), using as only auxiliary predicate a* 4*-ary predicate* $\Delta(x_1, y_1, x_2, y_2)$ *defined by:*

> *"There exists a path* ϖ_1 *from* x_1 *to* y_1 *with a label* λ_1 *and there exists a path* ϖ_2 *from* x_2 *to* y_2 *with a label* λ_2 *such that* $\lambda_1 \cdot \lambda_2$ *is a Dyck word".*

4.2.2 Reduction

We fix an MSO formula φ, and let $\mathcal{A}_{\varphi,4\kappa+3} = \langle Q, \Sigma, \iota, Q_{\mathsf{end}}, \delta\rangle$, which we simply name $\mathcal{A}$ in the rest of this section, be the (deterministic, bottom-up) tree automaton of Theorem 5.

We now describe our transformation of any subgraph G of $G_\star = \langle V, E_\star\rangle$ into an acyclic labeled graph Γ_G for the Dyck reachability problem. Let $\mathcal{D}_\star = \langle \mathcal{T}_\star, \mathbf{T}_\star\rangle$, where $\mathcal{T}_\star = \langle \mathcal{N}_\star, \mathcal{E}_\star\rangle$ be a tree decomposition of $G_\star$ of width $4\kappa + 3$, as defined in Lemma 6, and let $\mathcal{N}_\star$ be the set of nodes of $\mathcal{T}_\star$. In addition, let χ be a $\mathcal{D}_\star$-coloring function of V, and let $(\mathcal{S}_i)_{0\leqslant i\leqslant N}$ be the bottom-up progression of $\mathcal{T}_\star$.

Let $G = \langle V, E\rangle$ be a subgraph of $G_\star$. Observe that $\mathcal{D}_\star$ is also a tree decomposition of G, since $E \subseteq E_\star$. Hence, there exists a labeling function $\Lambda_G \colon \mathcal{N}_\star \to \Sigma$ that identifies $\mathcal{T}_\star$ with a $(\chi, \mathcal{D}_\star)$-succinct binary tree decomposition $\mathcal{T}_G$ of G. In particular, $(\mathcal{S}_i)_{0\leqslant i\leqslant N}$ is also the bottom-up progression of $\mathcal{T}_G$. Let ρ_G be the run of $\mathcal{A}$ over $\mathcal{T}_G$. We want to construct a graph Γ_G in order to identify the state progression $(\rho_G\lceil_{\mathcal{S}_i})_{0\leqslant i\leqslant N}$ with a (Dyck) path in Γ_G.

A naive construction. As a first try, we let the vertices of this graph be all pairs (i, π), where $0 \leq i \leq N$, and $\pi \colon \mathcal{S}_i \to Q$ is intended to represent the state progression at step i. Following the local computation described page 7, we include an edge $(i-1, \pi) \to (i, \pi')$ when $\pi' = \Pi_i(\pi, \Lambda_G(n_i))$, where n_i is the unique node in $\mathcal{S}_i \setminus \mathcal{S}_{i-1}$. Then, obviously, if π_{init} is the unique function from $\emptyset$ to Q, the path ρ_G is accepting if and only if, in this naive graph, the unique path from $(0, \pi_{\mathsf{init}})$ to (N, π) is such that π maps the unique element of $\mathcal{S}_N$ (namely, the root of the tree) onto Q_{end}.

While this naive construction is correct, it is not suitable in a dynamic complexity perspective: adding or removing an edge in G may affect many edges in the above graph. However, we show below that it can only affect edges at a single level (index i); using Dyck constraints, we then adapt the construction above to have updates of G only impact one edge of our graph.

The idea is illustrated on Figure 4. Assume that both upper (naive) graphs represent a parameterized function f with parameters γ (left) and γ' (right): there is an edge $\alpha_x \to \beta_y$ in the left graph whenever $\beta_y = f(\alpha_x, \gamma)$, and an edge $\alpha_x \to \beta_y$ in the right graph whenever $\beta_y = f(\alpha_x, \gamma')$. Replacing the value of γ with γ', i.e., transforming the left graph into the right one, requires many edge deletions and insertions.

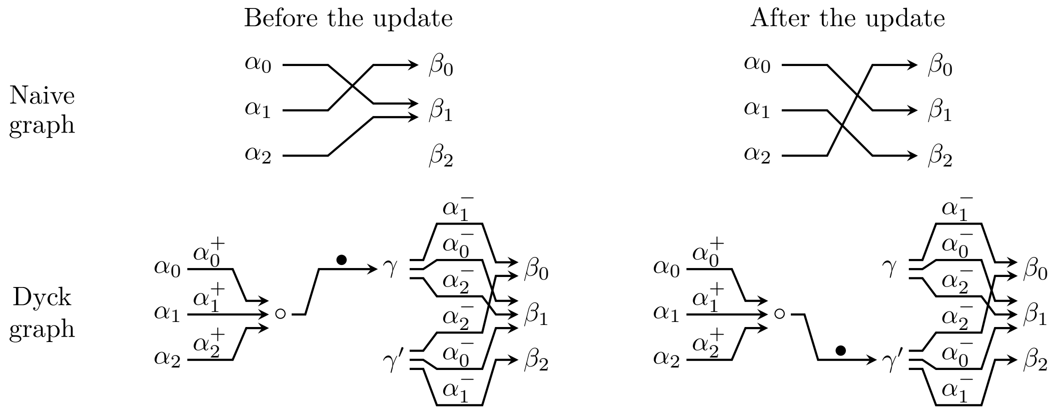


Figure 4 Using Dyck paths saves many changes when the input graph is updated

We circumvent this problem by using Dyck paths: the value of $f(\cdot,\gamma)$ is computed thanks to the Dyck path labeled with $\alpha_x^+ \cdot \bullet \cdot \alpha_x^-$, which goes from α_x to $\beta_y = f(\alpha_x,\gamma)$. Hence, changing the value of γ to γ' amounts to replacing the edge $\circ \xrightarrow{\bullet} \gamma$ with the new edge $\circ \xrightarrow{\bullet} \gamma'$.

The refined construction. A "nominal" vertex of the graph Γ_G is a pair (i,π), where $0 \leqslant i \leqslant N$ and π is a function $\pi\colon \mathcal{S}_i \to Q$, intended to represent the state progression at step i. Labels of Γ_G are the pairs $(i,\pi)^+$ and $(i,\pi)^-$, and the neutral label $\bullet$. We write π_{init} for the unique function from $\emptyset$ to Q. Now, for $1 \leq i \leq N$, let n_i be the unique node in $\mathcal{S}_i \setminus \mathcal{S}_{i-1}$, and recall that $\rho_G {\restriction}_{\mathcal{S}_i} = \Pi_i(\rho_G {\restriction}_{\mathcal{S}_{i-1}}, \Lambda_G(n_i))$. We therefore add the following edges and vertices:

- vertices $(i-1)$ and $(i-1,\gamma)$ for all $\gamma \in \Sigma$;
- edges $(i-1,\pi) \xrightarrow{(i-1,\pi)^+} (i-1)$ and $(i-1,\gamma) \xrightarrow{(i-1,\pi)^-} (i,\pi')$ for all $\gamma \in \Sigma$ and all $\pi\colon \mathcal{S}_{i-1} \to Q$, where $\pi'\colon \mathcal{S}_i \to Q$ is such that $\pi' = \Pi_i(\pi,\gamma)$;
- one neutral edge $(i-1) \xrightarrow{\bullet} (i-1,\gamma)$, where $\gamma = \Lambda_G(n_i)$.

Finally, observe that n_N is the root of $\mathcal{T}_\star$, that $\mathcal{S}_N = \{n_N\}$, and that the run ρ_G is accepting if, and only if, $\rho_G(n_N) \in Q_{\mathsf{end}}$. Hence, we complete the construction of Γ_G by adding a last state $\top$ and neutral edges $(N,\pi) \xrightarrow{\bullet} \top$ for those functions $\pi\colon \mathcal{S}_N \to Q$ such that $\pi(n_N) \in Q_{\mathsf{end}}$.

This construction is both sound and complete, and well-behaved under modifications of G, as outlined by the following results.

▶ **Proposition 12.** *The automaton $\mathcal{A}$ accepts the labeled tree $\mathcal{T}_G$ if, and only if, there exists a Dyck path in Γ_G from the vertex $(0,\pi_{\mathsf{init}})$ to the vertex $\top$.*

Proof. A path from $(0,\pi_{\mathsf{init}})$ to a vertex (N,ϖ), where ϖ is a function $\mathcal{S}_N \to Q$, is Dyck if, and only if, it uses only sub-paths of the form $(i-1,\pi) \xrightarrow{(i-1,\pi)^+} (i-1) \xrightarrow{\bullet} (i-1,\gamma) \xrightarrow{(i-1,\pi)^-} (i,\pi')$ with $\gamma = \Lambda_G(n_i)$ and $\pi' = \Pi_i(\pi,\gamma)$. Hence, such a Dyck path exists if, and only if, $\varpi = \rho_G(n_N)$, where ρ_G is the run of $\mathcal{A}$ on $\mathcal{T}_G$, in which case the intermediate vertices of the form (i,π) are the vertices $(i,\rho_G {\restriction}_{\mathcal{S}_i})$. The result follows immediately. ◀

▶ **Proposition 13.** *Let $e = (v,w)$ be an edge of the maximal graph $G_\star$, and let G and G' be two subgraphs of $G_\star$ such that G' is obtained by adding (resp. deleting) the edge e to G. The graph $\Gamma_{G'}$ is obtained by deleting an edge e_1 from Γ_G and inserting another edge e_2*

instead. Furthermore, both edges e_1 and e_2 are FO*-definable in terms of e, of Γ_G, and of some auxiliary precomputed predicates.*

Proof. We only deal here with insertion of an edge. We associate with the edge $e = \langle v, w\rangle$ of $G_\star$ a mapping $\mathsf{add}_e \colon \Sigma \to \Sigma$ defined by $\mathsf{add}_e \colon \langle \chi(A), \chi(B), \chi(C)\rangle \to \langle \chi(A), \chi(B), \chi(C) \cup \{\langle \chi(v), \chi(w)\rangle\}\rangle$. Then, let n be the top-most node of $\mathcal{T}_\star$ such that both v and w belong to $\mathbf{T}(n)$, and let $i = \mathsf{post}(n)$. Lemma 8 states that $n \in \mathcal{S}_i \setminus \mathcal{S}_{i-1}$. Hence, the labeling functions Λ_G and $\Lambda_{G'}$ coincide on all nodes $m \neq n$, and we have $\Lambda_{G'}(n) = \mathsf{add}_e(\Lambda_G(n))$. Consequently, the graph $\Gamma_{G'}$ is obtained from Γ_G in two consecutive steps:

1. we delete the only outgoing edge, of the vertex $(i-1)$, which is a neutral edge of the form $(i-1) \xrightarrow{\bullet} (i-1, \gamma)$;
2. we add the new edge $(i-1) \xrightarrow{\bullet} (i-1, \mathsf{add}_e(\gamma))$.

The case of deletion is analogous, and requires using a mapping del_e similar to add_e. Since the mappings $e \mapsto i$, $(e, \gamma) \mapsto \mathsf{add}_e(\gamma)$ and $(e, \gamma) \mapsto \mathsf{del}_e(\gamma)$ can be precomputed, it follows that both the edge e_1 that we deleted from Γ_G and the edge e_2 that we inserted instead can be computed with FO formulas. ◀

5 Overall complexity analysis

In this section, we analyze the complexity of our dynamic algorithm. While adequate notions of reduction do exist in dynamic complexity (see e.g. [19, 14]), our reduction does not satisfy all criteria, so we need to compute the complexity of our algorithm by hand.

First, denoting by $\mathcal{V}$ and $\mathcal{L}$ the vertex set and the label set of Γ_G, Lemma 11 states that the Dyck reachability problem in Γ_G can be solved by using FO update formulas *over the universe* $\mathcal{V} \uplus \mathcal{L}$. However, we need FO formulas over the universe V of our MSO model checking problem, i.e., V is the vertex set of the input graph. Hence, we must embed $\mathcal{V} \uplus \mathcal{L}$ into a set of tuples of elements of V of finite arity.

Lemma 6 states that $\mathcal{T}_\star$ is of height at most $c(\kappa) \cdot (\log_2(N) + 1)$, where $N = |V|$, and Lemma 8 proves that $|\mathcal{S}_i| \leqslant 2c(\kappa) \cdot (\log_2(N) + 1)$ for all $i \leqslant N$. It follows that $|\mathcal{V}| = 1 + \sum_{i=0}^{N} |Q|^{|\mathcal{S}_i|} = \mathcal{O}(N^{2c(\kappa)\log_2(|Q|)+1})$, which is polynomial in $|V|$. Likewise, $|\mathcal{L}|$ is polynomial in $|V|$, and therefore $\mathcal{V} \uplus \mathcal{L}$ can be embedded into some set V^k, where k is a large enough integer (which depends only on κ and on the MSO formula φ).

In the end, during the precomputation phase, the algorithm successively computes:

1. a binary rooted tree decomposition $\mathcal{D}_\star = \langle \mathcal{T}_\star, \mathbf{T}_\star\rangle$ of the maximal graph $G_\star = \langle V, E_\star\rangle$, of width $4\kappa + 3$, such as described in Lemma 6;
2. a (bottom-up, deterministic) tree automaton $\mathcal{A}_{\varphi, 4\kappa+3}$ such as defined in Courcelle's theorem;
3. a $\mathcal{D}_\star$-coloring function χ, a $(\chi, \mathcal{D}_\star)$-succinct tree decomposition of $G_\star$, and a bottom-up progression $(\mathcal{S}_i)_{0 \leqslant i \leqslant N}$ of $\mathcal{T}_\star$;
4. the vertices, labels and edges of the graph Γ_{G_E}, where G_E is the initial input graph;
5. an embedding $\mathcal{V} \uplus \mathcal{L} \mapsto V^k$;
6. mappings $e \mapsto i$, $(e, \gamma) \mapsto \mathsf{add}_e(\gamma)$ and $(e, \gamma) \mapsto \mathsf{del}_e(\gamma)$ mentioned in the proof of Proposition 13;
7. the value of the auxiliary predicate Δ (mentioned in Lemma 11) on Γ_{G_E}.

▶ **Lemma 14.** *Each of these 7 steps can be performed in* LOGSPACE.

Proof. The formula φ and the integer κ are fixed. Hence, Lemma 6 proves that the step 1 can be performed in LOGSPACE, and the step 2 is completed in constant time. Since performing the steps 3–6 in LOGSPACE is straightforward, it remains to deal with the step 7.

Let ϖ be a path with label λ in Γ_G. We say that ϖ is a *Dyck prefix* path if λ is a prefix of a Dyck word (which may be λ itself) and if its proper prefixes are not Dyck words; that ϖ is a *Dyck suffix* path if λ is a suffix of a Dyck word and if its proper suffixes are not Dyck words; that ϖ is a *minimal Dyck* path if ϖ is both a Dyck prefix and a Dyck suffix path.

Minimal non-empty Dyck paths are paths of the form $(i-1) \xrightarrow{\bullet} (i-1,\gamma)$, $(\ell,\pi) \xrightarrow{\bullet} \top$, and $(i-1,\pi) \xrightarrow{(i-1,\pi)^+} (i-1) \xrightarrow{\bullet} (i-1,\gamma) \xrightarrow{(i-1,\pi)^-} (i,\pi')$. Since Dyck prefix paths are the prefixes of minimal Dyck paths, and Dyck suffix paths are the suffixes of minimal Dyck paths, they have length at most 2. Furthermore, every Dyck path is a product of non-empty minimal Dyck paths. In addition, if ϖ_1 and ϖ_2 are paths with labels λ_1 and λ_2 such that $\lambda_1 \cdot \lambda_2$ is a Dyck word, then there exists factorisations $\varpi_1 = \varpi_1^{\mathsf{init}} \cdot \varpi_1^{\mathsf{end}}$ and $\varpi_2 = \varpi_2^{\mathsf{init}} \cdot \varpi_2^{\mathsf{end}}$ such that ϖ_1^{init} and ϖ_2^{end} are Dyck paths, ϖ_1^{end} is a Dyck prefix and ϖ_2^{init} is a Dyck suffix.

Hence, the predicate $\Delta(x_1,y_1,x_2,y_2)$ holds if, and only if, there exists vertices z_1 and z_2 and paths ϖ_1 (from z_1 to y_1) and ϖ_2 (from x_2 to z_2) with labels λ_1 and λ_2 such that:

- ϖ_1 and ϖ_2 are of length at most 2, and $\lambda_1 \cdot \lambda_2$ is a Dyck word;
- there exists Dyck paths from x_1 to z_1 and from z_2 to y_2.

Finally, note that every vertex of Γ_G is the source of at most one minimal Dyck path. Consequently, for any two vertices x and y of Γ_G, checking if there exists a Dyck path from x to y can be done in LOGSPACE, and Δ can be computed in LOGSPACE too. ◀

We sum up the above results as follows. First, we perform a LOGSPACE precomputation, and construct a graph Γ_G whose vertex, label and edge sets can be represented as predicates of finite arity on the universe V. Then, during the update phases, whenever introducing or deleting an edge e in G, we replace one edge of Γ_G by another one, and these edges are identified by precomputed FO formulas taking the edge e into account, as stated in Proposition 13. Consequently, and since the Dyck reachability problem is in DynFO, updating the edge-membership predicate of Γ_G and the auxiliary predicate Δ, which is useful for solving the Dyck reachability problem in Γ_G, can be done with FO formulas. Finally, deciding whether G satisfies the formula φ, i.e., whether there exists a suitable Dyck path in Γ_G, can be done using directly the auxiliary predicate Δ, which completes the proof of Theorem 3.

6 Conclusion

We developed a dynamic algorithm for checking a (fixed) MSO formula over (evolving) subgraphs of a given graph of bounded tree-width. A natural extension of this work would consist in getting rid of the hypothesis that there exists a maximal graph $G_\star$ of which the graphs under scrutiny are subgraphs. There are two main obstacles for this to be achieved in our approach: first, we would need to be able to dynamically compute tree decompositions of "moderate" width of our dynamic graphs; then, we would have to adapt the structure of our graph Γ_G to take into account these evolving tree decompositions.

Another direction of research, which was successfully put into practice in [4] when dealing with the particular case of parity games, would consist, given an input formula $\varphi = \exists X.\, \varphi'(X)$ (starting with an existential quantifier), to compute a witness X of the satisfiability of φ'.

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