Frequency domain local tests

for change detection

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Context :

Quality control in mechanical engineering

- Pairs of scalar input and output frequency domain data,
- Nominal input/output transfer functions available.

Wanted:

- Detect small deviations in those transfer functions,

  using nonparametric techniques.
- Needed: limit theorems (LLN, CLT) for Fourier analysis.

Contents

1. Some useful formulas

$$N = 2^q \; ; \quad \omega = e^{2i\pi n/N},\; n = 0, \ldots, N-1$$

$$Y^N(\omega) \triangleq \frac{1}{\sqrt{N}} \Sigma_{k=1}^N \, y_k \; e^{-i\omega k}$$

DFT (de)correlations $\quad y_n, z_n$ stationary

$$\begin{aligned}
\mathrm{E}\left(Y^N(-\omega)\; Z^N(\omega) \right) &= \frac{1}{N} \Sigma_{k,l=1}^N \, \mathrm{E}(y_k \; z_l) \; e^{-i\omega(l-k)} \\
&= \frac{1}{N} \Sigma_{r=1-N}^N \, e^{-i\omega r} \; (N-r) \; R_r^{yz} \\
&= S^{yz} \; (e^{i\omega}) + O\left(\frac{1}{N}\right) \qquad (a)
\end{aligned}$$

$$\mathrm{E}\left(Y^N(-\xi)\; Z^N(\omega) \right) = O\left(\frac{1}{N}\right) \qquad \text{for } \xi \neq \omega \qquad (b)$$

For $x_n = G(z)\ u_n$

$$| \ X^N(\omega) - G(e^{i\omega})\ U^N(\omega)\ | \le O\left(\frac{\text{Cst}_G\ \text{Cst}_u}{\sqrt{N}}\right) \qquad (c)$$

where $\text{Cst}_G \triangleq 2\ \Sigma_{k=1}^{\infty}\ k\ |g(k)| < \infty$

and $|u_n| < \text{Cst}_u$

2. Limit theorems

- No useful limit theorem when $N \to \infty$

  (N estimated frequencies)

- Idea: average over K successive blocks of size N

  (instead of windowing)

$$\left(Y_k^N(\omega),\ U_k^N(\omega) \right), \qquad k = 1, \ldots, K$$

1. For K large: $\frac{1}{K}\ \Sigma_{k=1}^{K}\ Y_k^N(-\omega)\ Z_k^N(\omega) = S^{yz}(e^{i\omega}) + O\left(\frac{1}{N}\right) \qquad (1)$

2. If y_n and z_n independent, and $S^{yy}(e^{i\omega})S^{zz}(e^{i\omega}) \neq 0$:
   using CLT for triangular arrays of random variables:

$$\frac{\Sigma_{k=1}^{K}\ Y_k^N(-\omega)\ Z_k^N(\omega)}{\sqrt{K}\ \sqrt{S^{yy}(e^{i\omega})S^{zz}(e^{i\omega})}} = O\left(\frac{\sqrt{K}}{N}\right) + \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)\ \mathcal{N}(0,1)$$

$$(\sqrt{K}/N \to 0)\quad \frac{1}{\sqrt{K}}\Sigma_{k=1}^{K}\ Y_k^N(-\omega)\ Z_k^N(\omega) \sim \mathcal{N}\left(0,\ S^{yy}(e^{i\omega})\ S^{zz}(e^{i\omega})\right) \qquad (2)$$

3. For $\xi \neq \omega$:

$$\mathrm{E}\left(\frac{1}{\sqrt{K}}\Sigma_{k=1}^{K}\ Y_k^N(-\xi)Z_k^N(\xi)\right)\left(\frac{1}{\sqrt{K}}\Sigma_{k=1}^{K}\ Y_k^N(-\omega)Z_k^N(\omega)\right) = O\left(\frac{1}{N}\right) \qquad (3)$$

4. $(K, N \to \infty)$ $\qquad \frac{1}{\sqrt{K}}\ \Sigma_{k=1}^{K}\ Y_k^N(-\omega)\ \sim\ \mathcal{N}(\ 0,\ S^{yy}(e^{i\omega})) \qquad (4)$

3. Local approach

$$y_n = G(z)\ u_n + v_n\ , \qquad S^{uu}(e^{i\omega})\ S^{vv}(e^{i\omega}) \neq 0$$

$$G\ =\ G_0\ +\ \frac{1}{\sqrt{K}}\ \tilde{G}$$

Frequency-wise complex residual:

$$\zeta_K^N(G_0, \omega) \triangleq \frac{1}{\sqrt{K}}\ \Sigma_{k=1}^{K}\ U_k^N(-\omega)\ \left(\ Y_k^N(\omega) - G_0(e^{i\omega})\ U_k^N(\omega)\ \right)$$

Theorem: $(\sqrt{K}/N \to 0)$

$$\zeta_K^N(G_0, \omega) \sim \mathcal{N}\left(\ S^{uu}(e^{i\omega})\ \tilde{G}(e^{i\omega}),\ \ S^{uu}(e^{i\omega})\ S^{vv}(e^{i\omega})\ \right)$$

For $\xi \neq \omega$: $\quad \mathrm{E}\ \left(\ \zeta_K^N(G_0, \xi)\ \zeta_K^N(G_0, \omega)\ \right) = 0 \quad$ (use (3))

Hints for the CLT:

$$Y_k^N(\omega) = G(e^{i\omega})\, U_k^N(\omega) + V_k^N(\omega) + O\left(\frac{1}{\sqrt{N}}\right)$$

$$\begin{aligned}
\zeta_K^N(G_0,\omega) &= \tilde{G}(e^{i\omega})\, \frac{1}{K}\, \Sigma_{k=1}^K \mid U_k^N(\omega) \mid^2 \quad \text{(use (1))} \\
&+ \frac{1}{\sqrt{K}}\, \Sigma_{k=1}^K\ U_k^N(-\omega)\ V_k^N(\omega) \quad \text{(use (2))} \qquad (\dagger) \\
&+ O\left(\frac{1}{\sqrt{N}}\right)\ \frac{1}{\sqrt{K}}\, \Sigma_{k=1}^K\, U_k^N(-\omega) \quad \text{(use (4))}
\end{aligned}$$

4. Testing

$$\mathrm{H}_0 : G = G_0 \quad \text{against} \quad \mathrm{H}_1 : G = G_0 + \frac{1}{\sqrt{K}}\, \tilde{G}$$

$$\chi_K^N\ (G_0,\omega) = \frac{\mid \zeta_K^N(G_0,\omega) \mid^2}{\hat{S}^{uu}(e^{i\omega})\ \hat{S}^{vv}(e^{i\omega})}\ , \qquad \chi^2(2) \text{ variable}$$

Noncentrality parameter:

$$\frac{S^{uu}(e^{i\omega})}{S^{vv}(e^{i\omega})} \mid \tilde{G}(e^{i\omega}) \mid^2$$

Estimates of the spectra:

$$\begin{aligned}
\hat{S}^{uu}(e^{i\omega}) &= \frac{1}{K}\, \Sigma_{k=1}^K \mid U_{0,k}^N(\omega) \mid^2 \\
\hat{S}^{vv}(e^{i\omega}) &= \frac{1}{K}\, \Sigma_{k=1}^K \mid Y_{0,k}^N(\omega) - G_0(e^{i\omega})\ U_{0,k}^N(\omega) \mid^2
\end{aligned}$$

5. Implementations issues

How many blocks?

$$K, N \to \infty \quad \text{such that} \quad \sqrt{K}/N \to 0$$

Closer look at $(\dagger)$

Distance between $\zeta_K^N(G_0,\omega)$ and its asymptotic behavior:

$$O\left(\frac{1}{\sqrt{K}} + \frac{1}{N} + \sqrt{\frac{K}{N}}\right)$$

$$\min_K \Longrightarrow K \sim \sqrt{N}$$