Advanced numerical computation of χ^2 tests for fault detection and isolation

Qinghua Zhang and Michèle Basseville

IRISA (INRIA & CNRS)
Rennes, France

Introduction

- Typical fault detection and isolation (FDI) procedure:
  - residual generation
  - residual evaluation
- Evaluation of Gaussian residuals
  - for parametric change in linear systems
  - for small parametric change in nonlinear systems (local asymptotic approach to change detection)
- Changes in the mean of a Gaussian vector — χ^2-tests

Gaussian residual evaluation — χ^2-tests

Consider m-dimensional residual $\mathbf{z} \sim \mathcal{N}(M\eta, \Sigma)$
with $M \in \mathbb{R}^{m\times n}$, $\eta \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{m\times m}$, $m \geq n$.

Fault detection
$H_0 : \eta = 0$ against $H_1 : \eta \neq 0$

Fault isolation for some partition $\eta = \begin{bmatrix} \eta_a \\ \eta_b \end{bmatrix}$
$H_0 : \eta_a = 0$ against $H_1 : \eta_a \neq 0$

These hypothesis testing problems lead to χ^2-tests.

Basic formulas for the χ^2-tests

$$\mathbf{z} \sim \mathcal{N}(M\eta, \Sigma) \quad \mathbf{I} = M^T\Sigma^{-1}M \quad \eta = \begin{bmatrix} \eta_a \\ \eta_b \end{bmatrix} \quad M = [M_a \; M_b] \quad \mathbf{I} = \begin{bmatrix} \mathbf{I}_{aa} & \mathbf{I}_{ab} \\ \mathbf{I}_{ba} & \mathbf{I}_{bb} \end{bmatrix}$$

Fault detection (global test)

$$t = \mathbf{z}^T\Sigma^{-1}M(M^T\Sigma^{-1}M_a)^{-1}M_a^T\Sigma^{-1}\mathbf{z}$$

Fault isolation by sensitivity test

$$\tilde{t}_a = \mathbf{z}^T\Sigma^{-1}M_a(M_a^T\Sigma^{-1}M_a)^{-1}M_a^T\Sigma^{-1}\mathbf{z}$$

or equivalently

$$\tilde{\zeta}_a = M_a^T\Sigma^{-1}\mathbf{z}$$
$$\tilde{t}_a = \tilde{\zeta}_a^T\mathbf{I}_{aa}^{-1}\tilde{\zeta}_a$$

Basic formulas of the χ^2-tests (contd.)

Fault isolation by min-max test

$$\begin{aligned}
\tilde{\zeta}_a &= M_a^T \Sigma^{-1} \mathbf{z} \\
\tilde{\zeta}_b &= M_b^T \Sigma^{-1} \mathbf{z} \\
\zeta_a^* &= \tilde{\zeta}_a - \mathbf{I}_{ab} \mathbf{I}_{bb}^{-1} \tilde{\zeta}_b \\
\mathbf{I}_a^* &= \mathbf{I}_{aa} - \mathbf{I}_{ab} \mathbf{I}_{bb}^{-1} \mathbf{I}_{ba} \\
t_a^* &= \zeta_a^{*T} \mathbf{I}_a^{*-1} \zeta_a^*
\end{aligned}$$

Numerical difficulties: when the matrices to be inverted are badly conditioned, these basic formulas can lead to large numerical errors. It is thus important to develop advanced numerical algorithms.

Advanced numerical computation – Global test

$$t = \mathbf{z}^T \Sigma^{-1} M (M^T \Sigma^{-1} M)^{-1} M^T \Sigma^{-1} \mathbf{z}$$

- Use pseudo-inverse for Σ^{-1} if badly conditioned
- Compute t as a square : $t = \|(M^T \Sigma^{-1} M)^{-\frac{1}{2}} M^T \Sigma^{-1} \mathbf{z}\|^2$
- Avoid the inverse involving M.

Proposed solution:

- let $\Gamma = \Sigma^{-\frac{1}{2}}$ (with pseudo-inverse if necessary),
- QR decomposition of ΓM: $\Gamma M = QR$ with $Q^T Q = I$,
- Then

$$\begin{aligned}
t &= \mathbf{z}^T \Gamma Q R (R^T Q^T Q R)^{-1} R^T Q^T \Gamma \mathbf{z} \\
&= \mathbf{z}^T \Gamma Q R (R^T R)^{-1} R^T Q^T \Gamma \mathbf{z} \\
&= \|Q^T \Gamma \mathbf{z}\|^2
\end{aligned}$$

Advanced numerical computation – Sensitivity test

$$\tilde{t}_a = \mathbf{z}^T \Sigma^{-1} M_a (M_a^T \Sigma^{-1} M_a)^{-1} M_a^T \Sigma^{-1} \mathbf{z}$$

Same as global test (M being replaced by M_a).

Advanced numerical computation:

QR decomposition of ΓM_a: $\Gamma M_a = Q_a R_a$ with $Q_a^T Q_a = I$, then

$$\tilde{t}_a = \|Q_a^T \Gamma \mathbf{z}\|^2$$

Advanced numerical computation – Minmax test

More computations are involved.

QR decompositions:

$$\begin{aligned}
\Gamma M_a &= Q_a R_a \\
\Gamma M_b &= Q_b R_b \\
(I - Q_b Q_b^T) Q_a &= Q_c R_c
\end{aligned}$$

then

$$t_a^* = \|Q_c^T \Gamma \mathbf{z}\|^2$$

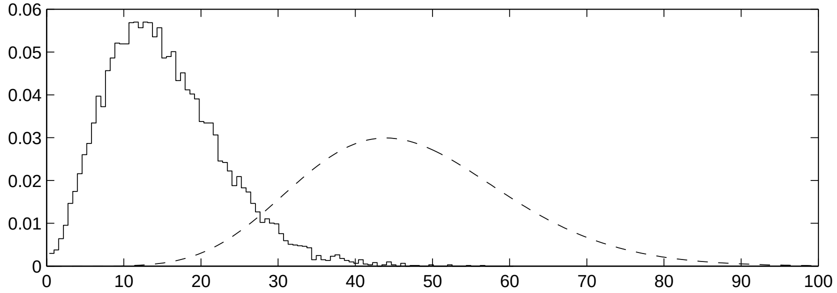
A non-trivial step for deriving the algorithm:

$$(I - Q_a^T Q_b Q_b^T Q_a) = Q_a^T (I - Q_b Q_b^T)(I - Q_b Q_b^T) Q_a$$

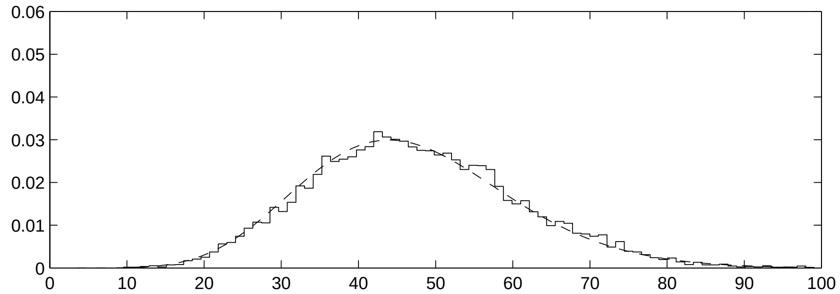
Remark: SVD can be used instead of QR decomposition.

A numerical example

Gaussian vector: $\dim(\mathrm{z}) = 9, \dim(\eta) = 5, \dim(\eta_a) = 2.$
Condition number of Σ: 3.4×10^{10}.
Histograms are based on 10000 random realizations.



Basic minmax test (solid line) and $\chi'^2(2, 44.955)$ density function (dashed line).



Advanced minmax test (solid line) and $\chi'^2(2, 44.955)$ density function (dashed line).

Conclusion

- χ^2-tests are frequently used for residual evaluation.
- Advanced numerical algorithms can significantly improve the numerical accuracy of badly conditioned problems.