Tight bounds for rumor spreading in graphs of a given conductance*

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Introduction

Epidemic algorithms are a prominent tool for scalable and robust information dissemination in networks. *Randomized rumor spreading* is a basic and well-studied family of such algorithms: A *rumor* spreads throughout the network by means of each node choosing a random neighbor to communicate with in every round.

Randomized rumor spreading algorithms have proven very efficient for various network topologies. Further, abstract graph properties of networks that guarantee efficient rumor spreading have been investigated. One such property yielding fast rumor spreading is high *conductance* – a standard measure of expansion in graphs.

We present some results on the relation between conductance and rumor spreading. Our main result is a tight upper bound on the speed of the classic PUSH-PULL algorithm. This bound improves a recent result by Chierichetti et al [2].

Randomized rumor spreading

We assume that the network is modeled by a connected and undirected graph G. Initially, an arbitrary vertex knows a *rumor*, and the goal is that every vertex learns the rumor. We consider the following classic rumor spreading algorithms. The algorithms proceed in rounds.

PUSH algorithm: In each round every *informed* vertex (i.e., every vertex that knows the rumor) chooses a random neighbor in G and sends the rumor to it.

PULL algorithm: In each round every *uninformed* vertex chooses a random neighbor, and if that neighbor knows the rumor it sends it to the uninformed vertex.

PUSH-PULL algorithm: In each round every vertex chooses a random neighbor to send the rumor to, or to request the rumor from.

Graph conductance

The *conductance* of a connected graph $G = (V, E)$ is a real $0 < \phi \leq 1$ defined as

$$\phi = \min_{\substack{U \subseteq V, \\ vol(U) \leq vol(V)/2}} \frac{|E(U, V \setminus U)|}{vol(U)},$$

where $vol(U)$ is the volume of U, i.e., the sum of the degrees of the vertices in U; and $E(U, V \setminus U)$ is the set of crossing edges of the cut $\{U, V \setminus U\}$. (See also **Figure 1**.)

* This work was published in [3].

Previously known bounds

PUSH or PULL:

- In any *regular* graph it takes $O(\phi^{-1} \log n)$ rounds until all vertices are informed w.h.p. [5].
- There are *non-regular* graphs with constant ϕ for which $\Omega(n)$ rounds are needed (e.g., a star). So, large ϕ *does not* imply fast PUSH or PULL.

PUSH-PULL:

- In *any* graph, it takes $O((\log \phi^{-1})^2 \phi^{-1} \log n)$ rounds until all vertices are informed w.h.p. [2].
- There are graphs for which $\Omega(\phi^{-1} \log n)$ rounds are needed [1].

gap

Our contribution

Our main result is that we close the above gap between upper and lower bounds for PUSH-PULL.

Theorem 1. *In any graph, PUSH-PULL takes* $O(\phi^{-1} \log n)$ *rounds w.h.p.*

For PUSH and PULL we provide optimal sufficient conditions for rumor spreading in $O(\phi^{-1} \log n)$ rounds in general graphs. By Δ (δ) we denote the max (min) graph degree.

Theorem 2. *In any graph, PULL takes* $O(\phi^{-1} \log n)$ *rounds w.h.p., if some of the next conditions hold:*

(a) *The rumor starts at a vertex of degree* $\Omega(\Delta(\phi + \delta^{-1}))$; *or*

(b) $\phi = O(1/\Delta)$ *(for any start vertex).*

Since $\phi + \delta^{-1} = O(1)$, we have the following interesting corollary.

Corollary 3. *Every graph contains a vertex such that PULL takes* $O(\phi^{-1} \log n)$ *rounds w.h.p. if the rumor starts at that vertex. E.g., a max-degree vertex is such a vertex.*

not true for PUSH!

Theorem 4. *In any graph, PUSH takes* $O(\phi^{-1} \log n)$ *rounds w.h.p. if* $\delta = \Omega(\Delta(\phi + \delta^{-1}))$ *or* $\phi = O(1/\Delta)$.

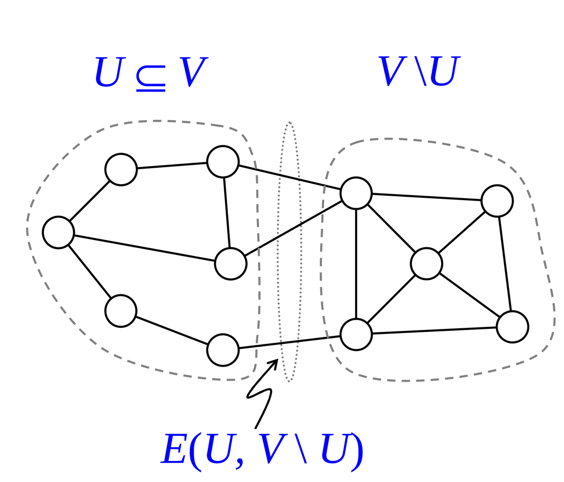


Figure 1. The above graph $G = (V, E)$ has conductance $\phi = 1/5$, since the cut $\{U, V \setminus U\}$ shown has the smallest ratio $|E(U, V \setminus U)| / vol(U) = 3/15$, over all sets $U \subseteq V$ with $vol(U) \leq vol(V)/2$. Typically, larger values of ϕ mean "better-knit" graphs. E.g., a path of length n has $\phi = \Theta(1/n)$; and the clique K_n has $\phi = \Theta(1)$.

Analysis

Key observation: It suffices to analyze PULL. The results for PUSH follow then by the symmetry between PUSH and PULL; and the bound for PUSH-PULL follows by combining results for PUSH and PULL.

Proof sketch of Theorem 1. By Corollary 3,

PULL spreads a rumor from a max-degree vertex ξ to all vertices in $O(\phi^{-1} \log n)$ rounds w.h.p.

From this and a symmetry argument,

PUSH spreads a rumor from any vertex to ξ in $O(\phi^{-1} \log n)$ rounds w.h.p.

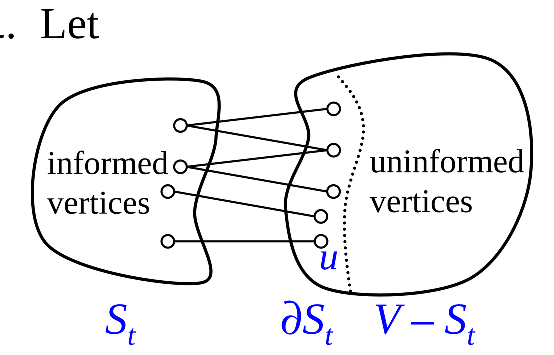
Thus, PUSH-PULL spreads the rumor to ξ in $O(\phi^{-1} \log n)$ rounds (just by "push" operations), and from ξ to all other vertex in $O(\phi^{-1} \log n)$ more rounds (by "pull" operations).

Intuition for the analysis of PULL. Let

S_t : set of informed vertices after round t;

∂S_t : outer boundary of S_t;

$\gamma(u)$, $u \in \partial S_t$: number of neighbors of u in S_t.



The *expected* increase of $vol(S_t)$ in a single round is

$$\sum_{u \in \partial S_t} \deg(u) \frac{\gamma(u)}{\deg(u)} = \sum_{u \in \partial S_t} \gamma(u) = |E(S_t, V \setminus S_t)| \geq \phi \cdot vol(S_t),$$

if $vol(S_t) \leq vol(V)/2$. Thus,

$$E[vol(S_{t+1})] \geq (1 + \phi) \cdot E[vol(S_t)].$$

So, *if the process behaved as in expectation*, it would take $O(\phi^{-1} \log[vol(V)]) = O(\phi^{-1} \log n)$ rounds until $vol(S_t) > vol(V)/2$. Similarly, once $vol(S_t) > vol(V)/2$, it would take $O(\phi^{-1} \log n)$ more rounds until $vol(V \setminus S_t) = 0$.

We turn this intuition into a rigorous proof by using a martingale argument.

Related problems

- Relation between rumor spreading and *vertex expansion* – another standard measure of expansion in graphs. Recent results show that similar bounds as with conductance hold [6,4].
- General *lower bounds* for rumor spreading time. Standard expansion measures are not sufficient, as there exist graphs with bad expansion where rumor spreading is fast.

References

[1] F. Chierichetti, S. Lattanzi and A. Panconesi. Rumour spreading and graph conductance. *SODA*, 2010.

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