The Effect of Power-Law Degrees on the Navigability of Small Worlds*

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Abstract

We analyze decentralized routing in small-world networks that combine a wide variation in node degrees with a notion of spatial embedding. Specifically, we consider a variation of Kleinberg's augmented-lattice model (STOC 2000), where the number of long-range contacts for each node is drawn from a power-law distribution. This model is motivated by the experimental observation that many "real-world" networks have power-law degrees. In such networks, the exponent α of the power law is typically between 2 and 3. We prove that, in our model, for this range of values, $2 < \alpha < 3$, the expected number of steps of greedy routing from any source to any target is $\mathrm{O}(\log^{\alpha-1} n)$ steps. This bound is tight in a strong sense. Indeed, we prove that the expected number of steps of greedy routing for a uniformly-random pair of source–target nodes is $\Omega(\log^{\alpha-1} n)$ steps. We also show that for $\alpha < 2$ or $\alpha \geq 3$, greedy routing performs in $\Theta(\log^2 n)$ expected steps, and for $\alpha = 2$, $\Theta(\log^{1+\varepsilon} n)$ expected steps are required, where $1/3 \leq \varepsilon \leq 1/2$.

To the best of our knowledge, these results are the first to formally quantify the effect of the power-law degree distribution on the navigability of small worlds. Moreover, they show that this effect is significant. In particular, as α approaches 2 from above, the expected number of steps of greedy routing in the augmented lattice with *power-law degrees* approaches the square-root of the expected number of steps of greedy routing in the augmented lattice with *fixed degrees*, although both networks have the same *average degree*.

1 Introduction

1.1 Navigability of small worlds

It has been observed that many "real-world" networks, such as social, information, technological, and biological networks, exhibit the *small-world* property; i.e., they are locally clustered, and (yet) short paths exist between almost all pairs of nodes (see [2, 9, 19] and the references therein). It is also well-established that many small-world networks (e.g., the network of acquaintances between individuals) are easy to *navigate*, provided that the nodes are able to estimate the distances to other nodes with respect to some underlying metric (e.g., geography, professions, etc.) [8, 18]. *Navigability* refers to the ability of nodes to route messages efficiently in a decentralized manner, using local information only. The most prominent example of such a routing scheme is *greedy* routing: a node handling a message destined to some target node forwards the message to its neighbor that is closest to the target, according to the underlying metric. The first formal analysis of greedy routing in a plausible model of small worlds was presented in [13]. The model studied there was the *augmented lattice*: Consider the n-node d-dimensional lattice that wraps around, where $d \geq 1$. A node has links to its $2d$ lattice-neighbors, and also to $k \geq 1$ other nodes, its *long-range contacts*. Each of the long-range contacts of a node u is chosen using an

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independent random trial following the d-harmonic distribution: the probability that node v is chosen in a given trial is

$$p_{u,v} \propto 1/(\text{dist}(u,v))^d, \tag{1.1}$$

where $\text{dist}(u,v)$ is the lattice-distance between u and v. In [13] it was shown that, in this model, greedy routing requires $\mathrm{O}\big(\frac{1}{k}\log^2 n\big)$ expected number of steps, for any source–target pair. (This complexity was later shown to be tight [17].) It was also shown that *any* decentralized routing algorithm performs poorly if the d-dimensional lattice is augmented using the h-harmonic distribution, for any $h \neq d$. Specifically, $\Omega(n^\gamma)$ expected steps are required, for some $\gamma > 0$ that depends on h and d.[1]

Despite its simplicity, the augmented-lattice model seems to capture successfully the small-world and navigability properties of real-world networks. Note that in the d-dimensional lattice the d-harmonic distribution is equivalent to the "natural" distribution $p_{u,v} \propto 1/|B_u(\text{dist}(u,v))|$, where $B_u(r)$ is the ball centered at u of radius r; this latter distribution was used in [10, 23] to extend the results of [13] to graphs of bounded ball growth, and to graphs of bounded doubling dimension. Also, the d-harmonic distribution is equivalent in the lattice to the rank-based distribution $p_{u,v} \propto 1/r_u(v)$, where $r_u(v)$ is the rank of v when nodes are sorted in increasing distance from node u; this latter distribution was used in [15] to extend the results of [13] to non-uniform population densities. In fact, it was experimentally demonstrated that two-thirds of friendships are geographically distributed this way: the probability of befriending a particular person is inversely proportional to the number of people closer to you [16]. Finally, it was recently shown that the d-harmonic distribution of the long-range links might as well be an inherent byproduct of node mobility [4]. See also [6, 20] for other dynamics yielding the d-harmonic distribution in the lattice. Therefore, there is now a consensus that the augmented-lattice model is an appropriate framework for analyzing small-world navigability.

1.2 Power-law degree distribution

The augmented-lattice model, however, fails to capture another commonly observed property of real-world networks, the *heavy-tailed degree distribution*. Such a distribution is well approximated by a *power law*

$$\mathbb{Pr}[\deg(u) = k] \propto 1/k^\alpha, \tag{1.2}$$

where α is a real, typically between 2 and 3 [2, 9, 19]. Nevertheless, it is straightforward to reconcile the augmented-lattice model with a power-law distribution for the node degrees, simply by drawing the number of long-range links added to each node independently at random from a power-law distribution [14]. It is reasonable to expect that this modification would reduce the lengths of shortest paths between nodes and the network diameter, since the (few) high-degree nodes should provide short-cuts between most nodes. This is typically the case in networks with power-law degree sequences [3, 5]. However, it is unclear how decentralized routing could benefit from the existence of these high-degree nodes [14].

Utilizing the heavy-tailed degree distribution in the design of decentralized routing algorithms was suggested in [1, 11, 12, 21]. In all these works, the routing algorithms only have access to information about the degrees of neighboring nodes, not to any embedding of the graph. Although some performance improvements are observed compared to routing algorithms oblivious to the node degrees, the expected number of steps remains polynomial in the network size. Also, [22] proposed a heuristic decentralized algorithm for routing in a variance of the augmented lattice where nodes have widely varying degrees. This heuristic assumes that nodes have access both to the locations of theirs neighbors, and to their degrees. Simulations showed that this algorithm performs better than decentralized algorithms using only one of these two sources of information. However, no formal analysis was provided.

[1] It was recently shown [7] that for $d = 1$, the augmentation using the 1-harmonic distribution is essentially optimal in the sense that for *any* augmentation distribution with k *expected* long-range contacts per node, greedy routing requires $\Omega\big(\frac{1}{k}\log^2 n\big)$ expected steps.

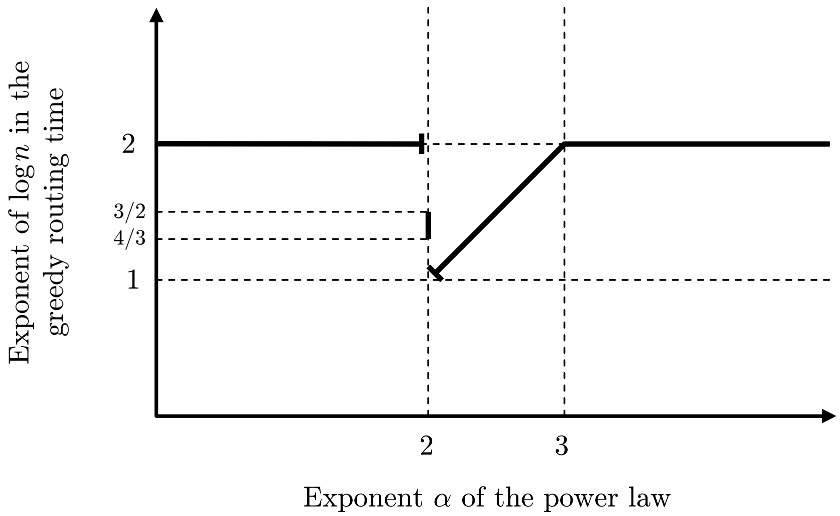

Figure 1: Summary of the results.

1.3 Our framework

We consider the following variance of the augmented-lattice model. As in the original model, the long-range links are drawn independently at random according the harmonic distribution with exponent equal to the dimensionality of the lattice (cf. Eq. 1.1). Unlike the original model, however, the number of long-range contacts each node has is not fixed, but it is drawn independently at random from the power-law distribution with exponent $\alpha \geq 0$ (cf. Eq. 1.2). This distribution is scaled so that its expectation is constant and each node has at least one long-range contact.[2] We then remove the orientation of each of the long-range links to get a non-directed network. We study the performance of greedy routing in this network.

1.4 Our results

In this section, we ignore $\mathrm{O}(\log\log n)$ multiplicative factors in the statement of the asymptotic bounds. The precise bounds are described in Section 2.3.

We prove that for $2 < \alpha < 3$, which is the case for most real-world networks, the expected number of steps of greedy routing *from any source to any target* is $\mathrm{O}(\log^{\alpha-1} n)$ steps. Thus, for this range of values for α, the effect of the power-law degree distribution is significant. In particular, when α approaches 2, the expected number of steps of greedy routing in the augmented lattice with *power-law degrees* approaches the square-root of the expected number of steps of greedy routing in the augmented lattice with *fixed degrees*, although both networks have the same *average degree*. For both $\alpha < 2$ and $\alpha \geq 3$, we show that the expected number of steps of greedy routing from any source to any target is $\mathrm{O}(\log^2 n)$ steps, which is the same order of magnitude as the performance of greedy routing in the augmented lattice with fixed degrees. For the critical value $\alpha = 2$, we prove that the expected number of steps of greedy routing from any source to any target is $\mathrm{O}(\log^{3/2} n)$ steps.

All these upper bounds are tight (but, perhaps, for $\alpha = 2$). For $\alpha > 2$, the upper bounds are even tight in a strong sense. Indeed, we prove that the expected number of steps of greedy routing for a *uniformly-random* pair of source–target nodes is $\Omega(\log^{\alpha-1} n)$ steps if $2 < \alpha < 3$, and $\Omega(\log^2 n)$ steps if $\alpha \geq 3$. For $\alpha < 2$, we prove that there exists a source–target pair for which greedy routing requires $\Omega(\log^2 n)$ expected steps. For $\alpha = 2$, we show that the expected number of steps for a uniformly-random source–target pair is $\Omega(\log^{4/3} n)$.

We formally prove the above results for the case of the 1-dimensional lattice, i.e., the ring. Nevertheless, none of the arguments we use is specifically tied to the ring, and the *exact* same results can be easily shown for

[2]For $\alpha > 2$, even without the scaling, the expectation is constant and, with constant probability, each node has at least one long-range contact. So, the scaling makes a difference only for $\alpha \leq 2$.

d-dimensional lattices, for constant values of d. Note that unlike the results in [13], where the critical value of the exponent depends on the dimensionality d of the lattice, our results do not depend on d.

To the best of our knowledge, these results are the first to formally quantify the effect of the power-law degree distribution on the navigability of small worlds.

The following picture emerges from our analysis. For $\alpha \geq 3$, almost all nodes are of small degree, and the nodes of higher degree are too few to contribute significantly. Hence greedy routing performs essentially the same as when the degrees are fixed.

For $2 < \alpha < 3$, there are still very few nodes of high degree. However, nodes of degree roughly $\log n$ are relatively abundant, and there are more and more such nodes as α approaches 2. It is the contribution of these nodes that reduces the routing time from $\log^2 n$ to $\log^{\alpha-1} n$.

The case $\alpha = 2$ is special. All "degree scales" are present, and each is equally likely to contribute. On the one hand, this results in greater routing speed than in the case $2 < \alpha < 3$ when the current node is far from the target, since there are many high-degree nodes between the current node and the target in the lattice. On the other hand, the balance in the degree scales means that as we get closer to the target the number of high-degree nodes available decreases faster than in the case $2 < \alpha < 3$; and when we get at distance sub-polynomial from the target (essentially at distance less than $e^{\sqrt{\ln n}}$), greedy routing performs the same as when the degrees are fixed.

Finally, for $\alpha < 2$, there are many nodes of high degree, and the role of the cut-off point $k_{\max}$ of the power law becomes critical. We assumed that $k_{\max} \sim n^{\gamma}$, for some $0 < \gamma \leq 1$. In this setting, only the contribution of nodes with degree close to $k_{\max}$ is significant. However, when the current node is at distance less than $k_{\max}$ from the target, it is very likely that greedy routing will not find a node of such degree, and from that distance it starts performing the same as when the degrees are fixed. Note that for $\alpha < 2$, nodes that are further away from the target may, in expectation, require fewer steps to reach the target than nodes closer to the target, which is not the case when $\alpha > 2$.

2 Model and main results

2.1 Network model

We will use the notation $[i..j] = \{k \in \mathbb{Z} : i \leq k \leq j\}$ and $[i..j) = [i..j-1]$, for $i, j \in \mathbb{Z}$. (If $i > j$ then $[i..j] = \emptyset$.) Also, whenever we treat a real number x as an integer we will mean $\lfloor x \rfloor$.

In our analysis we will focus on the 1-dimensional lattice case. Let $\mathbf{G}_n$ be the class of all directed graphs with set of nodes $[0..n)$ that contain as a subgraph the n-node *ring*, i.e., the graph with set of nodes $[0..n)$, and set of edges $\{(u, u \pm 1 \bmod n) : u \in [0..n)\}$.[3] Let G be a graph in $\mathbf{G}_n$, and E be the set of edges of G. The *out-neighbors* (*in-neighbors*) of a node u of G are all nodes v such that $(u, v) \in E$ ($(v, u) \in E$). More specifically, the nodes $u \pm 1 \bmod n$ are called the *ring-neighbors* of u, and the remaining out-neighbors (in-neighbors) of u are its *out-contacts* (*in-contacts*). For any two subsets of nodes A and B, we will write $A \to B$ to denote that a node in B is an out-contact of a node in A (or, equivalently, a node in A is an in-contact of a node in B). When $|A| = 1$, say $A = \{a\}$, we will often write $a \to B$, instead; and the same convention is used when $|B| = 1$. The *ring-distance* between nodes u and v, denoted $\delta(u, v)$, is the minimum number of ring edges between them in either clockwise or counter-clockwise direction, i.e.,

$$\delta(u, v) = \min\{u - v \bmod n,\, v - u \bmod n\}.$$

So, if v is a ring-neighbor of u then $\delta(u, v) = 1$, and if it is an in-contact or out-contact of u then $\delta(u, v) \geq 2$. We will write $\|u\|$ to denote $\delta(u, 0)$.

Random-graph models: We study two random-graph models. Each of them is parameterized by the size n of the graph, and the exponent $\alpha \geq 0$ of a power-law distribution. In the first model, denoted $\mathcal{G}(n, \alpha)$, a random

[3] For a graph in $\mathbf{G}_n$, the underlying ring will be used to compute distances between nodes. Also, when we refer to nodes we will mean their integer labels.

element of $\mathbf{G}_n$ is generated by choosing the out-contacts of the nodes as follows. For each node u, we draw an integer D_u from $[1..k_{\max}]$ independently at random, such that

$$(1)\ \mathbb{Pr}[D_u = k] \propto 1/k^\alpha,\ \text{for } k \neq 1, \quad \text{and} \quad (2)\ \mathbb{E}[D_u] = 2.$$

We assume that the "cut-off point" $k_{\max}$ of the power-law distribution is $\Theta(n^\gamma)$, for some constant $0 < \gamma \leq 1$. Then, we perform D_u independent identical random trials, such that in each trial a node $v \neq u$ is chosen with probability

$$\propto 1/\delta(u, v).$$

The out-contacts of u are all the *distinct* nodes chosen by these D_u trials that are not ring-neighbors of u; formally, if v_i is the node chosen in the i-th trial then the out-contacts of u are the elements of the set $\bigcup_{1 \leq i \leq D_u} \{v_i\} \setminus \{u \pm 1 \bmod n\}$. The second random-graph model we consider, denoted $\mathcal{U}(n, \alpha)$, is the model in which a random graph is obtained by first generating a random graph in $\mathcal{G}(n, \alpha)$, and then taking its underlying *undirected graph*; in fact, for each (directed) edge of $\mathcal{G}(n, \alpha)$ we also add its opposite-directed edge, if it does not already exist. Formally, if E is the set of edges in $\mathcal{G}(n, \alpha)$ then the set of edges in $\mathcal{U}(n, \alpha)$ is $\{(u, v) : (u, v) \in E \text{ or } (v, u) \in E\}$.

Discussion: Recall that the long-range contacts of a node in $\mathcal{G}(n, \alpha)$ are selected using independent trials *with replacement*. This assumption simplifies the analysis, but it has the side effect that the out-degree D_u^+ of node u in $\mathcal{G}(n, \alpha)$ can be smaller than D_u, and also the distribution of D_u^+ is not *exactly* a power law. Nevertheless, the use of trials with replacement gives essentially the same results as the trials *without replacement*. This is because the discrepancy between D_u^+ and D_u is significant only for large values of D_u (e.g., of order $\Omega(\sqrt{n})$). And our analysis shows that the effect of such high-degree nodes is negligible when $\alpha > 2$; while for $\alpha < 2$ our proof actually holds even if trials are without replacement. For the in-degree D_u^- of u in $\mathcal{G}(n, \alpha)$, it is easy to see that its distribution is close to a Poisson with constant expectation (see the Appendix); so, the distribution of the (total) degree $D_u^+ + D_u^-$ of u in $\mathcal{U}(n, \alpha)$ is essentially the same as that of D_u^+ for all but very small values.

Next, recall that $D_u \leq k_{\max} = \Theta(n^\gamma)$ and $D_u \neq 0$. For $\alpha > 2$, the exact same asymptotic results hold with or without these constrains. The rest of the discussion is for the case $\alpha \leq 2$. Note that it must be $k_{\max} < \infty$, otherwise the expectation of D_u is ∞. Also, a value for $k_{\max}$ that is polynomial in n is consistent with real-world networks. It can be shown that if $k_{\max}$ is poly-log in n then greedy routing performs in logarithmic time. On the lower side, a minimum value is imposed on D_u because otherwise D_u would be 0 with overwhelming probability. It can be shown that, in that case, greedy routing would require polynomial time if $\alpha < 2$, and poly-log time if $\alpha = 2$.

2.2 Greedy routing

We consider the following routing algorithm for graphs in $\mathbf{G}_n$. When a node u receives a message for a target node $t \neq u$, u forwards the message to an out-neighbor that is closest to t, with respect to the ring-distance. We call this routing algorithm GREEDY. We are interested in the performance of GREEDY in $\mathcal{G}(n, \alpha)$ and $\mathcal{U}(n, \alpha)$. Specifically, we study two performance measures: the *expected delivery time* of GREEDY, and the GREEDY *diameter*. Let $l_{u,v}$ be the *expected* length of the GREEDY routing path from u to v in the random graph. The *expected delivery time* is the *average* of $l_{u,v}$, taken over all possible source–target pairs, i.e.,

$$\text{Expected delivery time of GREEDY} = \frac{1}{n^2} \sum_{u,v} l_{u,v}.$$

The GREEDY *diameter* is the corresponding *maximum*, i.e.,

$$\text{GREEDY diameter} = \max_{u,v} l_{u,v}.$$

Note that the GREEDY diameter is always greater or equal to the corresponding expected delivery time. All the lower bounds we prove, except for the model $\mathcal{U}(n, \alpha)$ with $\alpha < 2$, are for the expected delivery time of GREEDY; whereas all the upper bounds are for the GREEDY diameter.

Throughout our analysis of GREEDY in $\mathcal{G}(n,\alpha)$ and $\mathcal{U}(n,\alpha)$, we will assume that the target node is node 0. We can make this assumption without loss of generality because of the symmetry of the random-graph models. Also, for the analysis in $\mathcal{U}(n,\alpha)$, instead of considering the graph $\mathcal{U}(n,\alpha)$ directly, we will consider $\mathcal{G}(n,\alpha)$, and for the purposes of routing we will ignore the direction of the links. So, whenever we refer to the in-/out-contacts, in-/out-links, etc., of a node, we will mean in $\mathcal{G}(n,\alpha)$; the same convention is used for the '$\rightarrow$' notation.

In $\mathcal{G}(n,\alpha)$, the GREEDY path from a fixed source to a fixed target is a Markov chain; the next node in the path depends only on the last node visited. However, this is not the case in $\mathcal{U}(n,\alpha)$, where the next node depends on all the previously visited nodes, and also on their in- and out-links. Specifically, if $\langle Y_0, Y_1, \ldots\rangle$ is the routing path from node Y_0 to 0 then for any node v with $\|v\| < \|Y_i\|$, $v \not\rightarrow \{Y_0, \ldots, Y_{i-1}\}$; hence, the values of $Y_0, \ldots, Y_{i-1}$ affect the distribution of the out-contacts of v. More importantly, the distribution of the out-contacts of Y_i is affected by whether $Y_{i-1} \rightarrow Y_i$ or not; e.g., if $Y_{i-1} \not\rightarrow Y_i$ and Y_i is not a ring-neighbor of Y_{i-1} then $Y_i \rightarrow Y_{i-1}$, which, for some values of α, changes the a-priory distribution of the out-degree of Y_i significantly.

2.3 Statement of the results

In all the results below, the asymptotic notation is as $n \rightarrow \infty$, and α is not a function of n.

Theorem 2.1. *The expected delivery time of* GREEDY *in* $\mathcal{G}(n,\alpha)$ *is* $\Omega(\ln^2 n)$.

Theorem 2.2. *The* GREEDY *diameter of* $\mathcal{G}(n,\alpha)$ *is* $\mathrm{O}(\ln^2 n)$.

Theorem 2.3. *The expected delivery time of* GREEDY *in* $\mathcal{U}(n,\alpha)$ *is*

$$\begin{cases} \Omega(\ln^{4/3} n), & \textit{if } \alpha = 2; \\ \Omega(\ln^{\alpha-1} n), & \textit{if } 2 < \alpha < 3; \\ \Omega(\ln^2 n/\ln\ln n), & \textit{if } \alpha = 3; \\ \Omega(\ln^2 n), & \textit{if } \alpha > 3. \end{cases}$$

Also, for $0 \le \alpha < 2$*, the* GREEDY *diameter is* $\Omega(\ln^2 n)$.

Theorem 2.4. *The* GREEDY *diameter of* $\mathcal{U}(n,\alpha)$ *is*

$$\begin{cases} \mathrm{O}(\ln^2 n), & \textit{if } 0 \le \alpha < 2; \\ \mathrm{O}(\ln^{3/2} n), & \textit{if } \alpha = 2; \\ \mathrm{O}(\ln^{\alpha-1} n \ln\ln n), & \textit{if } 2 < \alpha < 3; \\ \mathrm{O}(\ln^2 n), & \textit{if } \alpha \ge 3. \end{cases}$$

3 Definitions and basic facts

3.1 Distribution of out-contacts

Recall that the out-contacts of a node u in $\mathcal{G}(n,\alpha)$ are chosen using independent identical trials, such that, in each trial, node $v \ne u$ is picked with probability $\propto 1/\delta(u,v)$. So, the probability that v is picked in a given trial is

$$\frac{1}{\nu\,\delta(u,v)}, \quad \text{where } \nu = \sum_{v \ne u} \frac{1}{\delta(u,v)} = 2\ln n + \mathrm{O}(1).$$

Also, the number D_u of trials used is chosen independently at random from $[1..k_{\max}]$, where $k_{\max} = \Theta(n^\gamma)$, such that $\mathbb{Pr}[D_u = k] \propto 1/k^\alpha$, for $k \ne 1$, and $\mathbb{E}[D_u] = 2$. Let

$$q_k = \mathbb{Pr}[D_u = k] = \begin{cases} \frac{1}{\beta k^\alpha}, & \text{if } k \ne 1; \\ 1 - \sum_{j \ne 1} q_j, & \text{if } k = 1, \end{cases}$$

where β is the normalizing factor such that $\mathbb{E}[D_u] = \sum_k kq_k = 2$. It is easy to see that

$$\beta = \sum_k \frac{k-1}{k^\alpha} = \begin{cases} \Theta(1), & \text{if } \alpha > 2; \\ \Theta(\ln n), & \text{if } \alpha = 2; \\ \Theta(k_{\max}^{2-\alpha}), & \text{if } 0 \le \alpha < 2. \end{cases}$$

Also, the probability that $D_u = 1$ is $q_1 = \Theta(1)$, if $\alpha > 2$, and $q_1 = 1 - \mathrm{o}(1)$, if $0 \le \alpha \le 2$.

3.2 Simple facts about $\mathcal{G}(n, \alpha)$

We now state some simple facts that we use repeatedly in the analysis. Their proofs can be found in the Appendix. In all these facts, the underlying graph is $\mathcal{G}(n, \alpha)$ and $u, v \in [0..n)$.

Fact 3.1. *If $U \subseteq [0..n)$ and $p = \mathbb{Pr}[u \to U \mid D_u = 1]$ then*

$$\tfrac{1}{2}\min\{1, kp\} \le \mathbb{Pr}[u \to U \mid D_u = k] \le \min\{1, kp\}.$$

Fact 3.2. *If v is not a ring-neighbor of u*[4] *then*

$$\tfrac{1}{\nu\delta(u,v)} \le \mathbb{Pr}[u \to v] \le \tfrac{2}{\nu\delta(u,v)}.$$

Fact 3.3. *If $U = \{u + d \bmod n \,:\, d \in [a..b]\}$ or $U = \{u - d \bmod n \,:\, d \in [a..b]\}$, where $2 \le a \le b \le n/2$, then*

$$\tfrac{1}{\nu}\ln\tfrac{b+1}{a} \le \mathbb{Pr}[u \to U \mid D_u = 1] \le \tfrac{1}{\nu}\ln\tfrac{b}{a-1}.$$

Fact 3.4. *If $U_1, U_2 \subseteq [0..n)$ and $U_1 \cap U_2 = \emptyset$ then*

$$\mathbb{Pr}[u \to U_1 \mid \{D_u = k\} \cap \{u \not\to U_2\}] = \frac{\mathbb{Pr}[u \to U_1 \mid D_u = k]}{\mathbb{Pr}[u \not\to U_2 \mid D_u = 1]}.$$

For the next fact we need to introduce some notation, which we also use throughout the analysis. Let R_x be the set of all nodes at ring-distance at most x from 0; i.e.,

$$R_x = \{u \,:\, \|u\| \le x\}.$$

By $\mathbf{H}_u$ we denote the set of all sets $H \subseteq [0..n) \setminus R_{\|u\|}$, such that for any two distinct $v_1, v_2 \in H$, $\|v_1\| \ne \|v_2\|$. Note that for any graph in $\mathbf{G}_n$ and node s, every prefix path of the routing path from s to 0 that contains no nodes in $R_{\|u\|}$ belongs to $\mathbf{H}_u$.

Fact 3.5. *If $H \in \mathbf{H}_u$ and $d = \min_{v \in H} \delta(u, v)$ then (a) $\mathbb{Pr}[u \to H \mid D_u = k] \le \mathbb{Pr}[0 \to [d..d + |H|) \mid D_0 = k]$; (b) $\mathbb{Pr}[u \not\to H \mid D_u = k] \ge 1/2^k$; and (c) $\mathbb{Pr}[u \not\to H] \ge q_1/2$.*

4 Proof of the lower bounds

We begin with an auxiliary lemma that bounds from below the average length of any process that approaches 0 with jumps that follow a distribution of a specific form. We use this result in the proofs of all the lower bounds. We prove the lower bound for $\mathcal{G}(n, \alpha)$ in Section 4.1, and for $\mathcal{U}(n, \alpha)$ with $\alpha > 2$, $\alpha < 2$, and $\alpha = 2$ in Sections 4.2–4.4, respectively.

The next lemma provides a lower bound on the expected number of steps of an arbitrary process on the non-negative integers, which is non-increasing, and the length of the jump in each step is bounded by a distribution of a certain form. We will use this result in the proofs of all the lower bounds.

[4] If v *is* a ring-neighbor of u then $\mathbb{Pr}[u \to v] = 0$, by the definition of '$\to$'.

Lemma 4.1. *If $\langle X_0, X_1, \ldots\rangle$ is a non-increasing, non-negative, integer-valued random process with $X_0 > \rho \geq 1$, such that for all j with $\rho < j \leq X_0$,*

$$\mathbb{Pr}[X_{i+1} = j' \mid X_i = j] \leq \begin{cases} c\frac{(j/j')^\epsilon}{\rho(j-j')}, & \text{if } 0 < j' \leq j-2; \\ c\frac{1}{\rho \ln j}, & \text{if } j' = 0, \end{cases} \tag{4.1}$$

where $0 \leq \epsilon < 1$, then the expected number of steps to reach 0 is at least $c'\rho\ln(X_0/\rho)$, where $c' = c'(c,\epsilon) > 0$.

The proof of Lemma 4.1 is similar to the proof of the lower bound for the augmented lattice with fixed degrees, described in [17] (Theorem 7). Roughly, we consider the sequence of $\ln X_i$, show that the average reduction in each step is at most $c''(c,\epsilon)/\rho$, and use an expectation argument to obtain the lower bound. The full proof can be found in the Appendix.

4.1 Proof of Theorem 2.1

It is a straightforward application of Lemma 4.1. Let $\langle Y_0, Y_1, \ldots\rangle$ be the routing path from Y_0 to 0 in $\mathcal{G}(n,\alpha)$. For all u, v with $\|v\| \leq \|u\| - 2$,

$$\mathbb{Pr}[Y_{i+1} = v \mid Y_i = u] \leq \mathbb{Pr}[u \to v] \leq \tfrac{2}{\nu\delta(u,v)},$$

by Fact 3.2. From this and Lemma 4.1, applied for $n/4 \leq X_0 \leq n/2$, $\rho = \nu$, $\epsilon = 0$, and $X_i = \|Y_i\|$, we obtain that the expected length of the routing path from u to 0 is $\Omega(\nu \ln n) = \Omega(\ln^2 n)$, for all u with $\|u\| \geq n/4$; the theorem then follows.

4.2 Proof of Theorem 2.3 case $\alpha > 2$

We describe a random process $\mathcal{N}$, which we prove approaches zero faster than GREEDY (Section 4.2.1), and we derive a lower bound on its expected length (Section 4.2.2). Combining these two results we obtain the theorem (Section 4.2.3). Unlike GREEDY, $\mathcal{N}$ is a Markov chain, so, it is easier to analyze.

4.2.1 Process $\mathcal{N}$

Process $\mathcal{N}$ is parameterized by n, α, and s, where $s \in [0..n)$, and it resembles GREEDY routing in $\mathcal{U}(n,\alpha)$ from source s to target 0. Roughly speaking, $\mathcal{N}$ differs from GREEDY mainly in that: (1) each time the message is forwarded to an *in-contact*, say v, of the current node, the message is next forwarded to an *out-neighbor* of v closest to 0, and these *two* forwardings count as a *single* step of $\mathcal{N}$; and (2) the random graph is regenerated in each step of $\mathcal{N}$. In addition, instead of the contacts of the current node, say u, the out-contacts of a node a_1 and the in-contacts of a (possibly different) node a_2 are used to determine the next node. The a_i are functions on u, have $\|a_i\| \geq \|u\|$, and they are such that they minimize the expected length of $\mathcal{N}$. We introduce $\mathcal{N}$ because its expected length is a lower bound for the expected steps GREEDY takes to route a message from s to 0, and because $\mathcal{N}$ is a Markov chain, hence, it is easier to analyze than GREEDY. Another useful property of $\mathcal{N}$ is that its expected length is (provably) a non-decreasing function of $\|s\|$.

We now define $\mathcal{N}$ formally. Let $a_1 : [0..n) \to [0..n)$, $A_1 : [0..n) \to 2^{[0..n)}$, $a_2 : [0..n)^2 \to [0..n)$, and $A_2 : [0..n)^2 \to 2^{[0..n)}$ be functions such that for all nodes u, r,

$$\begin{aligned} \|a_1(u)\| &\geq \|u\|, & A_1(u) &\in \mathbf{H}_{a_1(u)}, \\ \|a_2(u,r)\| &\geq \|u\|, & A_2(u,r) &\in \mathbf{H}_{a_2(u,r)}. \end{aligned}$$

Recall from Section 3.2 that for any graph in $\mathbf{G}_n$ and node u', $\mathbf{H}_u$ contains every prefix path of the routing path from u' to 0 such that no node in this prefix path is in $R_{\|u\|} = \{v : v \leq \|u\|\}$. The a_i and A_i should also satisfy an additional condition, which we specify later.

Let $u \neq 0$ be the current node in $\mathcal{N}$. (Initially $u = s$, and $\mathcal{N}$ finishes when $u = 0$.) The next node, denoted N_u, is a node closest to 0 among the two ring-neighbors of u, and the nodes $N_{u,1}, N_{u,2}$ which are determined as follows. First we choose the out-contacts of $a_1(u)$ as in $\mathcal{G}(n, \alpha)$, conditioned on the event $\{a_1(u) \not\rightarrow A_1(u)\}$. We let $N_{u,1}$ be an out-contact of $a_1(u)$ that is closest to 0; or, if $a_1(u)$ has no out-contacts, $N_{u,1}$ is a randomly chosen node among u and the ring-neighbors of u (this ensures that $\mathbb{Pr}[N_{u,1} = v] > 0$, for all v). Suppose that $N_{u,1} = r$. $N_{u,2}$ is then chosen as follows. We choose the out-contacts of the nodes in $R_{\|a_2(u,r)\|-1}$ as in $\mathcal{G}(n, \alpha)$, conditioned on the event $\{R_{\|a_2(u,r)\|-1} \not\rightarrow A_2(u, r)\}$. (If $a_1(u) \in R_{a_2(u,r)-1}$ then the out-links of $a_1(u)$ generated earlier to determine $N_{u,1}$ are deleted, and replaced by new ones.) Let Z be the set of the in-contacts of $a_2(u, r)$ that are in $R_{\|a_2(u,r)\|-1}$ and are closest to 0 ($0 \leq |Z| \leq 2$). If $Z = \emptyset$, $N_{u,2} = a_2(u, r)$; if $Z = \{0\}$, $N_{u,2} = 0$; otherwise, $N_{u,2}$ is a node closest to 0 among the out-neighbors of the nodes in Z.

Functions a_i, A_i should satisfy the following optimization condition. Roughly speaking, this condition says that given the values of a_i and A_i for all u with $\|u\| < \|v\|$, their values for $u = v$ are such that they minimize the expected length of $\mathcal{N}$ when starting from $s = u$. Formally, let $L_u^{\mathcal{N}}$ denote the expected number of steps of $\mathcal{N}$ for $s = u$. The condition is described inductively as: for $\|u\| = 1, 2, \ldots,$

$$\begin{cases} \text{for all } r,\ a_2(u,r) \text{ and } A_2(u,r) \text{ are such that they minimize } \mathbb{E}[L_u^{\mathcal{N}} \mid N_{u,1} = r]; \\ a_1(u) \text{ and } A_1(u) \text{ are such that they minimize } \mathbb{E}[L_u^{\mathcal{N}}]. \end{cases} \tag{4.2}$$

The next two lemmata state the two properties of $\mathcal{N}$ we described at the beginning, that $\mathbb{E}[L_s^{\mathcal{N}}]$ is a non-decreasing function of $\|s\|$, and it is a lower bound for the expected value of the number of steps L_s that GREEDY requires to route a message from s to 0. The proofs are by induction, and can be found in the Appendix.

Lemma 4.2. *If* $\|u\| \geq \|u'\|$ *then* $\mathbb{E}[L_u^{\mathcal{N}}] \geq \mathbb{E}[L_{u'}^{\mathcal{N}}]$.

Lemma 4.3. $\mathbb{E}[L_s^{\mathcal{N}}] \leq \mathbb{E}[L_s]$.

4.2.2 Expected length of $\mathcal{N}$

The next lemma provides lower bounds on the expected length of $\mathcal{N}$, for $\alpha > 2$.

Lemma 4.4.

(a) If $\alpha > 3$ *then* $\mathbb{E}[L_{n/4}^{\mathcal{N}}] = \Omega(\ln^2 n)$.

(b) If $\alpha = 3$ *then* $\mathbb{E}[L_{n/4}^{\mathcal{N}}] = \Omega\big(\frac{\ln^2 n}{\ln\ln n}\big)$.

(c) If $2 < \alpha < 3$ *then for* $\lambda = e^{\ln^{\alpha-2} n}$, $\mathbb{E}[L_{\lambda}^{\mathcal{N}}] = \Omega\big(\frac{\ln^{\alpha-1} n}{\ln\ln n}\big)$.

Proof. (a) We show below that for all u, j such that $0 \leq j \leq \|u\| - 2$,

$$\mathbb{Pr}[\|N_u\| = j] = \mathrm{O}\big(\tfrac{1}{\nu(\|u\|-j)}\big). \tag{4.3}$$

From this and Lemma 4.1, applied for $X_0 = n/4$, $X_{i+1} = \|N_{X_i}\|$, $\rho = \nu$, and $\epsilon = 0$, it follows that $\mathbb{E}[L_{n/4}^{\mathcal{N}}] = \Omega(\ln^2 n)$. We now prove (4.3).

$$\mathbb{Pr}[\|N_u\| = j] \leq \mathbb{Pr}[\|N_{u,1}\| = j] + \max_r \mathbb{Pr}[\|N_{u,2}\| = j \mid N_{u,1} = r]. \tag{4.4}$$

Below we will write a_1 and A_1 instead of $a_1(u)$ and $A_1(u)$, respectively.

$$\mathbb{Pr}[\|N_{u,1}\| = j] \leq \mathbb{Pr}\left[a_1 \rightarrow \{j, n-j\} \,\middle|\, a_1 \not\rightarrow A_1\right] \leq \frac{\mathbb{Pr}[a_1 \rightarrow \{j, n-j\}]}{\mathbb{Pr}[a_1 \not\rightarrow A_1]} = \mathrm{O}\Big(\frac{1}{\nu(\|a_1\| - j)}\Big), \tag{4.5}$$

by Facts 3.2 and 3.5(c). Next we bound the second term on the right-hand side of (4.4). We will need the following definitions. Let S_v, for $v \neq 0$, be an *out-neighbor* of v in $\mathcal{G}(n, \alpha)$ that is closest to 0 (there may be two such nodes); and $S_0 = 0$. Let also Z_v be the set of the *in-contacts* of v that are in $R_{\|v\|-1}$ and are closest to 0 ($0 \le |Z_u| \le 2$).

$$\begin{aligned}
&\mathbb{P}\mathrm{r}[\|N_{u,2}\| = j \mid N_{u,1} = r] \\
&\qquad\le \sum_{v\,:\,j \le \|v\| < \|a_2\|} \sum_k \mathbb{P}\mathrm{r}\left[\{v \in Z_{a_2}\} \cap \{\|S_v\| = j\} \cap \{D_v = k\} \,\middle|\, R_{\|a_2\|-1} \not\to A_2\right],
\end{aligned} \tag{4.6}$$

where again we write a_2 and A_2 instead of $a_2(u, r)$ and $A_2(u, r)$, respectively. For $j + 2 \le \|v\| < \|a_2\|$,

$$\begin{aligned}
\mathbb{P}\mathrm{r}&\left[\|S_v\| = j \,\middle|\, \{v \in Z_{a_2}\} \cap \{D_v = k\} \cap \{R_{\|a_2\|-1} \not\to A_2\}\right] \\
&= \mathbb{P}\mathrm{r}\left[\|S_v\| = j \,\middle|\, \{D_v = k-1\} \cap \{v \not\to A_2\}\right] \\
&\le \mathbb{P}\mathrm{r}\left[v \to \{j,\, n-j\} \,\middle|\, \{D_v = k-1\} \cap \{v \not\to A_2\}\right] \\
&= \frac{\mathbb{P}\mathrm{r}\left[v \to \{j,\, n-j\} \,\middle|\, D_v = k-1\right]}{\mathbb{P}\mathrm{r}[v \not\to A_2 \mid D_v = 1]} \\
&= \mathrm{O}\Big(\frac{k-1}{\nu(\|v\| - j)}\Big),
\end{aligned} \tag{4.7}$$

where the second-to-last line was obtained using Fact 3.4, and the last using Facts 3.1 and 3.5(b); also,

$$\begin{aligned}
\mathbb{P}\mathrm{r}\left[v \in Z_{a_2} \,\middle|\, \{D_v = k\} \cap \{R_{\|a_2\|-1} \not\to A_2\}\right] &\le \mathbb{P}\mathrm{r}\left[v \to a_2 \,\middle|\, \{D_v = k\} \cap \{v \not\to A_2\}\right] \\
&= \mathrm{O}\Big(\frac{k}{\nu(\|a_2\| - \|v\|)}\Big),
\end{aligned} \tag{4.8}$$

similarly to (4.5); and

$$\mathbb{P}\mathrm{r}\left[D_v = k \,\middle|\, R_{\|a_2\|-1} \not\to A_2\right] = \mathbb{P}\mathrm{r}\left[D_v = k \,\middle|\, v \not\to A_2\right] = \mathrm{O}(q_k), \tag{4.9}$$

by Fact 3.5(c). Combining (4.6)–(4.9), we obtain

$$\begin{aligned}
\mathbb{P}\mathrm{r}[\|N_{u,2}\| = j \mid N_{u,1} = r] &= \mathrm{O}\bigg(\sum_{v\,:\,j+2 \le \|v\| < \|a_2\|} \sum_k \frac{k^{2-\alpha}}{\nu^2(\|v\| - j)(\|a_2\| - \|v\|)} + \frac{1}{\nu(\|a_2\| - j)}\bigg) \\
&= \mathrm{O}\Big(\frac{\ln(\|a_2\| - j)}{\nu^2(\|a_2\| - j)} + \frac{1}{\nu(\|a_2\| - j)}\Big) = \mathrm{O}\Big(\frac{1}{\nu(\|a_2\| - j)}\Big).
\end{aligned} \tag{4.10}$$

Applying (4.5) and (4.10) to (4.4), yields (4.3).

(b) We consider an "early-stopping" variance of $\mathcal{N}$ that differs from $\mathcal{N}$ as follows: Let $u \ne 0$ be the current node, suppose $N_{u,1} = r$, and let Z be the set of the in-contacts of $a_2(u, r)$ that are in $R_{\|a_2(u,r)\|-1}$ and are closest to 0 (see the definition of $\mathcal{N}$ in Section 4.2.1); if $D_v > \ln^2 n$ for some $v \in Z$ then the process jumps to node 0 in the next step. Let M_u denote the next node after node u in this new process, and L_u^M be the number of steps to reach 0 from u. Clearly, $\mathbb{E}[L_u^{\mathcal{N}}] \ge \mathbb{E}[L_u^M]$, so, it suffices to bound $\mathbb{E}[L_{n/4}^M]$. We show that for all u with $\|u\| \ge \rho = \frac{\ln n}{\ln \ln n}$,

$$\mathbb{P}\mathrm{r}[\|M_u\| = j] = \begin{cases} \mathrm{O}\big(\frac{1}{\rho(\|u\| - j)}\big), & \text{if } 0 < j \le \|u\| - 2; \\ \mathrm{O}\big(\frac{1}{\rho \ln n}\big), & \text{if } j = 0. \end{cases} \tag{4.11}$$

From this and Lemma 4.1, applied for $X_0 = n/4$, $X_{i+1} = \|M_{X_i}\|$, and $\epsilon = 0$, it follows that $\mathbb{E}[L_{n/4}^M] = \Omega\big(\frac{\ln^2 n}{\ln \ln n}\big)$. The proof of (4.11) is similar to that of (4.3), and can be found in the Appendix.

(c) We show that for all u with $\nu < \|u\| \le \lambda$,

$$\mathbb{P}\mathrm{r}[\|N_u\| = j] = \begin{cases} \mathrm{O}\big(\frac{(\|u\| j)^{3-\alpha}}{\nu(\|u\| - j)}\big), & \text{if } 0 < j \le \|u\| - 2; \\ \mathrm{O}\big(\frac{1}{\nu \ln \|u\|}\big), & \text{if } j = 0. \end{cases} \tag{4.12}$$

From this and Lemma 4.1, applied for $X_0 = \lambda$, $X_{i+1} = \|N_{X_i}\|$, $\rho = \nu$, and $\epsilon = 3 - \alpha$, it follows that $\mathbb{E}[L_\lambda^{\mathcal{N}}] = \Omega(\nu \ln \lambda) = \Omega(\ln^{\alpha-1} n)$. The derivation of (4.12) is similar to that of (4.3) — but more computationally involved. The main difference is that a more accurate bound is used in place of (4.7). The details are in the Appendix. ∎

4.2.3 Putting the pieces together

If $\alpha > 3$ then, by Lemmata 4.4(a), 4.2, and 4.3, $\mathbb{E}[L_u] \geq \mathbb{E}[L_u^{\mathcal{N}}] = \Omega(\ln^2 n)$, for all u with $\|u\| \geq n/4$. Hence, the expected delivery time is $\Omega(\ln^2 n)$. For the cases $\alpha = 3$ and $2 < \alpha < 3$ the theorem follows similarly, using Lemmata 4.4(b) and 4.4(c), respectively, in place of Lemma 4.4(a).

4.3 Proof of Theorem 2.3 case $\alpha < 2$

The theorem follows from the fact that for some λ that is polynomial in n, with probability $\Theta(1)$, all nodes u with $\|u\| \leq \lambda$ have out-degree (at most) 1. Specifically, the probability that $D_u = 1$ is

$$q_1 = \begin{cases} 1 - \Theta(1/k_{\max}^{2-\alpha}), & \text{if } 1 < \alpha < 2; \\ 1 - \Theta(\ln n / k_{\max}), & \text{if } \alpha = 1; \\ 1 - \Theta(1/k_{\max}), & \text{if } 0 \leq \alpha < 1. \end{cases} \tag{4.13}$$

Let $\lambda = \min\{\frac{n}{2}, \frac{1}{1-q_1}\}$, and $\mathcal{E} = \bigcap_{u \in R_\lambda} \{D_u = 1\}$. Then, $\mathbb{Pr}[\mathcal{E}] = q_1^{2\lambda+1} = \Theta(1)$. Let $\langle Y_0, Y_1, \ldots \rangle$ be the routing path from λ to 0. If $\|v\| \leq \|u\| - 2$,

$$\begin{aligned} &\mathbb{Pr}[Y_{i+1} = v \,|\, \{Y_i = u\} \cap \{\langle Y_j \rangle_{j=0}^{i-1} = H\} \cap \mathcal{E}] \\ &\qquad \leq \mathbb{Pr}[\{u \to v\} \cup \{v \to u\} \,|\, \{u, v \not\to H\} \cap \{D_u = D_v = 1\}] = \mathrm{O}\big(\tfrac{1}{\nu \delta(u,v)}\big), \end{aligned}$$

by Fact 3.5(b). This and Lemma 4.1, applied for $\rho = \nu$, $\epsilon = 0$, and $X_i = \|Y_i\| \,|\, \mathcal{E}$, yields $\mathbb{E}[L_\lambda \,|\, \mathcal{E}] = \Omega(\nu \ln \lambda)$. By (4.13) and the fact that $k_{\max}$ is polynomial in n, $\ln \lambda = \Theta(\ln k_{\max}) = \Theta(\ln n)$, so, $\mathbb{E}[L_\lambda \,|\, \mathcal{E}] = \Omega(\ln^2 n)$. And since $\mathbb{Pr}[\mathcal{E}] = \Theta(1)$, $\mathbb{E}[L_\lambda] \geq \mathbb{E}[L_\lambda \,|\, \mathcal{E}] \cdot \mathbb{Pr}[\mathcal{E}] = \Omega(\ln^2 n)$; hence, the GREEDY diameter is $\Omega(\ln^2 n)$.

4.4 Proof of Theorem 2.3 case $\alpha = 2$

The proof consists of two parts, which are roughly as follows. First we show that for any λ and node s with $\|s\| \geq \lambda$, with probability $\Theta(1)$, the routing path from s to 0 contains some node u such that $\lambda^{1/3} \leq \|u\| \leq \lambda$ and u has a small expected out-degree. Next, we show that if $\lambda = e^{\ln^{1/3} n}$ then the expected number of remaining steps from u to 0 is $\Omega(\ln^{4/3} n)$. The two lemmata we state below correspond to these two parts. Let $\langle Y_0, Y_1, \ldots \rangle$ be the routing path from node Y_0 to 0.

Lemma 4.5. *If $\|Y_0\| > \lambda = \omega(1)$ and $K = \min\{i : \|Y_i\| \leq \lambda\}$ then*

$$\mathbb{Pr}\left[\{\|Y_K\| \geq \lambda^{1/3}\} \cap \big(\{Y_K \not\to Y_{K-1}\} \cup \{D_{Y_K} = 1\}\big)\right] = \Theta(1).$$

Lemma 4.6. *For $\lambda = e^{\ln^{1/3} n}$ and all u with $\lambda^{1/3} \leq \|u\| \leq \lambda$,*

$$\mathbb{E}\left[L_{Y_i} \,\middle|\, Y_0, \ldots, Y_{i-1}, \{Y_i = u\} \cap \big(\{Y_i \not\to Y_{i-1}\} \cup \{D_{Y_i} = 1\}\big)\right] = \Omega(\ln^{4/3} n).$$

Let $\mathcal{E} = \{\|Y_K\| \geq \lambda^{1/3}\} \cap \big(\{Y_K \not\to Y_{K-1}\} \cup \{D_{Y_K} = 1\}\big)$. We prove Lemma 4.5 by showing that $\mathcal{E}$ occurs with probability $\Theta(1)$, for any fixed K and $Y_0, \ldots, Y_{K-1}$, and conditionally on the event that for all v with $\|v\| > \lambda$, $D_v \leq \|v\|$. Since this last event occurs with probability $\Theta(1)$, the lemma follows. The proof of Lemma 4.6 is analogous to that of Lemma 4.4(b). We analyze an early-stopping variation of GREEDY, where if in some step we visit a node $v \in R_{\|u\|}$ that has an in-contact $v' \in R_{\|v\|-1}$ with $D_{v'} > 1$ then we jump to 0 in the next step. The full proofs of Lemmata 4.5 and 4.6 can be found in the Appendix.

The theorem now follows easily. For $\lambda = e^{\ln^{1/3} n}$, K as in the statement of Lemma 4.6, and $\mathcal{E}$ as above,

$$\mathbb{E}[L_{Y_0}] \geq \mathbb{E}[L_{Y_0} \,|\, \mathcal{E}] \cdot \mathbb{Pr}[\mathcal{E}] \geq \mathbb{E}[L_{Y_K} \,|\, \mathcal{E}] \cdot \mathbb{Pr}[\mathcal{E}] = \Omega(\ln^{4/3} n),$$

by Lemmata 4.5 and 4.6.

5 Proof of the upper bounds

As in the proof of the lower bounds in Section 4, we start with a simple lemma that bound from above the length of any process that approaches 0 with jumps that follow a distribution of a specific form. In Section 5.1, we show that $\mathrm{O}(\ln^2 n)$ steps are required in all models. In Sections 5.2 and 5.3, we prove tighter upper bounds for $\mathcal{U}(n,\alpha)$, for $2<\alpha<3$ and $\alpha=2$, respectively.

Lemma 5.1(a) below is an analogue of Lemma 4.1, and we will use it in the proofs of all the upper bounds. Lemma 5.1(b) provides a with-high-probability bound for the length of the process; we will use it in Sections 5.2 and 5.3. The proof is straightforward and can be found in the Appendix.

Lemma 5.1. *If $\langle X_0, X_1, \ldots\rangle$ is a non-increasing, non-negative, integer-valued random process, such that for all j, $\mathbb{Pr}[X_{i+1} \le j/2 \,|\, X_0,\ldots,X_{i-1}, \{X_i = j\}] \ge 1/\rho$, then for $\kappa = \lceil \log(X_0+1)\rceil$,*

- *(a) The expected number of steps to reach 0 is at most $\rho\kappa$.*
- *(b) The number of steps to reach 0 is greater than $t \ge 4\rho\kappa$ with probability at most $e^{-\frac{t}{4\rho}}$.*

We will also use the following simple fact; its proof is in the Appendix.

Fact 5.2. *If $Q_1, Q_2, \ldots, Q_\kappa$ are independent 0–1 random variables and $Q=\sum_i Q_i$ then (a) $\mathbb{Pr}[Q=0] \le e^{-\mathbb{E}[Q]}$; and (b) if for all i, $\mathbb{E}[Q_i]\le 1/2$ then $\mathbb{Pr}[Q=0] \ge e^{-\frac{3}{2}\mathbb{E}[Q]}$.*

5.1 Proof of an $\mathrm{O}(\ln^2 n)$ bound for all models

Let $\langle Y_0, Y_1,\ldots\rangle$ be the routing path from node Y_0 to 0 in $\mathcal{G}(n,\alpha)$ or $\mathcal{U}(n,\alpha)$. We will show that for all u,

$$\mathbb{Pr}[\|Y_{i+1}\| \le \|u\|/2 \,|\, Y_0,\ldots,Y_{i-1},\{Y_i=u\}] = \Omega(1/\nu). \tag{5.1}$$

From this and Lemma 5.1(a), applied for $X_i = \|Y_i\|$ and $\rho=\Theta(\nu)$, we obtain that the expected length of the routing path from Y_0 to 0 is $\mathrm{O}(\nu\ln(\|Y_0\|+1)) = \mathrm{O}(\ln^2 n)$. We now prove (5.1). For $\|u\|\le 2$, it obviously holds; so, suppose that $\|u\|>2$.

- In $\mathcal{G}(n,\alpha)$, the left-hand side of (5.1) equals $\mathbb{Pr}[u \to R_{\|u\|/2}] \ge q_1\,\mathbb{Pr}[u\to R_{\|u\|/2}\,|\,D_u=1] = \Omega(1/\nu)$, by Fact 3.3.
- In $\mathcal{U}(n,\alpha)$, we have

$$\mathbb{Pr}[\|Y_{i+1}\|\le \|u\|/2 \,|\, \{Y_i=u\}\cap\{\langle Y_j\rangle_{j=0}^{i-1}=H\}] \ge \mathbb{Pr}[R_{\|u\|/2}\to u \,|\, R_{\|u\|/2}\not\to H]. \tag{5.2}$$

But for any $v\in R_{\|u\|/2}$,

$$\begin{aligned}\mathbb{Pr}[v\to u\,|\,R_{\|u\|/2}\not\to H] &= \mathbb{Pr}[v\to u\,|\,v\not\to H] \ge \mathbb{Pr}[\{v\to u\}\cap\{v\not\to H\}]\\ &\ge \mathbb{Pr}[\{v\to u\}\cap\{D_v=1\}] = q_1\,\mathbb{Pr}[v\to u\,|\,D_v=1]\\ &\ge \frac{q_1}{2\nu\|u\|}.\end{aligned}$$

So, $\sum_{v\in R_{\|u\|/2}}\mathbb{Pr}[v\to u\,|\,R_{\|u\|/2}\not\to H] \ge \frac{1}{4\nu}$; and since the events $\{v\to u\}$ are independent (conditionally on $R_{\|u\|/2}\not\to H$), we have, by Fact 5.2(a), that $\mathbb{Pr}[R_{\|u\|/2}\to u\,|\,R_{\|u\|/2}\not\to H] \ge 1-e^{-\frac{1}{4\nu}} = \Theta(1/\nu)$. Combining this and (5.2) yields (5.1).

5.2 Proof of Theorem 2.4 case $2 < \alpha < 3$

We will use the following result, which is analogous to Lemma 5.1(a).

Lemma 5.3. *If $\langle X_0, X_1, \ldots\rangle$ is a non-increasing, non-negative, integer-valued random process with $X_0 > \lambda \geq 1$, such that for all j with $\lambda < j \leq X_0$,*

$$\mathbb{Pr}[X_{i+1} \leq j/2 \,|\, X_0, \ldots, X_{i-1}, \{X_i = j\}] \geq \tfrac{\log j}{\rho} \tag{5.3}$$

then the expected number of steps until the process' value reduces to at most λ is at most $\rho(\ln \log X_0 + 1)$.

Proof. Let T_k, for $k \geq 0$, be the number of steps until the process' value is reduced to at most $2^k\lambda$; i.e., $T_k = \min\{i \,:\, X_i \leq 2^k\lambda\}$. (Note that smaller k correspond to larger T_k.) To prove the lemma we must show that $\mathbb{E}[T_0] \leq \rho(\ln \log X_0 + 1)$. For $k \geq \log X_0 - \log \lambda$, $T_k = 0$. For $0 \leq k < \log X_0 - \log \lambda$,

$$\begin{aligned}\mathbb{Pr}[T_k = i+1 \,|\, X_0, \ldots, X_i, \{T_{k+1} \leq i < T_k\}] \\ = \mathbb{Pr}[X_{i+1} \leq 2^k\lambda \,|\, X_0, \ldots, X_i, \{2^k\lambda < X_i \leq 2^{k+1}\lambda\}] \geq \tfrac{\log \lambda + k}{\rho},\end{aligned}$$

by (5.3). So, $T_k - T_{k+1}$ is stochastically smaller than a geometric random variable with probability parameter $\frac{\log \lambda + k}{\rho}$. Therefore,

$$\mathbb{E}[T_0] = \mathbb{E}\left[\sum_{k=0}^{\log X_0 - \log\lambda - 1} (T_k - T_{k+1})\right] \leq \sum_{k=0}^{\log X_0 - \log\lambda - 1} \frac{\rho}{\log \lambda + k} \leq \rho(\ln \log X_0 + 1). \qquad \blacksquare$$

Roughly, the proof of the theorem proceeds as follows. We show that in every three steps of GREEDY the ring-distance to 0 is halved with probability $\Omega\left(\frac{\ln \|u\|}{\ln^{\alpha-1} n}\right)$, provided that we are not too close to 0 and not too many steps have been taken so far. Also, by the analysis in Section 5.1, the ring-distance to 0 is halved with probability $\Omega(1/\ln n)$ in each step, independently of the previous steps. By combining these two results and applying Lemmata 5.1 and 5.3 we obtain the theorem.

The next lemma gives a lower bound on the speed of GREEDY when the length of the prefix of the routing path so far is much smaller than the current ring-distance to the target. Two steps at a time are considered instead of just one. Interestingly, this bound is obtained by counting only the contribution of nodes with out-degree $\Theta(\ln n)$. Let $\langle Y_0, Y_1, \ldots\rangle$ be the routing path from node Y_0 to 0.

Lemma 5.4. *If $\|u\| \geq 8(i^2+1)$ then*

$$\mathbb{Pr}[\|Y_{i+2}\| \leq \|u\|/2 \,|\, Y_0, \ldots, Y_{i-1}, \{Y_i = u\} \cap \{Y_i \not\to Y_{i-1}\}] = \Omega\Big(\frac{\ln \|u\|}{\ln^{\alpha-1} n}\Big).$$

Proof. We describe an event $\mathcal{E}$ such that if $\mathcal{E}$ occurs and $Y_i = u$ then $\|Y_{i+2}\| \leq \|u\|/2$, and we bound $\mathcal{E}$'s conditional probability instead. Informally, if $\mathcal{E} \cap \{Y_i = u\}$ occurs then the following statements are true about Y_{i+1}: (1) it is an in-contact of Y_i; (2) it has out-degree $\Theta(\ln n)$; (3) $\|u\|/2 < \|Y_{i+1}\| \leq \|u\| - \|u\|^{1/2}$; and (4) at least one of its out-contacts is in $R_{\|u\|/2}$. Formally, we define the following four events. Let

$$\mathcal{E}_0 = \{u \not\to R_{\|u\| - \|u\|^{1/2}}\}.$$

Define the sets $C = R_{\|u\| - \|u\|^{1/2}} \setminus R_{\|u\|/2}$ and $C^* = \{v \in C \,:\, \nu \leq D_v \leq 2\nu\}$, and let

$$\mathcal{E}_1 = \{C^* \to u\}, \quad \mathcal{E}_2 = \{R_{\|u\|-1} \setminus C^* \not\to u\}.$$

Last, if $\mathcal{E}_1$ occurs, let Z be the in-contact of u in C^* that is closest to 0 (if there are two such nodes then Z is the one that GREEDY would choose), and let

$$\mathcal{E}_3 = \{Z \to R_{\|u\|/2}\}.$$

We define

$$\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3.$$

It is easy to see that $\mathcal{E} \cap \{Y_i = u\} \subseteq \{\|Y_{i+2}\| \leq \|u\|/2\}$. So, to prove the lemma it suffices to show that

$$\mathbb{Pr}[\mathcal{E} \,|\, \{\langle Y_j \rangle_{j=0}^{i-1} = H\} \cap \{Y_i = u\} \cap \{Y_i \not\to Y_{i-1}\}] = \Omega\big(\tfrac{\ln \|u\|}{\ln^{\alpha-1} n}\big).$$

But the left-hand side is equal to $\mathbb{Pr}[\mathcal{E} \,|\, R_{\|u\|} \not\to H]$; so, we will show that $\mathbb{Pr}[\mathcal{E} \,|\, R_{\|u\|} \not\to H] = \Omega\big(\frac{\ln \|u\|}{\ln^{\alpha-1} n}\big)$. Let $\mathcal{A} = \{R_{\|u\|} \not\to H\}$. Since $\mathcal{E}_0$ is independent of the other three events,

$$\mathbb{Pr}[\mathcal{E} \,|\, \mathcal{A}] = \mathbb{Pr}[\mathcal{E}_0 \,|\, \mathcal{A}] \cdot \mathbb{Pr}[\mathcal{E}_1 \,|\, \mathcal{A}] \cdot \mathbb{Pr}[\mathcal{E}_2 \,|\, \mathcal{E}_1 \cap \mathcal{A}] \cdot \mathbb{Pr}[\mathcal{E}_3 \,|\, \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{A}]. \tag{5.4}$$

We compute lower bounds for the four probabilities on the right-hand side.

$$\begin{aligned}
\mathbb{Pr}[\mathcal{E}_0 \,|\, \mathcal{A}] &\geq \mathbb{Pr}[u \not\to R_{\|u\|-\|u\|^{1/2}} \cup H] \\
&\geq q_1\, \mathbb{Pr}[u \not\to R_{\|u\|-\|u\|^{1/2}} \cup H \,|\, D_u = 1] \\
&\geq q_1\, \mathbb{Pr}[0 \to [2..\|u\|^{1/2}) \,|\, D_0 = 1] + q_1\, \mathbb{Pr}[0 \to [|H| + 2..n/2 - \|u\|] \,|\, D_0 = 1] \\
&= \Omega\Big(\frac{1}{\nu} \ln \|u\|^{1/2} + \frac{1}{\nu} \ln \frac{n/2 - \|u\| + 1}{|H| + 1}\Big) \\
&= \Omega(1).
\end{aligned} \tag{5.5}$$

The third relation was obtained using Fact 3.5(a); the second-to-last was obtained using Fact 3.3; the last using the facts that $\|u\| + |H| < n/2 + 1$ and $\|u\| > i = |H|$. Next we bound $\mathbb{Pr}[\mathcal{E}_1 \,|\, \mathcal{A}]$. Let $\mathcal{D}_v$ denote the event $\{\nu \leq D_v \leq 2\nu\}$, and Q_v be the indicator random variable of the event $\mathcal{D}_v \cap \{v \to u\}$. For all $v \in C$,

$$\mathbb{E}[Q_v \,|\, \mathcal{A}] = \mathbb{Pr}[v \to u \,|\, \mathcal{D}_v \cap \{v \not\to H\}] \cdot \frac{\mathbb{Pr}[v \not\to H \,|\, \mathcal{D}_v]}{\mathbb{Pr}[v \not\to H]} \cdot \mathbb{Pr}[\mathcal{D}_v] = \Omega\Big(\frac{1}{\nu^{\alpha-1} \delta(u, v)}\Big), \tag{5.6}$$

where the last relation holds because: $\mathbb{Pr}[v \to u \,|\, \mathcal{D}_v \cap \{v \not\to H\}] \geq \mathbb{Pr}[v \to u \,|\, D_v = \nu] = \Theta\big(\frac{1}{\delta(u,v)}\big)$; and $\mathbb{Pr}[v \to H \,|\, \mathcal{D}_v] \leq |H| \cdot \frac{2\nu}{\nu \|u\|^{1/2}} \leq \frac{1}{\sqrt{2}}$, since $\|u\| > 8|H|^2$; and $\mathbb{Pr}[\mathcal{D}_v] = \Theta\big(\frac{1}{\nu^{\alpha-1}}\big)$. So, for $Q = \sum_{v \in C} Q_v$,

$$\mathbb{E}[Q \,|\, \mathcal{A}] = \sum_{v \in C} \mathbb{E}[Q_v \,|\, \mathcal{A}] = \Omega\Big(\frac{\ln \|u\|}{\nu^{\alpha-1}}\Big).$$

And since the Q_i are independent (conditionally on $\mathcal{A}$), we have, by Fact 5.2(a), that $\mathbb{Pr}[Q \neq 0 \,|\, \mathcal{A}] \geq 1 - e^{-\Omega\big(\frac{\ln \|u\|}{\nu^{\alpha-1}}\big)} = \Omega\big(\frac{\ln \|u\|}{\nu^{\alpha-1}}\big)$. Finally, since $\mathcal{E}_1 = \{Q \neq 0\}$,

$$\mathbb{Pr}[\mathcal{E}_1 \,|\, \mathcal{A}] = \Omega\big(\tfrac{\ln \|u\|}{\nu^{\alpha-1}}\big). \tag{5.7}$$

Next,

$$\mathbb{Pr}[\mathcal{E}_2 \,|\, \mathcal{E}_1 \cap \mathcal{A}] \geq \mathbb{Pr}[\mathcal{E}_2 \,|\, \mathcal{A}] \geq \mathbb{Pr}[R_{\|u\|-1} \not\to u \,|\, \mathcal{A}] = \Theta(1), \tag{5.8}$$

where the last relation is obtained as follows. For all $v \in R_{\|u\|-1}$,

$$\mathbb{Pr}[v \to u \,|\, \mathcal{A}] = \mathbb{Pr}[v \to u \,|\, v \not\to H] = \mathrm{O}(\mathbb{Pr}[v \to u]),$$

by Fact 3.5(c); so, $\sum_{v \in R_{\|u\|-1}} \mathbb{Pr}[v \to u \,|\, \mathcal{A}] = \mathrm{O}(\sum_v \mathbb{Pr}[v \to u]) = \mathrm{O}(1)$, since the expected number of in-contacts a node has is constant; and since the events $\{v \to u\}$ are independent, we have, by Fact 5.2(b), that $\mathbb{Pr}[R_{\|u\|-1} \not\to u \,|\, \mathcal{A}] \geq e^{-\mathrm{O}(1)} = \Theta(1)$. The last bound we need is

$$\mathbb{Pr}[\mathcal{E}_3 \,|\, \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{A}] \geq \mathbb{Pr}[u \to R_{\|u\|/2} \,|\, D_u = \nu] = \Theta(1), \tag{5.9}$$

by Facts 3.1 and 3.3. Combining (5.4), (5.5), (5.7), (5.8), and (5.9), yields $\mathbb{Pr}[\mathcal{E} \,|\, \mathcal{A}] = \Omega\big(\frac{\ln \|u\|}{\nu^{\alpha-1}}\big)$. ■

We will now use Lemma 5.4 to show that if $\|u\| > \lambda = e^{\ln^{\alpha-2} n}$ and $i = \mathrm{o}(\lambda^{1/2})$ then

$$\mathbb{Pr}[\|Y_{i+3}\| \leq \|u\|/2 \,|\, Y_0, \ldots, Y_{i-1}, \{Y_i = u\}] = \Omega\big(\tfrac{\ln \|u\|}{\nu^{\alpha-1}}\big). \tag{5.10}$$

By Lemma 5.4, it suffices to show that

$$\mathbb{Pr}[\|Y_{i+3}\| \leq \|u\|/2 \,|\, Y_0, \ldots, Y_{i-1}, \{Y_i = u\} \cap \{Y_i \to Y_{i-1}\}] = \Omega\big(\tfrac{\ln \|u\|}{\nu^{\alpha-1}}\big).$$

Let $\mathcal{H}$ be the σ-algebra generated by $Y_0, \ldots, Y_{i-1}, \{Y_i = u\} \cap \{Y_i \to Y_{i-1}\}$.

$$\begin{aligned}
&\mathbb{Pr}[\|Y_{i+3}\| \leq \|u\|/2 \,|\, \mathcal{H}] \\
&\quad \geq \mathbb{Pr}\left[\|Y_{i+3}\| \leq \|u\|/2 \,\middle|\, \mathcal{H}, \{Y_{i+1} \not\to Y_i\}\right] \cdot \mathbb{Pr}[Y_{i+1} \not\to Y_i \,|\, \mathcal{H}] \\
&\quad \geq \mathbb{Pr}\left[\|Y_{i+3}\| \leq \|u\|/2 \,\middle|\, \mathcal{H}, \{Y_{i+1} \not\to Y_i\} \cap \{\|Y_{i+1}\| > \|u\|/2\}\right] \cdot \mathbb{Pr}[R_{\|u\|-1} \not\to u \,|\, \mathcal{H}] \\
&\quad = \Omega\big(\tfrac{\ln \|u\|}{\nu^{\alpha-1}}\big),
\end{aligned}$$

because in the second-to-last line, the first probability is $\Omega\big(\frac{\ln(\|u\|/2)}{\ln^{\alpha-1} n}\big)$, by Lemma 5.4, and the second is $\Theta(1)$, similarly to the last relation in (5.8). We can now obtain the theorem as follows:

- If $Y_0 \leq \lambda$ then, by (5.1) and Lemma 5.1(a), applied for $X_i = \|Y_i\|$ and $\rho = \Theta(\ln n)$, we have $\mathbb{E}[L_{Y_0}] = \mathrm{O}(\ln n \ln \lambda) = \mathrm{O}(\ln^{\alpha-1} n)$.

- If $Y_0 > \lambda$, let T_1 be the number of steps from Y_0 until we reach a node within ring-distance λ of 0, and T_2 be the number of remaining steps to 0. Similarly to the case $Y_0 \leq \lambda$, $\mathbb{E}[T_2] = \mathrm{O}(\ln^{\alpha-1} n)$; so, we just need to show that $\mathbb{E}[T_1] = \mathrm{O}(\ln^{\alpha-1} n \ln \ln n)$. By (5.1) and Lemma 5.1(b), applied for $X_i = \|Y_i\|$, $\rho = \Theta(\ln n)$, and $t = 4\rho \ln n$, we have that $\mathbb{Pr}[T_1 \geq \ln^3 n] < 1/n$. Also, by (5.10) and Lemma 5.3, applied for $X_i = \|Y_{3i}\|$, if $i < \ln^3 n/3$, $X_i = 0$, if $i \geq \ln^3 n/3$, and $\rho = \Theta(\ln^{\alpha-1} n)$, we get that $\mathbb{E}[\min\{T_1, \ln^3 n\}] = \mathrm{O}(\ln^{\alpha-1} n \ln \ln n)$. From this, $\mathbb{E}[T_1 \,|\, T_1 < \ln^3 n] = \mathbb{E}[\min\{T_1, \ln^3 n\} \,|\, T_1 < \ln^3 n] \leq \mathbb{E}[\min\{T_1, \ln^3 n\}] = \mathrm{O}(\ln^{\alpha-1} n \ln \ln n)$. Therefore,

$$\begin{aligned}
\mathbb{E}[T_1] &= \mathbb{E}[T_1 \,|\, T_1 < \ln^3 n] \cdot \mathbb{Pr}[T_1 < \ln^3 n] + \mathbb{E}[T_1 \,|\, T_1 \geq \ln^3 n] \cdot \mathbb{Pr}[T_1 \geq \ln^3 n] \\
&\leq \mathbb{E}[T_1 \,|\, T_1 < \ln^3 n] + n\, \mathbb{Pr}[T_1 \geq \ln^3 n] \\
&= \mathrm{O}(\ln^{\alpha-1} n \ln \ln n).
\end{aligned}$$

5.3 Proof of Theorem 2.4 case $\alpha = 2$

It is similar to the proof of case $2 < \alpha < 3$. The next two lemmata are the analogues of Lemmata 5.3 and 5.4, respectively. Their proofs are can be found in the Appendix.

Lemma 5.5. *If $\langle X_0, X_1, \ldots \rangle$ is a non-increasing, non-negative, integer-valued random process with $X_0 > \lambda \geq 2$, such that for all j with $\lambda < j \leq X_0$, $\mathbb{Pr}[X_{i+1} \leq j^{1-\epsilon} \,|\, X_0, \ldots, X_{i-1}, \{X_i = j\}] \geq \frac{\log^2 j}{\rho}$, where $0 < \epsilon < 1$, then the expected number of steps until the process' value reduces to at most λ is at most $c\frac{\rho}{\log^2 \lambda}$, where $c = c(\epsilon)$.*

Let $\langle Y_0, Y_1, \ldots \rangle$ be the routing path from Y_0 to 0.

Lemma 5.6. *If $\|u\| \geq 4^6(i^6 + 1)$ then*

$$\mathbb{Pr}[\|Y_{i+2}\| \leq \|u\|^{2/3} \,|\, Y_0, \ldots, Y_{i-1}, \{Y_i = u\} \cap \{Y_i \not\to Y_{i-1}\}] = \Omega\big(\tfrac{\ln^2 \|u\|}{\ln^2 n}\big).$$

Note that, unlike in case $2 < \alpha < 3$ where only nodes of out-degree $\Theta(\ln n)$ contribute to routing significantly, now the contribution of nodes with out-degrees in a wider range is significant.

The rest of the proof is completely analogous to that of case $2 < \alpha < 3$. Instead of (5.10), we show (using Lemma 5.6) that if $\|u\| > \lambda = e^{\sqrt{\ln n}}$ and $i = \mathrm{o}(\lambda^{1/6})$ then

$$\mathbb{Pr}[\|Y_{i+3}\| \leq \|u\|^{2/3} \,|\, Y_0, \ldots, Y_{i-1}, \{Y_i = u\}] = \Omega\big(\tfrac{\ln^2 \|u\|}{\ln^2 n}\big);$$

and instead of Lemma 5.3, we use Lemma 5.5, for $\rho = \Theta(\ln^2 n)$ and $\epsilon = 1/3$.

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Appendix

A Appendix of Sections 2 and 3

A.1 In-degree distribution

Let D^- denote the in-degree of a node in $\mathcal{G}(n,\alpha)$. Then,

Lemma A.1. $\mathbb{Pr}[D^- = k] \le \frac{1}{(k/3e)^k}$.

Proof. Let X_u be the indicator random variable of the event $\{u \to 0\}$. The in-degree D^- of 0 is then $D^- = \sum_u X_u$. Note that X_u is stochastically smaller that a poisson random variable $\mathrm{Poi}(\mu_u)$ with expectation μ_u, such that $\mathbb{Pr}[\mathrm{Poi}(\mu_u) = 1] \ge \mathbb{E}[X_u]$. And since $\mathbb{Pr}[\mathrm{Poi}(\mu_u) = 1] = \mu_u e^{-\mu_u}$ and

$$\mathbb{E}[X_u] = \mathbb{Pr}[u \to 0] \le \frac{2}{\nu\|u\|},$$

by Fact 3.2, we can set $\mu_u = \frac{3}{\nu\|u\|}$. So, since the X_u are independent, D^- is stochastically smaller that $\sum_u \mathrm{Poi}(\mu_u)$, which is also a poisson random variable with expectation $\sum_u \mu_u = 3$. Hence,

$$\mathbb{Pr}[D^- = k] \le \mathbb{Pr}[D^- \ge k] \le \mathbb{Pr}[\mathrm{Poi}(3) \ge k] = \sum_{j \ge k} \frac{3^j}{j!} e^{-3} \le \frac{3^k}{k!} \le \left(\frac{3e}{k}\right)^k,$$

where the second-to-last relation follows by simple computations, and the last is obtained using the fact that $e^k = \sum_i k^i/i! \ge k^k/k!$. ∎

The following result is immediate from Lemma A.1.

Corollary A.2. *If* $\alpha > 2$ *then*

$$\mathbb{Pr}[D^- = k] = \begin{cases} \mathrm{O}(q_k), & \textit{if } k = \mathrm{O}(1); \\ \mathrm{o}(q_k), & \textit{if } k = \omega(1). \end{cases}$$

Also, there is a constant c *such that if* $\alpha = 2$ *then* $\mathbb{Pr}[D^- = k] = \mathrm{o}(q_k)$, *for* $k \ge c \ln\ln n$*; and if* $0 \le \alpha < 2$ *then* $\mathbb{Pr}[D^- = k] = \mathrm{o}(q_k)$, *for* $k \ge c \ln n$.

A.2 Proofs of Facts 3.1–3.5

Proof of Fact 3.1. The right relation follows from the union bound. For the left relation, we have

$$\mathbb{Pr}[u \to U \mid D_u = k] = 1 - (1-p)^k \ge 1 - e^{-kp} \ge \begin{cases} 1 - e^{-1} \ge \frac{1}{2}, & \text{if } kp \ge 1; \\ kp - \frac{(kp)^2}{2} \ge \frac{kp}{2}, & \text{if } kp < 1/2, \end{cases}$$

where the second relation was obtained using the fact that $1 - x \le e^x$; and the second case of the third relation using the fact that for $x \ge 0$, $e^{-x} \le 1 - x + x^2/2$. ∎

Proof Sketch of Fact 3.2. The left relation holds because $\mathbb{Pr}[u \to v] \ge \mathbb{Pr}[u \to v \mid D_u = 1]$, since $D_u \ge 1$. The right relation holds because $\mathbb{Pr}[u \to v] \le 2\,\mathbb{Pr}[u \to v \mid D_u = 1]$, since $\mathbb{E}[D_u] = 2$. ∎

Proof of Fact 3.3. By direct computation. Note that, by symmetry, the two cases for U are equivalent. ∎

Proof of Fact 3.4. By Baye's rule, the left-hand side equals

$$\mathbb{Pr}[u \to U_1 \mid D_u = k] \cdot \frac{\mathbb{Pr}[u \not\to U_2 \mid \{D_u = k\} \cap \{u \to U_1\}]}{\mathbb{Pr}[u \not\to U_2 \mid D_u = k]},$$

and

$$\frac{\mathbb{Pr}[u \not\to U_2 \mid \{D_u = k\} \cap \{u \to U_1\}]}{\mathbb{Pr}[u \not\to U_2 \mid D_u = k]} = \frac{\mathbb{Pr}[u \not\to U_2 \mid D_u = k-1]}{\mathbb{Pr}[u \not\to U_2 \mid D_u = k]} = \frac{1}{\mathbb{Pr}[u \not\to U_2 \mid D_u = 1]},$$

where the last relation holds because $\mathbb{Pr}[u \not\to U_2 \mid D_u = k] = \mathbb{Pr}[u \not\to U_2 \mid D_u = k-1] \cdot \mathbb{Pr}[u \not\to U_2 \mid D_u = 1]$. ■

Proof Sketch of Fact 3.5. It is easy to show, by induction on i, that if v_i is the (unique) i-th furthest from 0 node in H then $\delta(u, v_i) \geq d + i - 1$. Assume without loss of generality that $u \leq n/2$. Then, by replacing v_i by $u + 1 + i$, for $i = 1, \ldots, |H|$, we obtain

$$\mathbb{Pr}[u \to H \mid D_u = k] \leq \mathbb{Pr}[u \to [u + d..u + d + H) \mid D_u = k] = \mathbb{Pr}[0 \to [d..d + |H|) \mid D_0 = k].$$

We can now derive (b) and (c) as follows.

$$\mathbb{Pr}[u \not\to H \mid D_u = k] \geq \mathbb{Pr}[0 \not\to [d..d + |H|) \mid D_0 = k] \geq \mathbb{Pr}[0 \not\to [2..n/2] \mid D_0 = k] \leq 1/2^k,$$

since $\mathbb{Pr}[0 \to [2..n/2] \mid D_0 = 1] \leq \mathbb{Pr}[0 \to [1..n/2 - 1] \mid D_0 = 1] \leq 1/2$, by symmetry. And

$$\mathbb{Pr}[u \not\to H] \geq q_1 \, \mathbb{Pr}[u \not\to H \mid D_u = 1] \geq q_1/2.$$ ■

B Appendix of Section 4

B.1 Proof of Lemma 4.1

Let

$$K = \min\{i \,:\, X_i \leq \rho\}.$$

We show that $\mathbb{E}[K] \geq c'\rho \ln(X_0/\rho)$, for some $c' = c'(c, \epsilon) > 0$. And since K is upper bounded by the number of steps until the process reaches 0, the lemma follows. Let $\langle W_0, W_1, \ldots \rangle$ be the sequence obtained by taking the logarithms of $X_0, \ldots, X_K$, and letting $W_j = W_K$, for $j > K$. Formally, for $i \geq 0$,

$$W_i = \ln \max\{X_i,\, X_K,\, 1\},$$

where the '1' is needed for the case where $X_K = 0$. We first show that for all i,

$$\mathbb{E}[W_i - W_{i+1}] \leq c_1/\rho, \quad \text{where } c_1 = c_1(c, \epsilon). \tag{B.1}$$

Since $\mathbb{E}[W_i - W_{i+1} \mid X_i \leq \rho] = 0$, it suffices to show that $\mathbb{E}[W_i - W_{i+1} \mid X_i = j] \leq c_1/\rho$, for $\rho < j \leq X_0$.

$$\mathbb{E}[W_i - W_{i+1} \mid X_i = j] \leq \ln j \cdot \mathbb{Pr}[X_{i+1} = 0 \mid X_i = j] + \sum_{1 \leq j' \leq j-2} \ln \frac{j}{j'} \cdot \mathbb{Pr}[X_{i+1} = j' \mid X_i = j]$$
$$+ \ln \frac{j}{j-1} \cdot \mathbb{Pr}[X_{i+1} = j - 1 \mid X_i = j],$$

so, by (4.1),

$$\mathbb{E}[W_i - W_{i+1} \mid X_i = j] \leq \frac{c \ln j}{\rho \ln j} + \frac{c}{\rho} \sum_{1 \leq j' \leq j-2} \ln \frac{j}{j'} \cdot \frac{(j/j')^{\epsilon}}{(j - j')} + \ln \frac{j}{j-1} \leq \frac{c}{\rho} + \frac{c}{\rho} \cdot c_2(\epsilon) + \frac{1}{\rho}$$

where the last relation holds because $\ln \frac{j}{j-1} \le \frac{1}{j-1} \le \frac{1}{\rho}$, and

$$\begin{aligned}\sum_{1\le j'\le j-2} \ln\frac{j}{j'}\cdot\frac{(j/j')^\epsilon}{(j-j')} &= \sum_{1\le j'\le j/2}\frac{(j/j')^\epsilon}{j-j'}\ln\frac{j}{j'} + \sum_{j/2<j'\le j-2}\frac{(j/j')^\epsilon}{j-j'}\ln\frac{j}{j'}\\ &\le \frac{2}{j^{1-\epsilon}}\sum_{1\le j'\le j/2}\frac{1}{j'^\epsilon}\ln\frac{j}{j'} + \sum_{j/2<j'\le j-2}\frac{2^\epsilon}{j-j'}\cdot\frac{j-j'}{j'}\\ &\le c_2(\epsilon).\end{aligned}$$

Therefore, $\mathbb{E}[W_i - W_{i+1}] \le c_1/\rho$, for $c_1 = c\cdot(1+c_2)+1$. We can now bound $\mathbb{E}[K]$ as follows. Let $\kappa = \frac{\rho}{2c_1}(\ln X_0 - \ln\rho)$.

$$\mathbb{Pr}[K\le\kappa] = \mathbb{Pr}[X_\kappa\le\rho] = \mathbb{Pr}[W_\kappa\le\ln\rho] = \mathbb{Pr}[W_0 - W_\kappa \ge \ln X_0 - \ln\rho],$$

so, by Markov's inequality,

$$\mathbb{Pr}[K\le\kappa] \le \frac{\mathbb{E}[W_0-W_\kappa]}{\ln X_0 - \ln\rho} = \frac{\mathbb{E}[\sum_{i=0}^{\kappa-1}(W_i-W_{i+1})]}{\ln X_0-\ln\rho} \le \frac{\kappa c_1/\rho}{\ln X_0-\ln\rho} = \frac{1}{2}.$$

From that,

$$\mathbb{E}[K] \ge \kappa\cdot\mathbb{Pr}[K\ge\kappa] \ge \frac{\kappa}{2} = c'\rho\ln\frac{X_0}{\rho}, \quad \text{for } c' = \frac{1}{4c_1}.$$

B.2 Proof of Lemma 4.2

We prove the following more general version of Lemma 4.2.

Lemma B.1. *If* $\|u\|\ge\|u'\|$ *and* $\|r\|\ge\min\{\|r'\|,\|u'\|-1\}$ *then*

$$\mathbb{E}[L_u^{\mathcal{N}}\mid N_{u,1}=r] \ge \mathbb{E}[L_{u'}^{\mathcal{N}}\mid N_{u',1}=r'], \tag{B.2}$$

$$\mathbb{E}[L_u^{\mathcal{N}}] \ge \mathbb{E}[L_{u'}^{\mathcal{N}}]. \tag{B.3}$$

Proof. We prove the two results simultaneously by induction on $\|u\|$. Clearly, both results hold if $u'=0$. Below we assume that $u'\ne 0$ and, thus, $u\ne 0$. The induction hypothesis is that for all v,v' with $\|u\|>\|v\|\ge\|v'\|$, and for all w,w' such that $\|w\|\ge\min\{\|w'\|,\|v'\|-1\}$,

$$\text{(IH1): } \mathbb{E}[L_v^{\mathcal{N}}\mid N_{v,1}=w] \ge \mathbb{E}[L_{v'}^{\mathcal{N}}\mid N_{v',1}=w'] \qquad \text{and} \qquad \text{(IH2): } \mathbb{E}[L_v^{\mathcal{N}}]\ge\mathbb{E}[L_{v'}^{\mathcal{N}}].$$

From this hypothesis it is immediate that

$$\text{(IH1}'\text{): } \mathbb{E}[L_v^{\mathcal{N}}\mid N_{v,1}=w] = \mathbb{E}[L_v^{\mathcal{N}}\mid N_{v,1}=\|w\|] \qquad \text{and} \qquad \text{(IH2}'\text{): } \mathbb{E}[L_v^{\mathcal{N}}]=\mathbb{E}[L_{\|v\|}^{\mathcal{N}}].$$

We derive (B.2) as follows. By (IH2$'$),

$$\mathbb{E}[L_u^{\mathcal{N}}\mid N_{u,1}=r] = 1+\sum_v \mathbb{E}[L^{\mathcal{N}}_{\min\{\|u\|-1,\|r\|,\|v\|\}}]\cdot\mathbb{Pr}[N_{u,2}=v\mid N_{u,1}=r].$$

Given that $N_{u',1}=r'$, suppose that we compute $L_{u'}^{\mathcal{N}}$ using $a_2(u,r)$ and $A_2(u,r)$ in place of $a_2(u',r')$ and $A_2(u',r')$, respectively, and let M be the resulting quantity. By the optimality of the a_2 and A_2 (condition (4.2)),

$$\mathbb{E}[L_{u'}^{\mathcal{N}}\mid N_{u',1}=r'] \le \mathbb{E}[M\mid N_{u',1}=r'] = 1+\sum_v\mathbb{E}[L^{\mathcal{N}}_{\min\{\|u'\|-1,\|r'\|,\|v\|\}}]\cdot\mathbb{Pr}[N_{u,2}=v\mid N_{u,1}=r].$$

Combining the above two results and applying (IH2), we obtain (B.2). We now derive (B.3).

$$\mathbb{E}[L_u^{\mathcal{N}}] = \sum_r \mathbb{E}[L_u^{\mathcal{N}} \,|\, N_{u,1} = r] \cdot \mathbb{Pr}[N_{u,1} = r].$$

Suppose that when computing $L_{u'}^{\mathcal{N}}$ we replace $a_1(u')$ and $A_1(u')$ by $a_1(u)$ and $A_1(u)$, respectively, and let M' denote the resulting quantity. By the optimality of the a_1 and A_1, and (IH1′) (for $v = u'$),

$$\mathbb{E}[L_{u'}^{\mathcal{N}}] \le \mathbb{E}[M'] = \sum_r \mathbb{E}[L_{u'}^{\mathcal{N}} \,|\, N_{u',1} = r] \cdot \mathbb{Pr}[N_{u,1} = r].$$

Combining the above two results and applying (IH1) (for $v = u$), yields (B.3). ■

B.3 Proof of Lemma 4.3

We show that if $H, H' \in \mathbf{H}_u$ then

$$\mathbb{E}[L_u \,|\, \{u \not\to H\} \cap \{R_{\|u\|-1} \not\to H'\}] \ge \mathbb{E}[L_u^{\mathcal{N}}]. \tag{B.4}$$

The lemma follows then by taking $u = s$ and $H, H' = \emptyset$. We prove (B.4) by induction on $\|u\|$. In the induction we also show that if $u \neq 0$ then

$$\mathbb{E}[L_u \,|\, \{S_u = r\} \cap \{R_{\|u\|-1} \not\to H\}] \ge \mathbb{E}[L_u^{\mathcal{N}} \,|\, N_{u,1} = r], \tag{B.5}$$

where S_v, for $v \neq 0$, is the *out-neighbor* of v in $\mathcal{G}(n, \alpha)$ that is closest to 0 (if there are two such out-neighbors then S_v is the one that GREEDY would choose); and $S_0 = 0$. Clearly, (B.4) holds if $u = 0$; so, suppose that $u \neq 0$. Let ζ_v be the *in-contact* of v that is in $R_{\|v\|-1}$ and is closest to 0; and $\zeta_v = v$, if no such node exists. (Again if there are two candidate nodes, ζ_v is the one that GREEDY would use.) Define $\mathcal{E} = \{R_{\|u\|-1} \not\to H\}$.

$$\begin{aligned}
&\mathbb{E}[L_u \,|\, \{S_u = r\} \cap \mathcal{E}] \\
&\quad \ge 1 + \mathbb{E}[L_r \,|\, \{\|\zeta_u\| > \|r\|\} \cap \mathcal{E}] \cdot \mathbb{Pr}[\|\zeta_u\| > \|r\| \,|\, \mathcal{E}] \\
&\qquad + \sum_{v,v' : \|v'\| < \|v\| < \|r\|} \mathbb{E}[L_v \,|\, \{\zeta_u = v\} \cap \{S_v = v'\} \cap \mathcal{E}] \cdot \mathbb{Pr}[\{\zeta_u = v\} \cap \{S_v = v'\} \,|\, \mathcal{E}] \\
&\qquad + \min\Big\{ \mathbb{E}[L_r \,|\, \{r \not\to u\} \cap \{\|\zeta_u\| = \|r\|\} \cap \mathcal{E}] \cdot \mathbb{Pr}[\|\zeta_u\| = \|r\| \,|\, \mathcal{E}], \\
&\qquad\qquad \sum_{v,v' : \|v'\| < \|v\| = \|r\|} \mathbb{E}[L_v \,|\, \{\zeta_u = v\} \cap \{S_v = v'\} \cap \mathcal{E}] \cdot \mathbb{Pr}[\{\zeta_u = v\} \cap \{S_v = v'\} \,|\, \mathcal{E}], \\
&\qquad\qquad\quad \mathbb{E}[L_r \,|\, \{r \not\to u\} \cap \{\|\zeta_u\| = \|r\|\} \cap \mathcal{E}] \cdot \mathbb{Pr}[\{\|\zeta_u\| = \|r\|\} \cap \{\zeta_u \neq r\} \,|\, \mathcal{E}] \\
&\qquad\qquad\qquad + \sum_{v' : \|v'\| < \|r\|} \mathbb{E}[L_r \,|\, \{\zeta_u = r\} \cap \{S_r = v'\} \cap \mathcal{E}] \cdot \mathbb{Pr}[\{\zeta_u = r\} \cap \{S_r = v'\} \,|\, \mathcal{E}] \Big\}.
\end{aligned}$$

(The term $\min\{\cdot\}$ is a bound for the case $\|\zeta_u\| = \|r\|$.) But

$$\mathbb{E}[L_r \,|\, \{\|\zeta_u\| > \|r\|\} \cap \mathcal{E}] = \mathbb{E}[L_r \,|\, \{r \not\to H \cup \{u\}\} \cap \{R_{\|r\|-1} \not\to H \cup \{u\}\}] \ge \mathbb{E}[L_r^{\mathcal{N}}],$$

by the induction hypothesis, and, similarly, $\mathbb{E}[L_r \,|\, \{r \not\to u\} \cap \{\|\zeta_u\| = \|r\|\} \cap \mathcal{E}] \ge \mathbb{E}[L_r^{\mathcal{N}}]$. Also, for $\|v'\| < \|v\| \le \|r\|$,

$$\begin{aligned}
\mathbb{E}[L_v \,|\, \{\zeta_u = v\} \cap \{S_v = v'\} \cap \mathcal{E}] &= \mathbb{E}[L_v \,|\, \{S_v = v'\} \cap \{R_{\|v\|-1} \not\to H \cup \{u\}\}] \\
&\ge \mathbb{E}[L_v^{\mathcal{N}} \,|\, N_{v,1} = v'] \ge \mathbb{E}[L_{v'}^{\mathcal{N}}],
\end{aligned}$$

where the first inequality in the second line holds because of the induction hypothesis, and the second follows from (B.2). By combining all the above and then applying (B.3), we obtain

$$\mathbb{E}[L_u \mid \{S_u = r\} \cap \mathcal{E}] \geq 1 + \mathbb{E}[L_r^{\mathcal{N}}] \cdot \mathbb{Pr}[\|\zeta_u\| > \|r\| \mid \mathcal{E}] + \sum_{v' : \|v'\| < \|r\|} \mathbb{E}[L_{v'}^{\mathcal{N}}] \cdot \mathbb{Pr}[\{\|\zeta_u\| \leq \|r\|\} \cap \{S_{\zeta_u} = v'\} \mid \mathcal{E}].$$

Let Z_v be the set of all the *in-contacts* of v that are in $R_{\|v\|-1}$ and are closest to 0 ($0 \leq |Z_v| \leq 2$). Using u and H in place of $a_2(u, r)$ and $A_2(u, r)$, respectively, when computing $L_u^{\mathcal{N}}$, and using also (B.3), we get

$$\begin{aligned}\mathbb{E}[L_u^{\mathcal{N}} \mid N_{u,1} = r] &\leq 1 + \mathbb{E}[L_r^{\mathcal{N}}] \cdot \mathbb{Pr}[\min_{w \in Z_u} \|S_w\| \geq \|r\| \mid \mathcal{E}] + \sum_{j < \|r\|} \mathbb{E}[L_j^{\mathcal{N}}] \cdot \mathbb{Pr}[\min_{w \in Z_u} \|S_w\| = j\} \mid \mathcal{E}] \\ &\leq 1 + \mathbb{E}[L_r^{\mathcal{N}}] \cdot \mathbb{Pr}[\|\zeta_u\| > \|r\| \mid \mathcal{E}] + \sum_{j < \|r\|} \mathbb{E}[L_j^{\mathcal{N}}] \cdot \mathbb{Pr}[\{\|\zeta_u\| \leq \|r\|\} \cap \{\|S_{\zeta_u}\| = j\} \mid \mathcal{E}].\end{aligned}$$

We obtained the last relation by decreasing the probabilities inside the sum, and respectively increasing the probability by which $\mathbb{E}[L_r^{\mathcal{N}}]$ is multiplied. Combining the last two results and applying (B.3), yields (B.5). We now derive (B.4).

$$\mathbb{E}[L_u \mid \{u \not\to H\} \cap \{R_{\|u\|-1} \not\to H'\}] = \sum_r \mathbb{E}[L_u \mid \{S_u = r\} \cap \{R_{\|u\|-1} \not\to H'\}] \cdot \mathbb{Pr}[S_u = r \mid u \not\to H].$$

Using u and H in place of $a_1(u)$ and $A_1(u)$, respectively, when computing $L_u^{\mathcal{N}}$, and using also (B.2), we get

$$\mathbb{E}[L_u^{\mathcal{N}}] \leq \sum_r \mathbb{E}[L_u^{\mathcal{N}} \mid N_{u,1} = r] \cdot \mathbb{Pr}[S_u = r \mid u \not\to H].$$

Combining the two results above and using (B.5), we obtain (B.4).

B.4 Proof of Lemma 4.4(b)

We consider an "early-stopping" variation of $\mathcal{N}$ that differs from the original process as follows. Let $u \neq 0$ be the current node, suppose that $N_{u,1} = r$, and let Z be the set of the in-contacts of $a_2(u, r)$ that are in $R_{\|a_2(u,r)\|-1}$ and are closest to 0 (recall the definition of $\mathcal{N}$ from Section 4.2.1). If $Z \neq \emptyset$, let D be the maximum number of out-contacts that any element of Z has, i.e., $D = \max_{v \in Z} D_v$. If $D > \ln^2 n$ then the process jumps to node 0 in the next step (and it finishes). We denote by M_u the next node after node u in this new process, and by L_u^M the number of steps to reach 0 from u. Clearly, $\mathbb{E}[L_u^{\mathcal{N}}] \geq \mathbb{E}[L_u^M]$, so, to prove the lemma it suffices to show that $\mathbb{E}[L_{n/4}^M] = \Omega(\ln^2 n / \ln \ln n)$. We will prove below that for all u with $\|u\| \geq \rho = \ln n / \ln \ln n$,

$$\mathbb{Pr}[\|M_u\| = j] = \begin{cases} \mathrm{O}\left(\frac{1}{\rho(\|u\|-j)}\right), & \text{if } 0 < j \leq \|u\| - 2; \\ \mathrm{O}\left(\frac{1}{\rho \ln n}\right), & \text{if } j = 0. \end{cases} \tag{B.6}$$

From this and Lemma 4.1, applied for $X_0 = n/4$, $X_{i+1} = \|M_{X_i}\|$, and $\epsilon = 0$, it follows that $\mathbb{E}[L_{n/4}^M] = \Omega(\ln^2 n / \ln \ln n)$. We prove (B.6) similarly to (4.3).

- CASE: $j \neq 0$.

$$\mathbb{Pr}[\|M_u\| = j] \leq \mathbb{Pr}[\|N_{u,1}\| = j] + \max_r \mathbb{Pr}[\{\|N_{u,2}\| = j\} \cap \{D \leq \ln^2 n\} \mid N_{u,1} = r]. \tag{B.7}$$

  Similarly to (4.6),

$$\begin{aligned}&\mathbb{Pr}[\{\|N_{u,2}\| = j\} \cap \{D \leq \ln^2 n\} \mid N_{u,1} = r] \\ &\qquad \leq \sum_{v : j \leq \|v\| < \|a_2\|} \sum_{k \leq \ln^2 n} \mathbb{Pr}\left[\{v \in Z_{a_2}\} \cap \{\|S_v\| = j\} \cap \{D_v = k\} \,\middle|\, R_{\|a_2\|-1} \not\to A_2\right],\end{aligned}$$

so, as in (4.10),

$$
\begin{aligned}
&\mathbb{Pr}[\{\|N_{u,2}\| = j\} \cap \{D \le \ln^2 n\} \mid N_{u,1} = r] \\
&\qquad = \mathrm{O}\left(\sum_{v\,:\, j+2 \le \|v\| < \|a_2\|} \sum_{k \le \ln^2 n} \frac{k^{-1}}{\nu^2(\|v\| - j)(\|a_2\| - \|v\|)} + \frac{1}{\nu(\|a_2\| - j)} \right) \\
&\qquad = \mathrm{O}\left(\frac{\ln(\|a_2\| - j)}{\nu^2(\|a_2\| - j)} \ln\ln n + \frac{1}{\nu(\|a_2\| - j)} \right) = \mathrm{O}\left(\frac{\ln\ln n}{\nu(\|a_2\| - j)} \right). \qquad \text{(B.8)}
\end{aligned}
$$

Combining (B.7), (4.5), and (B.8), yields the top part of (B.6).

- CASE: $j = 0$.

$$
\mathbb{Pr}[\|M_u\| = 0] \le \mathbb{Pr}[|N_{u,1}\| = 0] + \max_r \mathbb{Pr}[D > \ln^2 n \mid N_{u,1} = r],
$$

and

$$
\begin{aligned}
\mathbb{Pr}[D > \ln^2 n \mid N_{u,1} = r] &\le \sum_{v\,:\,\|v\| < \|a_2\|} \sum_{k > \ln^2 n} \mathbb{Pr}[\{v \to a_2\} \cap \{D_v = k\} \mid v \not\to A_2] \\
&\le \sum_{v\,:\,\|v\| < \|a_2\|} \sum_{k > \ln^2 n} \frac{2}{q_1} \mathbb{Pr}[\{v \to a_2\} \cap \{D_v = k\}] \\
&= \mathrm{O}\left(\sum_{v\,:\,\|v\| < \|a_2\|} \sum_{k > \ln^2 n} \frac{1}{k^3} \frac{k}{\nu(\|a_2\| - \|v\|)} \right) = \mathrm{O}\left(\frac{\ln \|a_2\|}{\nu \ln^2 n} \right).
\end{aligned}
$$

where for the second relation we used Fact 3.5(c), and for the third Fact 3.1. From the above two results and (4.5),

$$
\mathbb{Pr}[\|M_u\| = 0] = \mathrm{O}\left(\frac{1}{\nu\|a_1\|} + \frac{\ln \|a_2\|}{\nu \ln^2 n} \right) = \mathrm{O}\left(\frac{1}{\nu\rho} \right).
$$

B.5 Proof of Lemma 4.4(c)

We show that for all u with $\nu < \|u\| \le \lambda$,

$$
\mathbb{Pr}[\|N_u\| = j] = \begin{cases} \mathrm{O}\left(\frac{(\|u\| j)^{3-\alpha}}{\nu(\|u\| - j)} \right), & \text{if } 0 < j \le \|u\| - 2; \\ \mathrm{O}\left(\frac{1}{\nu \ln \|u\|} \right), & \text{if } j = 0. \end{cases} \qquad \text{(B.9)}
$$

From this and Lemma 4.1, applied for $X_0 = \lambda$, $X_{i+1} = \|N_{X_i}\|$, $\rho = \nu$, and $\epsilon = 3 - \alpha$ it follows that $\mathbb{E}[L_\lambda^{\mathcal{N}}] = \Omega(\nu \ln \lambda) = \Omega(\ln^{\alpha-1} n)$. The derivation of (B.9) is similar to that of (4.3) (through results (4.4)–(4.10)). Specifically, the only difference is that we use a more refined bound instead of (4.7), when $j \ne 0$, and we ensure that the probability bounds used in (4.10) do not exceed 1, when $j = 0$.

- CASE: $j \ne 0$. By replacing the upper bound in the third line of (4.7) by the exact quantity, we have

$$
\begin{aligned}
&\mathbb{Pr}\left[\|S_v\| = j \,\middle|\, \{v \in Z_{a_2}\} \cap \{D_v = k\} \cap \{R_{\|a_2\|-1} \not\to A_2\} \right] \\
&\qquad = \mathbb{Pr}\left[\{v \to \{j,\, n-j\}\} \cap \{v \not\to R_{j-1}\} \,\middle|\, \{D_v = k-1\} \cap \{v \not\to A_2\} \right] \\
&\qquad = \mathbb{Pr}\left[v \to \{j,\, n-j\} \,\middle|\, \{D_v = k-1\} \cap \{v \not\to A_2\} \right] \\
&\qquad\qquad \cdot \mathbb{Pr}\left[v \not\to R_{j-1} \,\middle|\, \{D_v = k-2\} \cap \{v \not\to A_2\} \right] \\
&\qquad = \mathrm{O}\left(\frac{k-1}{\nu(\|v\| - j)} \cdot \left(1 - \frac{1}{\nu} \ln \frac{\|v\| + j}{\|v\| - j + 1} \right)^{k-2} \right), \qquad \text{(B.10)}
\end{aligned}
$$

where the last relation was obtained using Fact 3.3. Let $\gamma = \left(1 - \frac{1}{\nu}\ln\frac{\|v\|+j}{\|v\|-j+1}\right)^{k-2}$. Similarly to (4.10), but using (B.10) in place of (4.7),

$$\begin{aligned}\mathbb{Pr}[\|N_{u,2}\| = j \mid N_{u,1} = r] &= \mathrm{O}\bigg(\sum_{v\,:\,j+2\le\|v\|<\|a_2\|}\sum_k \frac{k^{2-\alpha}\gamma}{\nu^2(\|v\|-j)(\|a_2\|-\|v\|)} + \frac{1}{\nu(\|a_2\|-j)}\bigg)\\ &= \mathrm{O}\bigg(\frac{(\|u\|/j)^{3-\alpha}}{\nu(\|u\|-j)}\bigg),\end{aligned} \tag{B.11}$$

because, as we show below,

$$\sum_{v\,:\,j<\|v\|<\|a_2\|}\sum_k \frac{k^{2-\alpha}\gamma}{(\|v\|-j)(\|a_2\|-\|v\|)} = \mathrm{O}\left(\frac{\nu}{\|u\|-j}\left(\frac{\|u\|}{j}\right)^{3-\alpha}\right). \tag{B.12}$$

Combining (4.4), (4.5), and (B.11) yields the top part of (B.9). It remains to prove (B.12). For $0 < j < \|v\| < \|a_2\|$,

$$\gamma \le \left(1 - \frac{1}{\nu}\ln\frac{\|a_2\|}{\|a_2\|-j}\right)^{k-2} \le \left(1 - \frac{j}{\|a_2\|}\right)^{\frac{k-2}{\nu}}$$

where for the last relation we used the fact that $1 - x \le e^x$. So, the left-hand side of (B.12) is at most

$$\sum_{v\,:\,j<\|v\|<\|a_2\|}\sum_k \frac{k^{2-\alpha}\left(1-j/\|a_2\|\right)^{\frac{k-2}{\nu}}}{(\|v\|-j)(\|a_2\|-\|v\|)} = \mathrm{O}\left(\frac{\ln(\|a_2\|-j)}{\|a_2\|-j}\sum_k k^{2-\alpha}\left(1-\frac{j}{\|a_2\|}\right)^{\frac{k-2}{\nu}}\right).$$

But $\sum_k k^{2-\alpha}(1-j/\|a_2\|)^{\frac{k-2}{\nu}} = \mathrm{O}\big(\big(\frac{\nu\|a_2\|}{j}\big)^{3-\alpha}\big)$, since only the first $\Theta(\kappa)$ terms of the sum are non-negligible, where κ is such that $\left(1-j/\|a_2\|\right)^{\frac{\kappa-2}{\nu}} = 1/2$, so, $\kappa \approx \frac{\nu\|a_2\|}{j}$. Therefore, the left-hand side of (B.12) is

$$\mathrm{O}\left(\frac{\ln(\|a_2\|-j)}{\|a_2\|-j}\left(\frac{\nu\|a_2\|}{j}\right)^{3-\alpha}\right) = \mathrm{O}\left(\frac{\ln\|u\|}{\|u\|-j}\left(\frac{\nu\|u\|}{j}\right)^{3-\alpha}\right) = \mathrm{O}\left(\frac{\nu}{\|u\|-j}\left(\frac{\|u\|}{j}\right)^{3-\alpha}\right),$$

where the first relation holds because $\frac{\ln x\cdot x^{3-\alpha}}{x-j}$ decreases as x increases; and the second relation holds because $\ln\|u\| \le \ln\lambda = \nu^{\alpha-2}$.

- CASE: $j = 0$. By rewriting the first relation in (4.10) so that the bounds used for the probabilities do not exceed 1, we obtain

$$\begin{aligned}&\mathbb{Pr}[\|N_{u,2}\| = 0 \mid N_{u,1} = r]\\ &\quad= \mathrm{O}\bigg(\sum_{v\,:\,2\le\|v\|<\|a_2\|}\sum_k \min\left\{1, \frac{k}{\nu\|v\|}\right\}\cdot\min\left\{1, \frac{k}{\nu(\|a_2\|-\|v\|)}\right\}\cdot\frac{1}{k^\alpha} + \frac{1}{\nu\|a_2\|}\bigg)\\ &\quad= \mathrm{O}\left(\frac{1}{\nu^{\alpha-1}\|a_2\|^{\alpha-2}} + \frac{1}{\nu\|a_2\|}\right) = \mathrm{O}\left(\frac{1}{\nu\ln\|u\|}\right),\end{aligned} \tag{B.13}$$

where the first relation in the last line holds because the double sum in the middle line is $\mathrm{O}\left(\frac{1}{\nu^{\alpha-1}a_2^{\alpha-2}}\right)$, as it is easy to show. Combining (4.4), (4.5), and (B.13), yields the bottom part of (B.9).

B.6 Proof of Lemma 4.5

Let $\mathcal{E} = \{\|Y_K\| \geq \lambda^{1/3}\} \cap \big(\{Y_K \not\to Y_{K-1}\} \cup \{D_{Y_K} = 1\}\big)$, and

$$\mathcal{P} = \bigcap_{v \notin R_\lambda} \{D_v \leq \|v\|\}.$$

We will show that $\mathbb{Pr}[\mathcal{E} \,|\, \mathcal{P}] = \Theta(1)$. This, together with the fact that $\mathbb{Pr}[\mathcal{P}] = \Theta(1)$, which is easy to show, yields the lemma. To prove that $\mathbb{Pr}[\mathcal{E} \,|\, \mathcal{P}] = \Theta(1)$ it suffices to show that for arbitrary κ and instantiation $\langle y_0, \dots, y_{\kappa-1}\rangle$ of $\langle Y_0, \dots, Y_{\kappa-1}\rangle$, and for $\mathcal{H} = \{K = \kappa\} \cap \{\langle Y_0, \dots, Y_{\kappa-1}\rangle = \langle y_0, \dots, y_{\kappa-1}\rangle\}$,

$$\mathbb{Pr}[\mathcal{E} \,|\, \mathcal{H} \cap \mathcal{P}] = \Theta(1). \tag{B.14}$$

Define the sets $R_x^{=1} = \{u \in R_x \,:\, D_u = 1\}$ and $R_x^{\neq 1} = \{u \in R_x \,:\, D_u \neq 1\}$. Define the events

$$\mathcal{E}_1 = \{R_\lambda^{\neq 1} \to Y_{K-1}\}, \qquad \mathcal{E}_2 = \{R_{\lambda^{1/3}} \to Y_{K-1}\},$$
$$\mathcal{E}_3 = \{Y_{K-1} \to R_{\lambda^{1/3}}\} \cap \{Y_{K-1} \not\to Y_{K-2}\}, \qquad \mathcal{E}_4 = \{Y_{K-1} \to R_{\lambda^{1/3}}\} \cap \{Y_{K-1} \to Y_{K-2}\}.$$

Clearly $\bar{\mathcal{E}} \subseteq \bigcup_{i=1}^4 \mathcal{E}_i$, so,

$$\mathbb{Pr}[\bar{\mathcal{E}} \,|\, \mathcal{H} \cap \mathcal{P}] \leq \sum_{i=1}^{4} \mathbb{Pr}[\mathcal{E}_i \,|\, \mathcal{H} \cap \mathcal{P}]. \tag{B.15}$$

We now compute upper bounds for the four probabilities on the right-hand side. Let $\mathcal{A} = \{R_\lambda \not\to \langle y_0, \dots, y_{\kappa-2}\rangle\}$.

$$\mathbb{Pr}[\mathcal{E}_1 \,|\, \mathcal{H} \cap \mathcal{P}] \leq \mathbb{Pr}[R_\lambda^{\neq 1} \to y_{\kappa-1} \,|\, \{R_\lambda \to y_{\kappa-1}\} \cap \mathcal{A}] = \frac{\mathbb{Pr}[R_\lambda^{\neq 1} \to y_{\kappa-1} \,|\, \mathcal{A}]}{\mathbb{Pr}[R_\lambda \to y_{\kappa-1} \,|\, \mathcal{A}]}.$$

It is easy to show that for any $u \in R_\lambda$, $\mathbb{Pr}[\{u \to y_{\kappa-1}\} \cap \{D_u \neq 1\} \,|\, \mathcal{A}] \leq \mathbb{Pr}[\{u \to y_{\kappa-1}\} \cap \{D_u = 1\} \,|\, \mathcal{A}]$, and, using this, that $\mathbb{Pr}[R_\lambda^{\neq 1} \to y_{\kappa-1} \,|\, \mathcal{A}] \leq \mathbb{Pr}[R_\lambda^{=1} \to y_{\kappa-1} \,|\, \mathcal{A}]$. Also, $\mathbb{Pr}[R_\lambda \to y_{\kappa-1} \,|\, \mathcal{A}] \geq \mathbb{Pr}[R_\lambda^{\neq 1} \to y_{\kappa-1} \,|\, \mathcal{A}] + \mathbb{Pr}[R_\lambda^{=1} \to y_{\kappa-1} \,|\, \mathcal{A}] - \mathbb{Pr}[R_\lambda^{\neq 1} \to y_{\kappa-1} \,|\, \mathcal{A}] \cdot \mathbb{Pr}[R_\lambda^{=1} \to y_{\kappa-1} \,|\, \mathcal{A}]$, because of the negative dependence between $\{R_\lambda^{\neq 1} \to y_{\kappa-1}\}$ and $\{R_\lambda^{=1} \to y_{\kappa-1}\}$. Finally, $\mathbb{Pr}[R_\lambda^{=1} \to y_{\kappa-1} \,|\, \mathcal{A}] = \mathrm{O}\big(\frac{1}{\nu} \ln \frac{\|y_{\kappa-1}\|}{\|y_{\kappa-1}\| - \lambda}\big) = \mathrm{O}(\frac{1}{\nu} \ln \lambda) = \mathrm{o}(1)$. Combining all the above we obtain

$$\mathbb{Pr}[\mathcal{E}_1 \,|\, \mathcal{H} \cap \mathcal{P}] \leq 1/2 + \mathrm{o}(1). \tag{B.16}$$

Next,

$$\mathbb{Pr}[\mathcal{E}_2 \,|\, \mathcal{H} \cap \mathcal{P}] \leq \mathbb{Pr}[R_{\lambda^{1/3}} \to y_{\kappa-1} \,|\, \{R_\lambda \to y_{\kappa-1}\} \cap \mathcal{A}] = \frac{\mathbb{Pr}[R_{\lambda^{1/3}} \to y_{\kappa-1} \,|\, \mathcal{A}]}{\mathbb{Pr}[R_\lambda \to y_{\kappa-1} \,|\, \mathcal{A}]},$$

and since for $m = \lambda$ or $\lambda^{1/2}$, $\mathbb{Pr}[R_m \to y_{\kappa-1} \,|\, \mathcal{A}] = \Theta\big(\frac{1}{\nu} \ln \frac{\|y_{\kappa-1}\|}{\|y_{\kappa-1}\| - m}\big)$,

$$\mathbb{Pr}[\mathcal{E}_2 \,|\, \mathcal{H} \cap \mathcal{P}] = \mathrm{O}\left(\frac{\ln \frac{n}{n-\lambda^{1/3}}}{\ln \frac{n}{n-\lambda}}\right) = \mathrm{O}(\lambda^{-2/3}) = \mathrm{o}(1). \tag{B.17}$$

Let $\mathcal{B} = \{y_{\kappa-1} \not\to \langle y_0, \dots, y_{\kappa-2}\rangle\} \cap \{D_{y_{\kappa-1}} \leq \|y_{\kappa-1}\|\}$.

$$\mathbb{Pr}[\mathcal{E}_3 \,|\, \mathcal{H} \cap \mathcal{P}] \leq \mathbb{Pr}[y_{\kappa-1} \to R_{\lambda^{1/3}} \,|\, \{y_{\kappa-1} \to R_\lambda\} \cap \mathcal{B}] = \frac{\mathbb{Pr}[y_{\kappa-1} \to R_{\lambda^{1/3}} \,|\, \mathcal{B}]}{\mathbb{Pr}[y_{\kappa-1} \to R_\lambda \,|\, \mathcal{B}]},$$

and since for $m = \lambda$ or $\lambda^{1/2}$, $\mathbb{Pr}[y_{\kappa-1} \to R_m \,|\, \mathcal{B}] = \Theta(\mathbb{Pr}[y_{\kappa-1} \to R_m]) = \Theta\big(\frac{1}{\nu} \ln \frac{\|y_{\kappa-1}\|}{\|y_{\kappa-1}\| - m}\big)$,

$$\mathbb{Pr}[\mathcal{E}_3 \,|\, \mathcal{H} \cap \mathcal{P}] = \mathrm{O}(\lambda^{-2/3}) = \mathrm{o}(1), \tag{B.18}$$

as before. It remains to bound $\mathbb{P}\mathrm{r}[\mathcal{E}_4 \,|\, \mathcal{H} \cap \mathcal{P}]$. Let $H = \langle y_0, \ldots, y_{\kappa-3} \rangle$, and $\mathcal{C} = \{y_{\kappa-1} \to y_{\kappa-2}\} \cap \{y_{\kappa-1} \not\to H\}$.

$$\mathbb{P}\mathrm{r}[\mathcal{E}_4 \,|\, \mathcal{H} \cap \mathcal{P}] \le \mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_{\lambda^{1/3}} \,|\, \{y_{\kappa-1} \to R_\lambda\} \cap \mathcal{C} \cap \mathcal{P}] = \frac{\mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_{\lambda^{1/3}} \,|\, \mathcal{C} \cap \mathcal{P}]}{\mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_\lambda \,|\, \mathcal{C} \cap \mathcal{P}]}.$$

For $m = \lambda$ or $\lambda^{1/2}$, and $k' = \min\{\|y_{\kappa-1}\|, k_{\max}\}$,

$$\begin{aligned}
&\mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_m \,|\, \mathcal{C} \cap \mathcal{P}] \\
&\quad = \sum_{k=2}^{k'} \mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_m \,|\, \mathcal{C} \cap \{D_{y_{\kappa-1}} = k\}] \cdot \mathbb{P}\mathrm{r}[D_{y_{\kappa-1}} = k \,|\, \mathcal{C} \cap \{D_{y_{\kappa-1}} \le \|y_{\kappa-1}\|\}] \\
&\quad = \sum_{k=2}^{k'} \mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_m \,|\, \{y_{\kappa-1} \not\to H\} \cap \{D_{y_{\kappa-1}} = k-1\}] \\
&\qquad\qquad \cdot \frac{q_k \,\mathbb{P}\mathrm{r}[y_{\kappa-1} \to y_{\kappa-2} \,|\, \{y_{\kappa-1} \not\to H\} \cap \{D_{y_{\kappa-1}} = k\}] \cdot \mathbb{P}\mathrm{r}[y_{\kappa-1} \not\to H \,|\, D_{y_{\kappa-1}} = k]}{\mathbb{P}\mathrm{r}[\mathcal{C} \cap \{D_{y_{\kappa-1}} \le \|y_{\kappa-1}\|\}]} \\
&\quad = \sum_{k=2}^{k'} \frac{\mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_m \,|\, D_{y_{\kappa-1}} = k-1]}{\mathbb{P}\mathrm{r}[y_{\kappa-1} \not\to H \,|\, D_{y_{\kappa-1}} = 1]} \cdot \frac{q_k \,\mathbb{P}\mathrm{r}[y_{\kappa-1} \to y_{\kappa-2} \,|\, D_{y_{\kappa-1}} = k] \cdot \mathbb{P}\mathrm{r}[y_{\kappa-1} \not\to H \,|\, D_{y_{\kappa-1}} = k]}{\mathbb{P}\mathrm{r}[y_{\kappa-1} \not\to H \,|\, D_{y_{\kappa-1}} = 1] \cdot \mathbb{P}\mathrm{r}[\mathcal{C} \cap \{D_{y_{\kappa-1}} \le \|y_{\kappa-1}\|\}]},
\end{aligned}$$

where the last relation was obtained by applying Fact 3.4 twice. So, $\mathbb{P}\mathrm{r}[\mathcal{E}_4 \,|\, \mathcal{H} \cap \mathcal{P}]$ is at most

$$\begin{aligned}
&\frac{\sum_{k=2}^{k'} \frac{1}{k^2} \mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_{\lambda^{1/3}} \,|\, D_{y_{\kappa-1}} = k-1] \cdot \mathbb{P}\mathrm{r}[y_{\kappa-1} \to y_{\kappa-2} \,|\, D_{y_{\kappa-1}} = k] \cdot \mathbb{P}\mathrm{r}[y_{\kappa-1} \not\to H \,|\, D_{y_{\kappa-1}} = k]}{\sum_{k=2}^{k'} \frac{1}{k^2} \mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_\lambda \,|\, D_{y_{\kappa-1}} = k-1] \cdot \mathbb{P}\mathrm{r}[y_{\kappa-1} \to y_{\kappa-2} \,|\, D_{y_{\kappa-1}} = k] \cdot \mathbb{P}\mathrm{r}[y_{\kappa-1} \not\to H \,|\, D_{y_{\kappa-1}} = k]} \\
&\qquad \le \frac{\sum_{k=2}^{k'} \frac{1}{k} \mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_{\lambda^{1/3}} \,|\, D_{y_{\kappa-1}} = k-1]}{\sum_{k=2}^{k'} \frac{1}{k} \mathbb{P}\mathrm{r}[y_{\kappa-1} \to R_\lambda \,|\, D_{y_{\kappa-1}} = k-1]},
\end{aligned}$$

by Fact B.2 below (it is easy to verify that the conditions of Fact B.2 are met). Finally, it is not hard to show that fraction in the last line is larger when $k_{\max} \ge \|y_{\kappa-1}\|$, and then the numerator is $\big(1/3 + \mathrm{o}(1)\big) \ln \lambda$ and the denominator is $(1 + \mathrm{o}(1)) \ln \lambda$. Therefore,

$$\mathbb{P}\mathrm{r}[\mathcal{E}_4 \,|\, \mathcal{H} \cap \mathcal{P}] \le 1/3 + \mathrm{o}(1). \tag{B.19}$$

Combining (B.15)–(B.19) yields $\mathbb{P}\mathrm{r}[\bar{\mathcal{E}} \,|\, \mathcal{H}] \le 5/6 + \mathrm{o}(1)$, which implies (B.14).

Fact B.2. *Let $\{a_i\}_{i=1}^k$, $\{x_i\}_{i=1}^k$, and $\{y_i\}_{i=1}^k$ be non-increasing, positive sequences such that $\{\frac{x_i}{y_i}\}_{i=1}^k$ is non-decreasing. Then*

$$\frac{\sum_{i=1}^k a_i x_i}{\sum_{i=1}^k a_i y_i} \le \frac{\sum_{i=1}^k x_i}{\sum_{i=1}^k y_i}.$$

Proof. Let

$$\varphi = \frac{\sum_j x_j}{\sum_j y_j} \quad \text{and} \quad i_0 = \max\Big\{ i \,:\, \frac{x_i}{y_i} \le \varphi \Big\}.$$

Note that i_0 is well defined, since $\varphi = \frac{\sum_j y_j (x_j/y_j)}{\sum_j y_j} \ge \frac{\sum_j y_j (x_1/y_1)}{\sum_j y_j} = \frac{x_1}{y_1}$. Normalize the a_i such that $a_i \ge 1$, if $i \le i_0$, and $a_i \le 1$, if $i > i_0$. By the facts that for any $\alpha, \beta, \gamma, \delta > 0$, if $\frac{\alpha}{\beta} \ge \frac{\gamma}{\delta}$ then $\frac{\alpha}{\beta} \ge \frac{\alpha+\gamma}{\beta+\delta}$, and if $\frac{\alpha}{\beta} \le \frac{\gamma}{\delta}$ and $\beta > \delta$ then $\frac{\alpha}{\beta} \ge \frac{\alpha-\gamma}{\beta-\delta}$, we have

$$\varphi \ge \frac{\sum_i x_i + \sum_{i \le i_0} (a_i - 1) x_i}{\sum_i y_i + \sum_{i \le i_0} (a_i - 1) y_i} \ge \frac{\sum_i x_i + \sum_{i \le i_0} (a_i - 1) x_i - \sum_{i > i_0} (1 - a_i) x_i}{\sum_i y_i + \sum_{i \le i_0} (a_i - 1) y_i - \sum_{i > i_0} (1 - a_i) y_i} = \frac{\sum_i a_i x_i}{\sum_i a_i y_i},$$

where the first relation holds because if $\sum_{i \le i_0} (a_i - 1) y_i \ne 0$ then $\frac{\sum_{i \le i_0} (a_i - 1) x_i}{\sum_{i \le i_0} (a_i - 1) y_i} \le \frac{x_{i_0}}{y_{i_0}} \le \varphi$; and the second relation holds because if $\sum_{i > i_0} (1 - a_i) y_i \ne 0$ then $\frac{\sum_{i > i_0} (1 - a_i) x_i}{\sum_{i > i_0} (1 - a_i) y_i} \ge \frac{x_{i_0+1}}{y_{i_0+1}} > \varphi \ge \frac{\sum_i x_i + \sum_{i \le i_0} (a_i - 1) x_i}{\sum_i y_i + \sum_{i \le i_0} (a_i - 1) y_i}$. ■

B.7 Proof of Lemma 4.6

We consider an early-stopping variation of GREEDY, where if in some step we visit a node $v \in R_{\|u\|}$ that has an in-contact $v' \in R_{\|v\|-1}$ with $D_{v'} > 1$ then we jump to 0 in the next step. Let L'_w denote the number of steps it takes in this process to reach 0 starting from node w. Let also $\langle Y'_0 = Y_0, Y'_1, Y'_2, \ldots \rangle$ be the sequence of nodes visited when starting from Y_0. Clearly, $L_u \geq L'_u$, so, to prove the lemma it suffices to show that $\mathbb{E}[L'_u \,|\, \mathcal{H}] = \Omega(\ln^{4/3} n)$, where $\mathcal{H}$ is the σ-algebra generated by $Y'_0, \ldots, Y'_{i-1}, \{Y'_i = u\} \cap \big(\{u \not\to Y'_{i-1}\} \cup \{D_u = 1\}\big)$. We will show below that for all $j \geq i$ and v with $\nu \leq \|v\| \leq \|u\|$,

$$\mathbb{P}\mathrm{r}[Y'_{j+1} = v' \,|\, \mathcal{H} \cap \{Y'_j = v\}] = \begin{cases} \mathrm{O}\left(\frac{1}{\nu\delta(v,v')}\right), & \text{if } 0 < \|v'\| \leq \|v\| - 2; \\ \mathrm{O}\left(\frac{1}{\nu \ln \|v\|}\right), & \text{if } v' = 0. \end{cases} \tag{B.20}$$

From this and Lemma 4.1, applied for $X_\imath = \|Y'_{i+\imath}\| \,|\, \mathcal{H}$, $\rho = \nu$, and $\epsilon = 0$, it follows that $\mathbb{E}[L'_u \,|\, \mathcal{H}] = \Omega(\nu \ln \lambda) = \Omega(\ln^{4/3} n)$. We now prove (B.20).

- CASE: $v' \neq 0$. The left-hand side of (B.20) is at most

  $$\max_{H \in \mathbf{H}_v} \mathbb{P}\mathrm{r}[v \to v' \,|\, v \not\to H] + \max_{H \in \mathbf{H}_v} \mathbb{P}\mathrm{r}[v' \to v \,|\, v' \not\to H] = \mathrm{O}\left(\frac{1}{\nu\delta(v,v')}\right),$$

  by Facts 3.2 and 3.5(c).

- CASE: $v' = 0$. The left-hand side of (B.20) is at most

  $$\max_{H \in \mathbf{H}_v} \mathbb{P}\mathrm{r}[v \to 0 \,|\, v \not\to H] + \max_{H \in \mathbf{H}_v} \mathbb{P}\mathrm{r}[0 \to v \,|\, v' \not\to H] + \sum_{r \in R_{\|v\|-1}} \max_{H \in \mathbf{H}_v} \mathbb{P}\mathrm{r}[\{r \to v\} \cap \{D_r \neq 1\} \,|\, r \not\to H] = \mathrm{O}\left(\frac{1}{\nu \ln \|v\|}\right),$$

  because the first two terms are $\mathrm{O}\left(\frac{1}{\nu\|v\|}\right)$, as before, and, by Fact 3.5(c), the third term is

  $$\mathrm{O}\Bigg(\sum_{r \in R_{\|v\|-1}} \mathbb{P}\mathrm{r}[\{r \to v\} \cap \{D_r \neq 1\}]\Bigg) = \mathrm{O}\Bigg(\sum_{r \in R_{\|v\|-1}} \sum_{k \neq 1} q_k \, \mathbb{P}\mathrm{r}[r \to v \,|\, D_r = k]\Bigg)$$
  $$= \mathrm{O}\Bigg(\sum_{r \in R_{\|v\|-1}} \sum_{k \neq 1} \frac{1}{\beta k^2} \cdot \min\left\{1, \frac{k}{\nu\delta(v,r)}\right\}\Bigg) = \mathrm{O}\Bigg(\frac{\ln^2 \|v\|}{\nu\beta}\Bigg) = \mathrm{O}\Bigg(\frac{1}{\nu \ln \|v\|}\Bigg),$$

  where the last relation holds because $\|v\| \leq \lambda$ and $\beta = \Theta(\nu)$.

C Appendix of Section 5

C.1 Proof of Lemma 5.1

Let Q_i be the indicator random variable of the event $\{X_i \leq X_{i-1}/2\}$. The number of steps to reach 0 is upper bounded by $T = \min\{j \,:\, \sum_{i=1}^{j} Q_i \geq \kappa\}$, because it takes at most κ "halvings" to get from X_0 to 0. Also, since $\mathbb{P}\mathrm{r}[X_i \leq j/2 \,|\, X_0, \ldots, X_{i-2}, \{X_{i-1} = j\}] \geq 1/\rho$, $\mathbb{E}[Q_i \,|\, Q_1, \ldots, Q_{i-1}] \geq 1/\rho$. Therefore, T is stochastically smaller than the sum of κ independent geometric random variables, each with probability parameter $1/\rho$, and, so, $\mathbb{E}[T] \leq \kappa\rho$. Hence, part (a) holds. For part (b), we observe that $\sum_{i=1}^{j} Q_i$ is stochastically larger than the binomial random variable $\mathrm{B}(j, 1/\rho)$; so,

$$\mathbb{P}\mathrm{r}[T > t] = \mathbb{P}\mathrm{r}\left[\sum_{i=1}^{t} Q_i < \kappa\right] \leq \mathbb{P}\mathrm{r}[\mathrm{B}(t, 1/\rho) < \kappa] \leq e^{-\frac{t}{2\rho}(1-\frac{\rho\kappa}{t})^2} \leq e^{-\frac{t}{4\rho}},$$

where the second-to-last relation is obtained using Chernoff's bounds, and the last holds because $t \geq 4\rho\kappa$.

C.2 Proof of Fact 5.2

Since the Q_i are independent, $\Pr[Q=0] = \prod_i (1 - \mathbb{E}[Q_i])$. So, by the fact that for all x, $1 - x \leq e^{-x}$,

$$\Pr[Q=0] \leq \prod_i e^{-\mathbb{E}[Q_i]} = e^{-\sum_i \mathbb{E}[Q_i]} = e^{-\mathbb{E}[Q]},$$

and, by the fact that for all $x \leq 1/2$, $1 - x \geq e^{-x-x^2}$,

$$\Pr[Q=0] \geq \prod_i e^{-(1+\mathbb{E}[Q_i])\cdot\mathbb{E}[Q_i]} \geq \prod_i e^{-\frac{3}{2}\mathbb{E}[Q_i]} = e^{-\frac{3}{2}\sum_i \mathbb{E}[Q_i]} = e^{-\frac{3}{2}\mathbb{E}[Q]}.$$

C.3 Proof of Lemma 5.5

It is similar to the proof of Lemma 5.3. Let $T_k = \min\{i : X_i \leq \lambda^{\sigma^k}\}$, where $\sigma = \frac{1}{1-\epsilon}$. We must show that $\mathbb{E}[T_0] \leq \frac{c\rho}{\ln^2 \lambda}$. For $k \geq \log_\sigma \frac{\ln X_0}{\ln \lambda}$, $T_k = 0$. For $0 \leq k < \log_\sigma \frac{\ln X_0}{\ln \lambda}$,

$$\begin{aligned}
&\Pr[T_k = i+1 \,|\, X_0, \ldots, X_i, \{T_{k+1} \leq i < T_k\}] \\
&\qquad = \Pr[X_{i+1} \leq \lambda^{\sigma^k} \,|\, X_0, \ldots, X_i, \{\lambda^{\sigma^k} < X_i \leq \lambda^{\sigma^{k+1}}\}] \geq \frac{\sigma^{2k}\log^2\lambda}{\rho}
\end{aligned}$$

So, $T_k - T_{k+1}$ is stochastically smaller than a geometric random variable with probability parameter $\frac{\sigma^{2k}\log^2\lambda}{\rho}$. Therefore,

$$\mathbb{E}[T_0] = \mathbb{E}\Big[\sum_k (T_j - T_{j+1})\Big] \leq \sum_k \frac{\rho}{\sigma^{2k}\log^2\lambda} \leq c(\sigma)\frac{\rho}{\log^2\lambda}.$$

C.4 Proof of Lemma 5.6

It is similar to the proof of Lemma 5.4. We define $\mathcal{E}$ such that if $\mathcal{E} \cap \{Y_i = u\}$ occurs then: (1) Y_{i+1} is an in-contact of Y_i; (2) it has out-degree between roughly $\|u\|^{1/3}$ and $\|u\|^{1/2}$; (3) $\|u\|^{2/3} < \|Y_{i+1}\| \leq \|u\| - \|u\|^{2/3}$; and (4) some of the out-contacts of Y_{i+1} is in $R_{\|u\|^{2/3}}$. Formally, let

$$C = R_{\|u\|-\|u\|^{2/3}} \setminus R_{\|u\|^{2/3}}, \qquad C^* = \{v \in C : \nu\|u\|^{1/3} \leq D_v \leq \nu\|u\|^{1/2}\},$$

define $\mathcal{E}_1$, $\mathcal{E}_2$, Z, and $\mathcal{E}_3$ as in the proof of Lemma 5.4 (using the above value for C^*), and let

$$\mathcal{E}_0 = \{u \not\to R_{\|u\|-\|u\|^{2/3}}\}.$$

Again we have $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ and $\mathcal{E} \cap \{Y_i = u\} \subseteq \{\|Y_{i+2}\| \leq \|u\|^{2/3}\}$, and we show that $\Pr[\mathcal{E} \,|\, \mathcal{A}] = \Omega\big(\frac{\ln^2\|u\|}{\ln^2 n}\big)$, where $\mathcal{A} = \{R_{\|u\|} \not\to H\}$. Equation (5.4) still applies and we bound the probabilities on its left-hand side. Similarly to (5.5), (5.8), and (5.9), we have that

$$\Pr[\mathcal{E}_0 \,|\, \mathcal{A}],\ \Pr[\mathcal{E}_2 \,|\, \mathcal{E}_1 \cap \mathcal{A}],\ \Pr[\mathcal{E}_3 \,|\, \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{A}] \in \Theta(1). \tag{C.1}$$

Also, similarly to (5.6), if $v \in C$ and Q_v is the indicator random variable of the event $\{v \to u\} \cap \{\nu\|u\|^{1/3} \leq D_v \leq \nu\|u\|^{1/2}\}$ then

$$\begin{aligned}
\mathbb{E}[Q_v \,|\, \mathcal{A}] &= \sum_{k=\nu\|u\|^{1/3}}^{\nu\|u\|^{1/2}} \Pr[v \to u \,|\, \{D_v = k\} \cap \{v \not\to H\}] \cdot \frac{\Pr[v \not\to H \,|\, D_v = k]}{\Pr[v \not\to H]} \cdot q_k \\
&= \Omega\Big(\sum_{k=\nu\|u\|^{1/3}}^{\nu\|u\|^{1/2}} \frac{k}{\nu\delta(u,v)} \cdot \frac{1}{\beta k^2}\Big) = \Omega\Big(\frac{\ln\|u\|}{\nu\beta\delta(u,v)}\Big),
\end{aligned}$$

where the second-to-last relation holds because: $\mathbb{Pr}[v \to u \,|\, \{D_v = k\} \cap \{v \not\to H\}] \geq \mathbb{Pr}[v \to u \,|\, D_v = k] = \Theta\big(\frac{k}{\nu\delta(u,v)}\big)$, since $k < \nu\delta(u,v)$; and $\mathbb{Pr}[v \to H \,|\, D_v = k] \leq |H| \cdot \frac{2k}{\nu\|v\|^{2/3}} \leq 1/2$, since $k \leq \nu\|u\|^{1/2}$ and $\|u\| > 4^6 i^6 = 4^6|H|^6$. So, $\mathbb{E}[\sum_{v\in C} Q_v \,|\, \mathcal{A}] = \Omega\big(\frac{\ln^2\|u\|}{\nu\beta}\big)$, and, similarly to (5.7), we obtain

$$\mathbb{Pr}[\mathcal{E}_1 \,|\, \mathcal{A}] = \Omega\Big(\frac{\ln^2\|u\|}{\nu\beta}\Big) = \Omega\Big(\frac{\ln^2\|u\|}{\ln^2 n}\Big). \tag{C.2}$$

Applying (C.1) and (C.2) to (5.4) yields $\mathbb{Pr}[\mathcal{E} \,|\, \mathcal{A}] = \Omega\big(\frac{\ln^2\|u\|}{\ln^2 n}\big)$.