

# Tight bounds for rumor spreading in graphs of a given conductance

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## Abstract

We study the connection between the rate at which a rumor spreads throughout a graph and the *conductance* of the graph—a standard measure of a graph’s expansion properties. We show that for any  $n$ -node graph with conductance  $\phi$ , the classical PUSH-PULL algorithm distributes a rumor to all nodes of the graph in  $O(\phi^{-1} \log n)$  rounds with high probability. This bound improves a recent result of Chierichetti, Lattanzi, and Panconesi [6], and it is tight in the sense that there exist graphs where  $\Omega(\phi^{-1} \log n)$  rounds of the PUSH-PULL algorithm are required to distribute a rumor.

We also explore the PUSH and the PULL algorithms, and derive optimal sufficient conditions for the above upper bound to hold for those algorithms as well. An interesting finding is that every graph contains a node such that  $O(\phi^{-1} \log n)$  rounds of the PULL algorithm suffice with high probability to distribute a rumor started at that node. In contrast, there exist graphs where the PUSH algorithm requires significantly more rounds for any start node.

## 1 Introduction

*Gossip-based algorithms*, also known as *epidemic algorithms*, have become a prominent paradigm for designing simple, scalable, and robust protocols for disseminating information in large networks. Perhaps the most basic and well-studied example of a gossip-based information dissemination algorithm is the so-called *randomized rumor spreading*. The algorithm proceeds in a sequence of synchronous rounds. Initially, in round 0, an arbitrary *start* node receives a piece of information, the *rumor*. This rumor is then spread iteratively to other nodes: In each round, every *informed* node (i.e., every node that received the rumor in a previous round) chooses a random neighbor to which it sends the rumor. This is the *PUSH* version of randomized rumor spreading. The *PULL* version is symmetric: In each round, every *uninformed* node chooses a random neighbor, and if that neighbor knows the rumor it sends it to the uninformed node. Finally, the *PUSH-PULL* algorithm is the combination of both strategies: In each round, every node chooses a random neighbor to send the rumor to, if the node knows the rumor, or to request the rumor from, otherwise.

The above rumor spreading algorithms were proposed by Demers et al. [8] for maintaining distributed replicated database systems. Subsequently, these algorithms and variations of them have found applications in various contexts, including failure detection [31], resource discovery [23], and data aggregation [3]. Also, the performance of these algorithms has been studied extensively, both theoretically and experimentally, and they have been shown to be very efficient for several network topologies. In particular, the number of rounds to inform all nodes is exponentially smaller than the network size for topologies ranging from basic networks, such as complete graphs and

hypercubes, to more complex structures, such as preferential attachment graphs modeling social networks (see the Related Work Section for more details).

In this paper, we investigate the intuitive relationship between the performance of randomized rumor spreading in an arbitrary network, and the expansion properties of the network. More precisely, we study the connection between the *broadcast time*, i.e., the number of rounds until all nodes get informed—a primary performance measure for rumor spreading algorithms—and the *conductance* of the network graph—a standard measure of graph expansion. Roughly speaking, the conductance of a connected graph is a value  $\phi \in (0, 1]$  defined as the minimum ratio of the edges leaving a set of vertices over the edges incident to that set (see Section 2 for the precise definition). Intuitively, the conductance  $\phi$  is large for graphs that are well connected (e.g., the complete graph), and small for graphs that are not (e.g., graphs with communication bottlenecks). Note that most of the networks for which randomized rumor spreading is known to be fast have large conductance. Further, some theoretical and empirical studies indicate that social networks have also large conductance [25, 26].

A connection between randomized rumor spreading and conductance has been observed in several works, e.g., in [1, 12, 29], where upper bounds on the broadcast time were obtained for various graph topologies based, essentially, on lower bounds on the conductance. The first explicit result relating rumor spreading to conductance was by Mosk-Aoyama and Shah [27], who showed a bound of  $O(\phi^{-1} \log n)$  rounds with high probability (*w.h.p.*) on the broadcast time of the PUSH protocol in any  $n$ -node *regular* graph.<sup>1</sup> A comparable bound involving the spectral gap was shown by Boyd et al. [3], however, similarly to the result of [27], it does not extend to non-regular graphs.

In [5], Chierichetti et al. posed the question whether randomized rumor spreading is fast in *all* graphs with large conductance. For the PUSH and the PULL algorithms the answer is negative [6], since a star with  $n$  vertices has constant conductance but the expected broadcast time for a random start node is  $\Omega(n)$  rounds. For the PUSH-PULL algorithm, however, the answer is positive! Chierichetti et al. [7] showed that for any graph and any start node, the broadcast time of the PUSH-PULL algorithm is  $O(\phi^{-6} \log^4 n)$  rounds w.h.p. This bound was subsequently improved by the same authors to  $O((\log \phi^{-1})^2 \phi^{-1} \log n)$  rounds w.h.p., in [6]. Further, it was shown that this bound is by at most a  $(\log \phi^{-1})^2$ -factor larger than the optimal bound. More precisely, it was shown that for any  $\phi \geq 1/n^{1-\epsilon}$ , there exist  $n$ -node graphs with conductance at least  $\phi$  and diameter  $\Omega(\phi^{-1} \log n)$ . Finally, the authors of [6] described a simple sufficient condition for their upper bound to hold for the PUSH and the PULL algorithms as well. This condition states that for any edge, the ratio of the degrees of its two endpoints is bounded by a constant.

Besides conductance, two other important measures of a graph’s expansion properties are *edge expansion* and *vertex expansion*. There are examples of graphs with large *edge* expansion where randomized rumor spreading is slow. Specifically, the authors of [5] described a graph with constant edge expansion in which the expected broadcast time of the PUSH-PULL algorithm for a random start node is  $\Omega(\sqrt{n})$  rounds. The question whether large *vertex* expansion always yields fast rumor spreading is largely open. In a very recent work [30], Sauerwald and Stauffer showed that for *regular* graphs this is true: the broadcast time of the PUSH algorithm is  $O(\alpha^{-1} \log^5 n)$  rounds w.h.p. for any regular graph with vertex expansion  $\alpha$ .

## Our Contributions

We mentioned above that an upper bound of  $O((\log \phi^{-1})^2 \phi^{-1} \log n)$  rounds w.h.p. is known to hold for the broadcast time of the PUSH-PULL algorithm in *all* graphs with  $n$  nodes and conductance  $\phi$ ;

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<sup>1</sup>The authors of [27] obtained results for non-regular graphs as well, but in this case neighbors are not chosen *uniformly* at random.

and that  $\Omega(\phi^{-1} \log n)$  rounds are required for *some* of those graphs, for any  $n$  and  $\phi \geq 1/n^{1-\epsilon}$ . Our first contribution is the following result, which closes the gap between these two bounds.

**Theorem 1.1.** *For any graph on  $n$  vertices with conductance  $\phi$ , and for any start vertex, the broadcast time of the PUSH-PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p.<sup>2</sup>*

Further, we observe that Theorem 1.1 is tight for any  $\phi = \Omega(1/n)$ —not just for  $\phi \geq 1/n^{1-\epsilon}$  as it was previously known. Clearly, the theorem is not tight for  $\phi = o(1/n)$ , as a general bound of  $O(n \log n)$  rounds w.h.p. is known for the broadcast time of the PUSH algorithm (and thus, of the PUSH-PULL algorithm) [17].

Our proof of Theorem 1.1 relies on an analysis of the PUSH and the PULL algorithms. We show that in any graph, the broadcast time of the PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p. *if the rumor starts at a node of degree  $\Delta$* , where  $\Delta$  is the maximum degree of the graph. Further, based on the symmetry between the PUSH and the PULL algorithms, we show that  $O(\phi^{-1} \log n)$  rounds of the PUSH algorithm suffice w.h.p. to inform a node of degree  $\Delta$ , for any start node. From these two results, it is immediate that w.h.p. after  $O(\phi^{-1} \log n)$  rounds of the PUSH-PULL algorithm a node of degree  $\Delta$  gets informed, and in  $O(\phi^{-1} \log n)$  additional rounds all the remaining nodes are informed.

Our analysis is different than previous approaches. Specifically, the analysis in [7] is based on an interesting connection between rumor spreading and a spectral sparsification process; and the proof in [6] analyzes the PUSH-PULL algorithm directly. Still, our analysis uses some ideas from [6]. In particular, our analysis of the PULL algorithm relies on showing that essentially the volume of the informed nodes (i.e., the sum of their degrees) increases exponentially with time, which is also what the analysis of the PUSH-PULL algorithm in [6] shows. However, the way the two proofs proceed to show that is different.

As mentioned earlier, the PUSH and the PULL algorithms cannot guarantee short broadcast times based solely on the assumption of large conductance—unlike the PUSH-PULL algorithm. Our second contribution is that we derive conditions guaranteeing a broadcast time of  $O(\phi^{-1} \log n)$  rounds w.h.p. for those algorithms as well. In the outline of the proof of Theorem 1.1 that we gave above, we mentioned that one such condition for the PULL algorithm is that the rumor start from a max-degree node. This condition follows from the next more general statement that we prove.

**Theorem 1.2.**

- (a) *For any graph on  $n$  vertices with conductance  $\phi$ , minimum degree  $\delta$  and maximum degree  $\Delta$ , if the rumor starts at a vertex of degree  $\Omega(\Delta \cdot (\phi + \delta^{-1}))$  then the broadcast time of the PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p.*
- (b) *If, in particular,  $\Delta = O(1/\phi)$  then the above bound on the broadcast time holds for any start vertex.*

Further, we show that the sufficient conditions described in Theorem 1.2 are optimal. That is, for any given  $\phi$ ,  $\delta$ ,  $\Delta$  and node degree  $d$ , if neither of the conditions  $\Delta = O(1/\phi)$  or  $d = \Omega(\Delta(\phi + \delta^{-1}))$  is true, then there exist graphs with those  $\phi$ ,  $\delta$  and  $\Delta$ , and with a node of degree  $d$  such that a rumor started at that node needs  $\omega(\phi^{-1} \log n)$  rounds of the PULL algorithm to spread to all nodes with non-negligible probability, i.e., with probability  $n^{-o(1)}$ .

Unlike the PULL algorithm, which achieves rumor spreading in  $O(\phi^{-1} \log n)$  rounds for at least one start node in every graph, the PUSH algorithm requires significantly more rounds in some graphs for any start node. E.g., the broadcast time of the PUSH algorithm is at least  $n - 1$  for any

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<sup>2</sup>By “with high probability (w.h.p.)” we mean with probability  $1 - O(n^{-c})$ , for an arbitrary constant  $c > 0$ .

start node in a star with  $n$  vertices, although it has constant conductance; whereas the broadcast time of the PULL algorithm is just one round when the rumor starts from the center of the star.

From Theorem 1.2(a) it follows that if  $\delta = \Omega(\Delta(\phi + \delta^{-1}))$  then the broadcast time of the PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p. for *all* start nodes. This statement and Theorem 1.2(b) are also true for the PUSH algorithm, by the same symmetry argument for PUSH and PULL that we use in the proof of Theorem 1.1.

Finally, we also tighten the result of [6] for the PUSH and the PULL algorithms. We show that if, for any edge, the ratio of the degrees of its endpoints is bounded by a constant, then the broadcast time of those algorithms is  $O(\phi^{-1} \log n)$  rounds w.h.p., for any start node.

## Related Work

The first works on randomized rumor spreading were by Frieze and Grimmett [21] and Pittel [28], and they provided a precise analysis of the broadcast time of the PUSH algorithm on the complete graph. A bound of  $O(\log n)$  rounds on the broadcast time was later established for the hypercube and for sufficiently dense random graphs by Feige et al. [17]. Other symmetric graphs similar to the hypercube in which rumor spreading takes  $O(\log n + D)$  rounds, where  $D$  is the graph diameter, were studied by Elsässer and Sauerwald [14, 15]. A refined analysis for random graphs proving that the broadcast time is essentially the same as that for the complete graph was provided by Fountoulakis et al. [18], and subsequently extended to random regular graphs by Fountoulakis and Panagiotou [19].

The PUSH-PULL algorithm was also first analyzed for the complete graph, by Karp et al. [24]. The motivation was to reduce the total number of transmissions of the rumor, as it was shown that the PUSH-PULL algorithm incurs only  $\Theta(n \log \log n)$  transmissions, whereas  $\Theta(n \log n)$  transmissions are needed for the PUSH algorithm. The broadcast time and the number of transmissions of the PUSH-PULL algorithm and variations of it were subsequently analyzed for random graphs by Elsässer [13] and Elsässer and Sauerwald [16], and for random regular graphs by Berenbrink et al. [1]. The results obtained were similar to those for the complete graph. Fraigniaud and Gakkoupis [20] proposed a variant of the PUSH-PULL algorithm with a feedback mechanism, for minimizing the total communication complexity of rumor spreading in the complete graph (i.e., the total number of bits exchanged throughout a run of the protocol).

Chierichetti et al. [5] studied randomized rumor spreading on preferential attachment graphs, and showed that the broadcast time of both the PUSH and the PULL algorithms is polynomial in  $n$ , whereas for the PUSH-PULL algorithm the broadcast time is just  $O(\log^2 n)$  rounds. The bound for the PUSH-PULL algorithm was recently improved to  $\Theta(\log n)$  by Doerr et al. [10]. These results highlight the advantage of the PUSH-PULL algorithm for rumor spreading in graphs with highly skewed degree distribution.

A quasi-random variant of randomized rumor spreading was proposed by Doerr et al. [11], as a means to reduce the amount of randomness used. In the quasi-random model, each node has a (cyclic) list of its neighbors in which it just chooses a random starting position, and in each round it contacts the next node in that list—instead of choosing a new random neighbor to contact in each round. Quasi-random rumor spreading was shown to be at least as efficient as randomized rumor spreading for several graph topologies (see [11, 12] for details). In fact, for some topologies, e.g., sparse random graphs, it is even superior. For additional results on reducing further the amount of randomness see [9, 22].

Mosk-Aoyama and Shah [27] considered the problem of computing separable functions of values located at the nodes of a network. As part of this problem they studied a generalization of the PUSH-PULL algorithm with non-uniform probabilities: in every round, each node  $v$  chooses its

neighbor  $u$  with probability  $p_{vu}$ , and with the remaining probability  $p_{vv} = 1 - \sum_u p_{vu}$  it does not contact any neighbor. They showed that if the matrix  $P = \{p_{vu}\}$  is doubly stochastic, the broadcast time is  $O(\phi_P^{-1} \log n)$  rounds w.h.p., where  $\phi_P$  is the *conductance of matrix  $P$*  (not of the graph). This result implies an  $O(\phi^{-1} \log n)$  bound for the standard PUSH-PULL algorithm for *regular graphs*. However, their result is incomparable to ours for non-regular graphs. In particular, as observed in [6], there exist graphs for which  $\phi_P \ll \phi$  for any possible way of choosing the doubly stochastic matrix  $P$ .

Recently, Censor-Hillel and Shachnai [4] introduced a refinement of conductance, called *weak conductance*, and they related this quantity to the time needed to inform a certain fraction of nodes using the PUSH-PULL algorithm. In particular, they showed that large weak conductance guarantees fast “partial” rumor spreading, even if the (standard) conductance is small.

**Paper organization.** We start with some definitions and notations in Section 2. Section 3 contains the analysis of the PULL algorithm, including the proof of Theorem 1.2; this section constitutes the largest part of the paper. In Section 4, we provide a lemma on the symmetry between the PUSH and PULL algorithms, and we use it to derive results for the PUSH algorithm from the results in Section 3 for the PULL algorithm. Finally, in Section 5, we consider the PUSH-PULL algorithm, and we prove Theorem 1.1 using results from the previous two sections.

## 2 Preliminaries

We consider an arbitrary network represented as a connected undirected graph  $G = (V, E)$ . The degree of a vertex  $v \in V$  is denoted  $d(v)$ . By  $\Delta$  we denote the maximum degree of  $G$ , i.e.,  $\Delta = \max_{v \in V} d(v)$ , and by  $\delta$  we denote the minimum degree. The *volume* of a subset of vertices  $S \subseteq V$  is the sum of the degrees of the vertices in  $S$ , i.e.,  $\text{vol}(S) = \sum_{v \in S} d(v)$ . Note that  $\text{vol}(V) = 2|E|$ . By  $\text{cut}(S, V - S)$  we denote the set of edges crossing the partition  $\{S, V - S\}$  of  $V$ , i.e.,  $\text{cut}(S, V - S) = \{\{v, u\} \in E : v \in S, u \in V - S\}$ . The *conductance*  $\phi$  of  $G$  is defined as

$$\phi = \min_{\substack{S \subseteq V, \\ 0 < \text{vol}(S) \leq \text{vol}(V)/2}} \frac{|\text{cut}(S, V - S)|}{\text{vol}(S)}.$$

It is easy to see that  $0 < \phi \leq 1$ . (It is  $\phi \neq 0$  because  $G$  is a connected graph.) The following statement is also immediate from the definition of conductance.

**Observation 2.1.** *For any  $S \subseteq V$ ,  $|\text{cut}(S, V - S)| \geq \lceil \phi \cdot \min\{\text{vol}(S), \text{vol}(V - S)\} \rceil$ .*

We will denote by  $S_i$  the set of informed vertices at the *end* of round  $i$  of the rumor spreading algorithm, and by  $U_i$  the set of uninformed vertices at that time, i.e.,  $U_i = V - S_i$ .  $S_0$  and  $U_0$  denote the corresponding sets initially. To simplify notation, we will assume for the analysis that  $S_0$  can be any non-empty subset of vertices—we do not require that  $|S_0| = 1$ .

## 3 PULL Algorithm

In Section 3.1 we establish an upper bound on the broadcast time of the PULL algorithm that holds for any initial set of informed vertices. The proof of this bound is at the heart of our analysis. Then, in Section 3.2, we build upon and refine this result to derive sufficient conditions for the PULL algorithm to achieve broadcast times of  $O(\phi^{-1} \log n)$  rounds. In more detail, in Section 3.2.1 we give the proof of Theorem 1.2; in Section 3.2.2 we show that the sufficient conditions that this

theorem specifies are optimal; and in Section 3.2.3 we show that a condition proposed in [6] also guarantees rumor spreading in  $O(\phi^{-1} \log n)$  rounds.

### 3.1 An Upper Bound on the Broadcast Time

In this section we prove the following high-probability bound on the broadcast time of the PULL algorithm for an arbitrary initial set  $S_0$  of informed vertices. Recall that  $\Delta$  is the maximum degree of  $G$ .

**Lemma 3.1.** *For any initial set of informed vertices  $S_0 \subseteq V$  and any fixed constant  $\beta > 0$ , all vertices get informed in at most  $48(\beta + 2) \log n (\phi^{-1} + \Delta / \lceil \phi \text{vol}(S_0) \rceil)$  rounds of the PULL algorithm with probability  $1 - O(n^{-\beta})$ .*

Note that if  $\Delta / \lceil \phi \text{vol}(S_0) \rceil = O(1/\phi)$  then the above lemma implies that the broadcast time is  $O(\phi^{-1} \log n)$  rounds w.h.p.

Before we present the proof we give some simple intuition as to why  $O(\phi^{-1} \log n)$  rounds is the correct bound (under certain conditions). Fix some round  $i$  and the set  $S_{i-1}$  of informed vertices before round  $i$ . Let us consider how the expected *volume* of the informed vertices changes in this round. Suppose that the current volume  $\text{vol}(S_{i-1})$  is no greater than half of the total volume  $\text{vol}(V) = 2|E|$  of the graph. For each uninformed vertex  $u \in U_{i-1}$  denote by  $\gamma(u)$  the number of its informed neighbors. The contribution of  $u$  to the expected increase of the volume  $\text{vol}(S_i) - \text{vol}(S_{i-1})$  is precisely  $d(u) \cdot (\gamma(u)/d(u)) = \gamma(u)$ , since  $u$  chooses some of its informed neighbors in round  $i$  with probability  $\gamma(u)/d(u)$ . Therefore,

$$\mathbf{E}[\text{vol}(S_i)] - \text{vol}(S_{i-1}) = \sum_{u \in U_{i-1}} d(u) \cdot \frac{\gamma(u)}{d(u)} = \sum_{u \in U_{i-1}} \gamma(u) = |\text{cut}(S_{i-1}, U_{i-1})| \geq \phi \cdot \text{vol}(S_{i-1}),$$

by the definition of conductance. Thus,  $\mathbf{E}[\text{vol}(S_i)] \geq (1 + \phi) \text{vol}(S_{i-1})$ . This means that *if the process behaved as in expectation*, then  $O(\phi^{-1} \log |E|) = O(\phi^{-1} \log n)$  rounds would suffice to increase the volume of informed vertices to  $|E|$ . A similar reasoning about the *decrease* in the volume of *uninformed* vertices, yields that  $O(\phi^{-1} \log n)$  additional rounds would suffice to reduce the volume of uninformed vertices from  $|E|$  to 0 (and thus, to inform all vertices). Of course, the rumor spreading process does *not* behave exactly as in expectation. However, the above argument provides the right intuition (except that it does not justify the condition  $\Delta / \lceil \phi \text{vol}(S_0) \rceil = O(1/\phi)$ ). We turn this intuition into a rigorous proof by considering the variance of  $\text{vol}(S_i)$  as well. Also, in the proof we have to argue about the accumulated result of a sequence of rounds, instead of about the outcome of just a single round.

We now proceed to the proof of Lemma 3.1. We divide the run of the algorithm into three phases: The first phase lasts until the total volume of the informed vertices  $\text{vol}(S_t)$  becomes at least  $\Delta$ ; the second phase lasts until this volume exceeds one half of the total volume of the graph; and the third phase lasts until all vertices get informed. We measure progress in the first two phases by the increase in the volume of informed vertices; and in the third phase by the decrease in the volume of uninformed vertices. For each phase, the next lemma gives upper bounds on the number of rounds until “significant” progress is made with constant probability. Roughly, it states that with probability  $1/2$  the first phase completes in  $O(\Delta / \lceil \phi \text{vol}(S_0) \rceil)$  rounds, and with the same probability  $O(1/\phi)$  rounds suffice to double the volume of informed vertices in the second phase or to halve the volume of uninformed vertices in the third phase.

**Lemma 3.2.**

- (a) If  $\text{vol}(S_0) < \Delta$  then  $\Pr(\text{vol}(S_i) \geq \Delta) \geq 1/2$ , for  $i \geq 4\Delta/\lceil\phi \text{vol}(S_0)\rceil$ .
- (b) If  $\Delta \leq \text{vol}(S_0) \leq |E|$  then  $\Pr(\text{vol}(S_i) \geq \min\{2 \text{vol}(S_0), |E| + 1\}) \geq 1/2$ , for  $i \geq 4/\phi$ .
- (c) If  $\text{vol}(S_0) > |E|$  then  $\Pr(\text{vol}(U_i) \leq \text{vol}(U_0)/2) \geq 1/2$ , for  $i \geq 6/\phi$ .

The proof of Lemma 3.2 proceeds as follows. Consider part (a)—for parts (b) and (c) the reasoning is similar. Consider a round  $i \geq 1$ . At the beginning of the round there are at least  $\phi \text{vol}(S_{i-1}) \geq \phi \text{vol}(S_0)$  crossing edges between informed and uninformed vertices. We fix  $\lceil\phi \text{vol}(S_0)\rceil$  of these edges arbitrarily before round  $i$  is executed, and then we count the total volume  $L_i$  of the vertices that get informed in round  $i$  by receiving the rumor *through some of those edges*. Clearly,  $L_i$  is a lower bound on the total volume of the vertices informed in round  $i$ , and thus,  $\text{vol}(S_0) + \sum_{k \leq i} L_k$  is a lower bound on  $\text{vol}(S_i)$ . Therefore, to prove (a) it suffices to show that  $\sum_{k \leq i} L_k \geq \Delta - \text{vol}(S_0)$  with probability at least  $1/2$ . Using a martingale formulation we compute the expectation and the variance of  $\sum_{k \leq i} L_k$ , and then we bound  $\sum_{k \leq i} L_k$  by applying Chebyshev's inequality.

The proof of Lemma 3.2 is the core of our analysis, and the same approach is also used for some of the proofs presented in Section 3.2.

Next we provide the detailed proof of Lemma 3.2 (in Section 3.1.1), and then we use this result to easily derive Lemma 3.1 (in Section 3.1.2).

### 3.1.1 Proof of Lemma 3.2

(a) Consider the collection of random variables  $L_1, L_2, \dots$ , where each  $L_i$ , for  $i \geq 1$ , is defined as follows. We distinguish two cases depending on whether or not  $\text{vol}(S_{i-1}) \leq |E|$ .

- $\text{vol}(S_{i-1}) \leq |E|$ : By Observation 2.1,  $|\text{cut}(S_{i-1}, U_{i-1})| \geq \lceil\phi \text{vol}(S_{i-1})\rceil \geq \lceil\phi \text{vol}(S_0)\rceil$ . Let  $E_i$  be an arbitrary subset of  $\text{cut}(S_{i-1}, U_{i-1})$  consisting of  $M = \lceil\phi \text{vol}(S_0)\rceil$  edges;  $E_i$  is fixed *before* round  $i$  is executed. We define  $L_i$  to be the total volume of the vertices that get informed in round  $i$  by receiving the rumor through edges in  $E_i$ . Formally, for each vertex  $u \in U_{i-1}$ , let  $L_{i,u}$  be the 0/1 random variable with  $L_{i,u} = 1$  if and only if in round  $i$  vertex  $u$  receives the rumor through at least one edge in  $E_i$ . Then,  $L_i = \sum_{u \in U_{i-1}} L_{i,u} \cdot d(u)$ .
- $\text{vol}(S_{i-1}) > |E|$ : We let  $L_i = M$ .

We will prove the following results for the expectation and the variance of the sum of  $L_i$ .

**Claim 3.3.**  $\mathbf{E}[\sum_{k \leq i} L_k] = iM$  and  $\mathbf{Var}[\sum_{k \leq i} L_k] \leq \Delta \cdot \mathbf{E}[\sum_{k \leq i} L_k]$ .

Using this claim, the lemma follows easily by Chebyshev's inequality: Suppose that  $i \geq 4\Delta/M$ . Let  $\mu = \mathbf{E}[\sum_{k \leq i} L_k] = iM$ , and note that  $\mu \geq 4\Delta$ . Then,

$$\begin{aligned} \Pr\left(\sum_{k \leq i} L_k < \Delta\right) &\leq \Pr\left(\left|\sum_{k \leq i} L_k - \mu\right| > \mu - \Delta\right) \leq \frac{\mathbf{Var}[\sum_{k \leq i} L_k]}{(\mu - \Delta)^2} \leq \frac{\mu\Delta}{(\mu - \Delta)^2} \\ &= \frac{\mu/\Delta}{(\mu/\Delta - 1)^2} = \frac{1}{\mu/\Delta - 1} + \frac{1}{(\mu/\Delta - 1)^2} \leq 4/9, \end{aligned} \quad (3.1)$$

since  $\mu \geq 4\Delta$ . By the definition of the  $L_k$ , if  $\text{vol}(S_i) < \Delta$  then  $\sum_{k \leq i} L_k$  cannot be larger than the total volume of the vertices informed since round 1 and thus  $\sum_{k \leq i} L_k \leq \text{vol}(S_i) - \text{vol}(S_0) < \Delta$ . Hence,  $\Pr(\text{vol}(S_i) < \Delta) \leq \Pr(\sum_{k \leq i} L_k < \Delta)$ , and by (3.1),  $\Pr(\text{vol}(S_i) < \Delta) \leq 4/9 < 1/2$ .

To complete the proof of Lemma 3.2(a) it remains to show Claim 3.3, which we do next.

*Proof of Claim 3.3.* For each  $i \geq 1$ , let  $Y_i = L_i - M$ . For each  $i \geq 0$ , let  $X_i = \sum_{k \leq i} Y_k$ , and let  $\mathcal{F}_i$  be the  $\sigma$ -algebra generated by all the choices of the algorithm in the first  $i$  rounds. We show that the sequence of random variables  $X_0, X_1, \dots$  is a martingale with respect to the filter  $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots$ , or equivalently that  $Y_1, Y_2, \dots$  is a martingale difference sequence, i.e.,

$$\mathbf{E}[Y_i \mid \mathcal{F}_{i-1}] = 0.$$

We distinguish two cases:

- If  $\text{vol}(S_{i-1}) \leq |E|$ , then by the definition of  $L_i$ ,

$$\begin{aligned} \mathbf{E}[Y_i \mid \mathcal{F}_{i-1}] &= \mathbf{E}[L_i \mid \mathcal{F}_{i-1}] - M = \mathbf{E}\left[\sum_{u \in U_{i-1}} L_{i,u} d(u) \mid \mathcal{F}_{i-1}\right] - M \\ &= \sum_{u \in U_{i-1}} \mathbf{E}[L_{i,u} \mid \mathcal{F}_{i-1}] \cdot d(u) - M. \end{aligned}$$

For any  $u \in U_{i-1}$ ,

$$\mathbf{E}[L_{i,u} \mid \mathcal{F}_{i-1}] = \Pr(L_{i,u} = 1 \mid \mathcal{F}_{i-1}) = \gamma_i(u)/d(u), \quad (3.2)$$

where  $\gamma_i(u)$  is the number of edges in  $E_i$  that are incident to  $u$ . Note that

$$\sum_{u \in U_{i-1}} \gamma_i(u) = |E_i| = M, \quad (3.3)$$

since each edge in  $E_i$  is incident to exactly one  $u \in U_{i-1}$ . Combining all the above yields  $\mathbf{E}[Y_i \mid \mathcal{F}_{i-1}] = \sum_{u \in U_{i-1}} \gamma_i(u) - M = M - M = 0$ .

- If  $\text{vol}(S_{i-1}) > |E|$  then  $\mathbf{E}[Y_i \mid \mathcal{F}_{i-1}] = 0$  as well, because  $Y_i = L_i - M = M - M = 0$ .

Therefore, the sequence  $X_0, X_1, \dots$  is a martingale. From this it follows that

$$\mathbf{E}[X_i] = \mathbf{E}[X_0] = 0,$$

and substituting the definition of  $X_i$  yields the desired formula for the expectation of  $\sum_{k \leq i} L_k$ , that  $\mathbf{E}[\sum_{k \leq i} L_k] = iM$ .

Next we prove the bound on the variance. We have

$$\mathbf{Var}\left[\sum_{k \leq i} L_k\right] = \mathbf{Var}\left[\sum_{k \leq i} Y_k\right] = \mathbf{Var}[X_i] = \mathbf{E}[X_i^2] - (\mathbf{E}[X_i])^2 = \mathbf{E}[X_i^2]. \quad (3.4)$$

Thus, it suffices to bound  $\mathbf{E}[X_i^2]$ . It is

$$\begin{aligned} \mathbf{E}[X_i^2 \mid \mathcal{F}_{i-1}] &= \mathbf{E}[(Y_i + X_{i-1})^2 \mid \mathcal{F}_{i-1}] \\ &= \mathbf{E}[Y_i^2 \mid \mathcal{F}_{i-1}] + X_{i-1}^2 + 2\mathbf{E}[Y_i \mid \mathcal{F}_{i-1}] \cdot X_{i-1} \\ &= \mathbf{E}[Y_i^2 \mid \mathcal{F}_{i-1}] + X_{i-1}^2, \end{aligned} \quad (3.5)$$

since  $\mathbf{E}[Y_i \mid \mathcal{F}_{i-1}] = 0$ . We bound  $\mathbf{E}[Y_i^2 \mid \mathcal{F}_{i-1}]$  as follows. We have two cases:



- If  $\text{vol}(S_{i-1}) \leq |E|$  then, by the definition of  $L_i$  and Equation (3.3),

$$\begin{aligned} \mathbf{E}[Y_i^2 \mid \mathcal{F}_{i-1}] &= \mathbf{E}[(L_i - M)^2 \mid \mathcal{F}_{i-1}] = \mathbf{E}\left[\left(\sum_{u \in U_{i-1}} L_{i,u}d(u) - \sum_{u \in U_{i-1}} \gamma_i(u)\right)^2 \mid \mathcal{F}_{i-1}\right] \\ &= \mathbf{E}\left[\left(\sum_{u \in U_{i-1}} Z_{i,u}\right)^2 \mid \mathcal{F}_{i-1}\right], \end{aligned}$$

where  $Z_{i,u} = L_{i,u}d(u) - \gamma_i(u)$ . Since the random variables  $L_{i,u}$ ,  $u \in U_{i-1}$ , are mutually independent conditionally on  $\mathcal{F}_{i-1}$ , the same is true for the random variables  $Z_{i,u}$ ,  $u \in U_{i-1}$ . Also, by (3.2), we have  $\mathbf{E}[Z_{i,u} \mid \mathcal{F}_{i-1}] = 0$ . From these two observations it follows that

$$\mathbf{E}\left[\left(\sum_{u \in U_{i-1}} Z_{i,u}\right)^2 \mid \mathcal{F}_{i-1}\right] = \sum_{u \in U_{i-1}} \mathbf{E}[Z_{i,u}^2 \mid \mathcal{F}_{i-1}].$$

For any  $u \in U_{i-1}$ ,

$$\begin{aligned} \mathbf{E}[Z_{i,u}^2 \mid \mathcal{F}_{i-1}] &= \mathbf{E}[(L_{i,u}d(u) - \gamma_i(u))^2 \mid \mathcal{F}_{i-1}] \\ &= \mathbf{E}[(L_{i,u}d(u))^2 \mid \mathcal{F}_{i-1}] + (\gamma_i(u))^2 - 2\mathbf{E}[L_{i,u}d(u) \mid \mathcal{F}_{i-1}] \cdot \gamma_i(u) \\ &\stackrel{(3.2)}{=} \mathbf{E}[(L_{i,u}d(u))^2 \mid \mathcal{F}_{i-1}] - (\gamma_i(u))^2 \leq \mathbf{E}[(L_{i,u}d(u))^2 \mid \mathcal{F}_{i-1}] \\ &= \mathbf{E}[L_{i,u} \mid \mathcal{F}_{i-1}] \cdot (d(u))^2 \stackrel{(3.2)}{=} \gamma_i(u)d(u). \end{aligned}$$

Combining all the above we obtain

$$\mathbf{E}[Y_i^2 \mid \mathcal{F}_{i-1}] \leq \sum_{u \in U_{i-1}} \gamma_i(u)d(u) \leq \Delta \cdot \sum_{u \in U_{i-1}} \gamma_i(u) \stackrel{(3.3)}{=} \Delta M. \quad (3.6)$$

- If  $\text{vol}(S_{i-1}) > |E|$  then  $\mathbf{E}[Y_i^2 \mid \mathcal{F}_{i-1}] = 0$ , since  $Y_i = 0$ .

Therefore, in both cases,  $\mathbf{E}[Y_i^2 \mid \mathcal{F}_{i-1}] \leq \Delta M$ . Applying this result to (3.5) gives  $\mathbf{E}[X_i^2 \mid \mathcal{F}_{i-1}] \leq M\Delta + X_{i-1}^2$ , and recursively we obtain  $\mathbf{E}[X_i^2] \leq iM\Delta$ . Combining this and (3.4) yields the desired bound for  $\mathbf{Var}[\sum_{k \leq i} L_k]$ . This completes the proof of Claim 3.3, and of Lemma 3.2(a).  $\square$

(b) We consider the same sequence  $L_1, L_2, \dots$  of random variables as in the proof of part (a). Similarly to (3.1), using Chebyshev's inequality and Claim 3.3 we obtain that for  $i \geq 4/\phi$  and  $\mu = \mathbf{E}[\sum_{k \leq i} L_k] = iM \geq (4/\phi) \cdot \lceil \phi \text{vol}(S_0) \rceil \geq 4 \text{vol}(S_0)$ ,

$$\Pr\left(\sum_{k \leq i} L_k < \text{vol}(S_0)\right) \leq \frac{\mu\Delta}{(\mu - \text{vol}(S_0))^2} \leq \frac{\mu \text{vol}(S_0)}{(\mu - \text{vol}(S_0))^2} \leq 4/9.$$

Observe that if  $\text{vol}(S_i) < \min\{2 \text{vol}(S_0), |E| + 1\}$  then  $\sum_{k \leq i} L_k \leq \text{vol}(S_i) - \text{vol}(S_0) < \text{vol}(S_0)$ . Thus,  $\Pr(\text{vol}(S_i) < \min\{2 \text{vol}(S_0), |E| + 1\}) \leq \Pr(\sum_{k \leq i} L_k < \text{vol}(S_0)) \leq 4/9$ , for  $i \geq 4/\phi$ , which implies Lemma 3.2(b).

(c) Unlike in parts (a) and (b), the set of uninformed vertices has now a smaller volume than the set of informed vertices. So, by Observation 2.1,  $|\text{cut}(S_i, U_i)| \geq \lceil \phi \text{vol}(U_i) \rceil$ . Consider the sequence  $L_1, L_2, \dots$  of random variables, where each  $L_i$  is defined as follows:

- If  $\text{vol}(U_{i-1}) > \text{vol}(U_0)/2$ , we fix an arbitrary subset  $E_i$  of  $\text{cut}(S_{i-1}, U_{i-1})$  consisting of  $M = \lceil \phi \text{vol}(U_0)/2 \rceil$  edges;  $E_i$  is fixed before round  $i$ . As before,  $L_i$  is the total volume of the vertices in  $U_i$  that receive the rumor in round  $i$  through edges in  $E_i$ .
- If  $\text{vol}(U_{i-1}) \leq \text{vol}(U_0)/2$ , then  $L_i = M$ .

Similarly to Claim 3.3, we can show that  $\mathbf{E}[\sum_{k \leq i} L_k] = iM$  and  $\mathbf{Var}[\sum_{k \leq i} L_k] \leq \text{vol}(U_0) \cdot \mathbf{E}[\sum_{k \leq i} L_k]$ . To obtain the variance result we use the fact that the degree of every vertex in  $U_i$  is at most  $\text{vol}(V_i) \leq \text{vol}(U_0)$ ; thus we can replace  $\Delta$  by  $\text{vol}(U_0)$  in (3.6). Next, as in (3.1), by using Chebyshev's inequality we obtain that for  $i \geq 6/\phi$  and  $\mu = \mathbf{E}[\sum_{k \leq i} L_k] = iM \geq 3 \text{vol}(S_0)$ ,

$$\Pr\left(\sum_{k \leq i} L_k < \text{vol}(U_0)/2\right) \leq \frac{\mu \text{vol}(U_0)}{(\mu - \text{vol}(U_0)/2)^2} \leq 12/25.$$

Finally, observing that if  $\text{vol}(U_i) > \text{vol}(U_0)/2$  then  $\sum_{k \leq i} L_k \leq \text{vol}(U_0) - \text{vol}(U_i) < \text{vol}(U_0)/2$  yields  $\Pr(\text{vol}(U_i) > \text{vol}(U_0)/2) \leq \Pr(\sum_{k \leq i} L_k < \text{vol}(U_0)/2) \leq 12/25 < 1/2$ , for  $i \geq 6/\phi$ . This completes the proof of the last case of Lemma 3.2.

### 3.1.2 Proof of Lemma 3.1

By Lemma 3.2(a), if  $\text{vol}(S_i) < \Delta$  then with probability at least  $1/2$ , a number of  $\lceil 4\Delta/\lceil \phi \text{vol}(S_i) \rceil \rceil \leq 5\Delta/\lceil \phi \text{vol}(S_i) \rceil$  rounds suffices to increase the total volume of informed vertices from  $\text{vol}(S_i)$  to at least  $\Delta$ . Thus, if  $\text{vol}(S_0) < \Delta$ , the probability that  $\text{vol}(S_i) \geq \Delta$  for  $t = 2\beta \ln n \cdot (5\Delta/\lceil \phi \text{vol}(S_0) \rceil)$  is at least  $1 - (1 - 1/2)^{2\beta \ln n} \geq 1 - e^{-(1/2)2\beta \ln n} = 1 - n^{-\beta}$ .

By Lemma 3.2(b), if  $\Delta \leq \text{vol}(S_i) \leq |E|$  then with probability at least  $1/2$ ,  $\lceil 4/\phi \rceil$  rounds increase the total volume of informed vertices to at least  $\min\{2 \text{vol}(S_i), |E| + 1\}$ . Divide now the run of the algorithm into *phases* of  $\lceil 4/\phi \rceil$  rounds each, starting from the end of the first round  $i$  with  $\text{vol}(S_i) \geq \Delta$ . A phase is *successful* if the total volume of the informed vertices at the end of the phase is at least  $\min\{2 \text{vol}(S), |E| + 1\}$ , where  $S$  is the set of informed vertices at the beginning of the phase. (Note that if  $\text{vol}(S) \geq |E| + 1$  then the phase is always successful.) Then, the probability that the  $k$ -th phase is successful is at least  $1/2$ , regardless of the outcome of the previous  $k - 1$  phases. From this, it follows (by a standard coupling argument) that the number of successful phases among the first  $k$  phases is (stochastically) greater or equal to the binomial random variable  $B(k, 1/2)$ . So, by Chernoff bounds, the probability that fewer than  $m = \log |E|$  of the first  $k = 2(\beta + 2)m$  phases are successful is at most equal to

$$\begin{aligned} \Pr(B(k, 1/2) < m) &= \Pr(k/2 - B(k, 1/2) > k/2 - m) \leq e^{-2(k/2 - m)^2/k} \\ &= e^{-\frac{(\beta+1)^2}{\beta+2} \cdot m} \leq e^{-\beta m} = O(n^{-\beta}), \end{aligned}$$

as  $m = \log |E|$  and  $|E| \geq n - 1$ . And since at most  $m$  successful phases are needed for the total volume of informed vertices to exceed  $|E|$ , it follows that with probability  $1 - O(n^{-\beta})$  at most  $k = 2(\beta + 2)m$  phases are needed, that is, at most  $k \cdot \lceil 4/\phi \rceil = 2(\beta + 2) \log(|E|) \lceil 4/\phi \rceil \leq 2(\beta + 2)(2 \log n)(5/\phi)$  rounds.

Finally, by Lemma 3.2(c), if  $\text{vol}(S_i) > |E|$  then with probability at least  $1/2$ ,  $\lceil 6/\phi \rceil$  rounds suffice to halve the total volume of uninformed vertices. A similar reasoning as above gives that once the volume of informed vertices has exceeded  $|E|$ , then  $2(\beta + 2)(2 \log n)(7/\phi)$  rounds suffice to inform all nodes with probability  $1 - O(n^{-\beta})$ .

Combing all the above and applying the union bound, we obtain that with probability  $1 - O(n^{-\beta})$  all vertices get informed within  $48(\beta + 2) \log n(\phi^{-1} + \Delta/\lceil \phi \text{vol}(S_0) \rceil)$  rounds.

## 3.2 Sufficient Conditions for Rumor Spreading in $O(\phi^{-1} \log n)$ Rounds

In this section we derive conditions that guarantee broadcast times of  $O(\phi^{-1} \log n)$  rounds for the PULL algorithm. In Section 3.2.1 we give the proof of Theorem 1.2; in Section 3.2.2 we show that the conditions that this theorem specifies are optimal; and in Section 3.2.3 we prove that a simple uniformity condition on the degrees also guarantees rumor spreading in  $O(\phi^{-1} \log n)$  rounds.

### 3.2.1 Derivation of Theorem 1.2

Theorem 1.2(b) follows immediately from Lemma 3.1: By Lemma 3.1, if  $\Delta/\lceil\phi \text{vol}(S_0)\rceil = O(1/\phi)$  then the broadcast time is  $O(\phi^{-1} \log n)$  rounds w.h.p. And since  $\Delta/\lceil\phi \text{vol}(S_0)\rceil \leq \Delta$ , it follows that the condition  $\Delta = O(1/\phi)$  also guarantees broadcast times of  $O(\phi^{-1} \log n)$  rounds w.h.p.

The following interesting result is also immediate from Lemma 3.1.

**Corollary 3.4.** *Every graph contains a vertex such that if a rumor starts at this vertex then the broadcast time of the PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p. In particular, a vertex of degree equal to the maximum degree  $\Delta$  is such a vertex.*

This statement is true because if the rumor starts at a vertex of degree  $\Delta$ , then  $\text{vol}(S_0) = \Delta$  and thus  $\Delta/\lceil\phi \text{vol}(S_0)\rceil \leq \Delta/\phi \text{vol}(S_0) = 1/\phi$ . Then, by Lemma 3.1, all vertices get informed in  $O(\phi^{-1} \log n)$  rounds w.h.p.

Note that Corollary 3.4 is a weaker version of Theorem 1.2(a), since  $\phi + \delta^{-1} = O(1)$  and thus  $\Delta = \Omega(\Delta(\phi + \delta^{-1}))$ . However, it will suffice for the purposes of proving Theorem 1.1 (in Section 5).

Next we describe the proof of Theorem 1.2(a). Recall that Lemma 3.1 holds for any initial set  $S_0$  of informed vertices. If  $S_0$  is an arbitrary subset of vertices, then the size of  $\text{cut}(S_0, U_0)$  can be as small as  $\lceil\phi \text{vol}(S_0)\rceil$ . However, if  $S_0$  consists of a single vertex then  $|\text{cut}(S_0, U_0)| = \text{vol}(S_0)$ , which can be much larger than  $\lceil\phi \text{vol}(S_0)\rceil$ . This observation is a key ingredient in our proof.

We begin by observing that if  $S_0$  consists of a single vertex, then the size of  $\text{cut}(S_i, U_i)$  is at least  $\text{vol}(S_0)/2$  for all  $i$  until  $\text{vol}(S_i)$  increases to at least  $\delta \text{vol}(S_0)/2$ , where  $\delta$  is the minimum graph degree. More precisely, suppose that  $S_0 = \{v\}$ ; so,  $\text{vol}(S_0) = |\text{cut}(S_0, U_0)| = d(v)$ . Since all the  $\text{vol}(S_0)$  edges of the start vertex  $v$  are initially incident to uniformed vertices, and since each new vertex that gets informed is incident to at most one of those edges, it follows that  $|\text{cut}(S_i, U_i)| \geq \text{vol}(S_0) - |S_i| + 1$ . Also,  $\text{vol}(S_i) \geq |S_i| \cdot \delta$  and thus  $|S_i| \leq \text{vol}(S_i)/\delta$ . Therefore,  $|\text{cut}(S_i, U_i)| \geq \text{vol}(S_0) - \text{vol}(S_i)/\delta$ , and the next statement follows.

**Observation 3.5.** *If  $|S_0| = 1$  and  $\text{vol}(S_i) \leq \delta \text{vol}(S_0)/2$  then  $|\text{cut}(S_i, U_i)| \geq \text{vol}(S_0)/2$ .*

We use this result in the proof of the following lemma, which is similar to Lemma 3.2(a).

**Lemma 3.6.** *Suppose that  $|S_0| = 1$ , and let  $D = \min\{\delta \text{vol}(S_0)/2, \Delta\}$ . Fix a round  $j \geq 0$  and the set  $S_j$ . If  $\text{vol}(S_j) < D$  then  $\Pr(\text{vol}(S_{j+i}) \geq D) \geq 1/2$ , for  $i \geq 8\Delta/\text{vol}(S_0)$ .*

*Proof.* As in the proof of Lemma 3.2(a), we consider a sequence  $L_1, L_2, \dots$  of random variables, where each  $L_i$  is defined as follows:

- If  $\text{vol}(S_{j+i-1}) \leq D$ , then at the beginning of round  $j+i$  we fix an arbitrary subset  $E_i$  of  $\text{cut}(S_{j+i-1}, U_{j+i-1})$  consisting of  $M = \lceil\text{vol}(S_0)/2\rceil$  vertices. (By Observation 3.5, we have  $|\text{cut}(S_{j+i-1}, U_{j+i-1})| \geq M$ .) As before,  $L_i$  is defined to be the total volume of the vertices in  $U_{j+i-1}$  that receive the rumor in round  $j+i$  through edges in  $E_i$ .
- If  $\text{vol}(S_{j+i-1}) > D$ , then  $L_i = M$ .

Similarly to Claim 3.3, we have  $\mathbf{E}[\sum_{k \leq i} L_k] = iM$  and  $\mathbf{Var}[\sum_{k \leq i} L_k] \leq \Delta \cdot \mathbf{E}[\sum_{k \leq i} L_k]$ . And similarly to (3.1), using Chebyshev's inequality we obtain that for  $i \geq 8\Delta/\text{vol}(S_0)$  and  $\mu = \mathbf{E}[\sum_{k \leq i} L_k] = iM \geq 4\Delta$ ,

$$\Pr\left(\sum_{k \leq i} L_k < D\right) \leq \frac{\mu\Delta}{(\mu - D)^2} \leq \frac{\mu\Delta}{(\mu - \Delta)^2} \leq 4/9.$$

Finally, since  $\text{vol}(S_{j+i}) < D$  implies  $\sum_{k \leq i} L_k \leq \text{vol}(S_{j+i}) - \text{vol}(S_j) < D$ , the lemma follows.  $\square$

We can now derive Theorem 1.2(a) similarly to Lemma 3.1.

*Proof of Theorem 1.2(a).* Let  $d = \Omega(\Delta(\phi + \delta^{-1}))$  be the degree of the start vertex; thus  $\text{vol}(S_0) = d$ .

By Lemma 3.6,  $c \ln(n) \cdot \lceil 8\Delta/d \rceil$  rounds increase the volume of informed vertices to at least  $D = \min\{\Delta, \delta d/2\}$  with provability at least  $1 - (1 - 1/2)^{c \ln n} \geq 1 - n^{-c/2}$ . This number of rounds is at most  $O(\phi^{-1} \ln n)$ , since  $d = \Omega(\Delta(\phi + \delta^{-1})) = \Omega(\phi\Delta)$ .

Once the total volume of informed vertices is at least  $D$ , we have by Lemma 3.1 that  $O(\log n(\phi^{-1} + \Delta/\lceil \phi D \rceil))$  additional rounds inform all vertices w.h.p. Again this number of rounds is at most  $O(\phi^{-1} \ln n)$ , because  $d = \Omega(\Delta(\phi + \delta^{-1})) = \Omega(\Delta/\delta)$  and thus  $D = \min\{\Delta, \delta d/2\} = \Omega(\Delta)$ .  $\square$

The following direct corollary of Theorem 1.2(a) gives a sufficient condition for rumor spreading in  $O(\phi^{-1} \log n)$  rounds for any start vertex.

**Corollary 3.7.** *If  $\delta = \Omega(\Delta(\phi + \delta^{-1}))$ , or equivalently,  $\delta = \Omega(\phi\Delta + \sqrt{\Delta})$  then the broadcast time of the PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p., for any start vertex.*

The condition  $\delta = \Omega(\Delta(\phi + \delta^{-1}))$  is equivalent to  $\delta = \Omega(\phi\Delta + \sqrt{\Delta})$  because it is equivalent to  $(\delta = \Omega(\phi\Delta)) \wedge (\delta = \Omega(\Delta/\delta))$ , which is equivalent to  $(\delta = \Omega(\phi\Delta)) \wedge (\delta = \Omega(\sqrt{\Delta}))$ .

### 3.2.2 Optimality of Theorem 1.2

The sufficient conditions for rumor spreading in  $O(\phi^{-1} \log n)$  rounds described in Theorem 1.2, i.e., that the degree of the start vertex be  $d = \Omega(\Delta(\phi + \delta^{-1}))$  or the maximum degree be  $\Delta = O(1/\phi)$ , are optimal in the following sense.

**Theorem 3.8.** *For any  $\phi, \delta, \Delta$  and degree  $d$  (that are functions of  $n$ ) such that  $d = o(\Delta(\phi + \delta^{-1}))$  and  $\Delta = \omega(1/\phi)$ , there exists an infinite sequence of graphs  $G_1, G_2, \dots$  that has the following properties:  $G_n$  has  $\Theta(n)$  vertices, conductance  $\Theta(\phi)$ , and maximum (minimum) degree  $\Theta(\Delta)$  (respectively  $\Theta(\delta)$ ), and it contains a vertex of degree  $\Theta(d)$  such that if a rumor starts at that vertex then  $\omega(\phi^{-1} \log n)$  rounds of the PULL algorithm are needed to inform all vertices w.h.p.*

*Proof.* We distinguish two cases. First we consider the case where  $d = o(\phi\Delta)$ . We give a construction of a graph  $G_n$  that possesses all the properties specified by the theorem. Take a  $\Delta$ -regular graph  $R_\Delta$  on  $n$  vertices with edge expansion  $\xi = \Theta(\Delta)$ . Such a graph exists since the edge expansion of a random  $\Delta$ -regular graph is  $\Theta(\Delta)$  w.h.p. [2]. The conductance of  $R_\Delta$  is obviously  $\xi/\Delta = \Theta(1)$ . Add a vertex  $s$  of degree  $d$  and a vertex  $v_{\min}$  of degree  $\delta$ , choosing their neighbors arbitrarily among the vertices of  $R_\Delta$ . Vertex  $s$  will be the start vertex, and  $v_{\min}$  is added just to achieve a minimum degree of  $\delta$ . Next we add to this graph a component that reduces the conductance to  $\phi$ : Take the complete graph on  $\Delta$  vertices  $K_\Delta$ , and let  $A$  be an arbitrary subset of the vertices of  $R_\Delta$  of size  $|A| = \lfloor \phi\Delta \rfloor$ . (It is  $A \neq \emptyset$  since, by the statement of the theorem,  $\Delta = \omega(1/\phi)$  and thus  $\phi\Delta = \omega(1)$ .) Then draw edges from every vertex of  $K_\Delta$  to all vertices in  $A$ . The resulting graph has the desired number of vertices, maximum and minimum degrees, and conductance. Also, since the degree of  $s$

is  $d = o(\phi\Delta)$ , the probability that no neighbor of  $s$  (each having degree at least  $\Delta$ ) chooses  $s$  in any of  $k = \lfloor \phi^{-1} \ln(n) \cdot (2/3) \sqrt{\phi\Delta/d} \rfloor = \omega(\phi^{-1} \ln n)$  rounds is at least

$$(1 - 1/\Delta)^{kd} \geq e^{-(3/2)kd/\Delta} \geq e^{-\ln n \sqrt{d/\phi\Delta}} = n^{-o(1)},$$

where for the first inequality we used the fact that  $1 - x \geq e^{-3x/2}$  for  $0 \leq x \leq 1/2$ . Thus, with probability  $n^{-o(1)}$  no vertex learns a rumor started at  $s$  in  $O(\phi^{-1} \ln n)$  rounds.

Next we consider the complementary case where  $d = \Omega(\phi\Delta)$ . Since also  $d = o(\Delta(\phi + \delta^{-1}))$  by the statement of the theorem, it follows that  $\phi = o(1/\delta)$  and  $d = o(\Delta/\delta)$ . Consider the following construction: Take the graph we described in the previous case and remove vertex  $s$  together with its incident edges. Take also  $\lceil d/\delta \rceil$  copies of  $K_\delta$ . Add a vertex  $s'$  of degree  $\Theta(d)$  with neighbors the vertices of the  $\lceil d/\delta \rceil$   $\delta$ -cliques, and the vertices in an arbitrary subset  $B$  of  $R_\Delta$ , with  $|B| = \lceil \phi d \delta \rceil$ . (It is  $|B| = O(d)$  since  $\phi = o(1/\delta)$  as we saw above.) The resulting graph has the desired number of vertices, maximum and minimum degrees, and conductance. Note that since  $d = o(\Delta/\delta)$  and  $\Delta = \omega(1/\phi)$ , we have  $|B| = \lceil \phi d \delta \rceil \leq \phi d \delta + 1 = o(\phi\Delta) + 1 = o(\phi\Delta)$ . Thus, the probability that no neighbor of  $s'$  in  $B$  receives the rumor from  $s'$  in  $k' = \lfloor \phi^{-1} \ln n \cdot (2/3) \sqrt{\phi\Delta/|B|} \rfloor = \omega(\phi^{-1} \ln n)$  rounds is at least

$$(1 - 1/\Delta)^{k'|B|} \geq e^{-(3/2)k'|B|/\Delta} \geq e^{-\ln n \sqrt{|B|/\phi\Delta}} = n^{-o(1)}.$$

Hence, with probability  $n^{-o(1)}$ , no vertex in  $B$  learns a rumor started at  $s'$  in  $O(\phi^{-1} \ln n)$  rounds.  $\square$

### 3.2.3 Bounded Ratio of the Degrees of Adjacent Vertices

Chierichetti et al. [6] showed that if the ratio of the degrees of any two adjacent vertices is bounded by a constant, then the broadcast time of the PULL algorithm is  $O((\log \phi^{-1})^2 \phi^{-1} \log n)$  rounds w.h.p., for any start vertex. Using a similar reasoning as for the proofs of Lemma 3.1 and Theorem 1.2(a), we show that in fact the above condition yields a broadcast time of  $O(\phi^{-1} \log n)$  rounds.

**Theorem 3.9.** *If for every edge  $\{v, u\}$ ,  $d(v)/d(u) = \Theta(1)$  then the broadcast time of the PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p., for any start vertex.*

To prove the theorem we use the next result, which is similar to Lemmata 3.2 and 3.6.

**Lemma 3.10.** *Suppose that for every edge  $\{v, u\}$ ,  $d(v)/d(u) \in [\alpha^{-1}, \alpha]$ . If  $\text{vol}(S_0) < \Delta$  then  $\Pr(\text{vol}(S_i) \geq \min\{2 \text{vol}(S_0), \Delta\}) \geq 1/2$ , for  $i \geq 4\alpha/\phi$ .*

*Proof.* We consider the sequence  $L_1, L_2, \dots$  of random variables, where  $L_i$  is defined as follows:

- If  $\text{vol}(S_{i-1}) < \Delta$ , we fix an arbitrary subset  $E_i$  of  $\text{cut}(S_{i-1}, U_{i-1})$  with  $M = \lceil \phi \text{vol}(S_0) \rceil$  edges (determined before round  $i$ ). Then  $L_i$  is the total volume of the vertices in  $U_{i-1}$  that receive the rumor in round  $i$  through edges in  $E_i$ .
- If  $\text{vol}(S_{i-1}) \geq \Delta$ , then  $L_i = M$ .

Similarly to Claim 3.3, we obtain  $\mathbf{E}[\sum_{k \leq i} L_k] = iM$  and  $\mathbf{Var}[\sum_{k \leq i} L_k] \leq \alpha \text{vol}(S_0) \cdot \mathbf{E}[\sum_{k \leq i} L_k]$ . To show the variance result we observe that if  $\text{vol}(S_j) < 2 \text{vol}(S_0)$  then every vertex  $v \in \bar{S}_j$  has degree at most  $\text{vol}(S_0)$ , and thus, every neighbor of  $v$  in  $U_j$  has degree at most  $\alpha \text{vol}(S_0)$ ; so, we can replace  $\Delta$  by  $\alpha \text{vol}(S_0)$  in (3.6). Next, as in (3.1), by using Chebyshev's inequality we get that for  $i \geq 4\alpha/\phi$  and  $\mu = \mathbf{E}[\sum_{k \leq i} L_k] \geq 4\alpha \text{vol}(S_0)$ ,

$$\Pr\left(\sum_{k \leq i} L_k < \text{vol}(S_0)\right) \leq \frac{\mu \alpha \text{vol}(S_0)}{(\mu - \text{vol}(S_0))^2} \leq \frac{\mu \alpha \text{vol}(S_0)}{(\mu - \alpha \text{vol}(S_0))^2} \leq 4/9.$$

And since  $\text{vol}(S_i) < \min\{2 \text{vol}(S_0), \Delta\}$  implies  $\sum_{k \leq i} L_k \leq \text{vol}(S_i) - \text{vol}(S_0) < \text{vol}(S_0)$ , the lemma follows.  $\square$

*Proof of Theorem 3.9.* By Lemma 3.10 and Chernoff bounds, we obtain that w.h.p.  $O(\phi^{-1} \log n)$  rounds suffice to increase the total volume of informed vertices to at least  $\Delta$ . And, by Lemma 3.1,  $O(\phi^{-1} \ln n)$  additional rounds suffice to inform all vertices w.h.p.  $\square$

## 4 PUSH Algorithm

We reduce the analysis of the PUSH algorithm to that of the PULL algorithm using the following duality lemma. This result is similar to Lemma 3 in [6].

**Lemma 4.1.** *Let  $\mathcal{E}_{\text{PUSH}}(v, u, t)$  be the event that vertex  $u$  learns a rumor started at vertex  $v$  after at most  $t$  rounds of the PUSH algorithm; and let  $\mathcal{E}_{\text{PULL}}(v, u, t)$  be defined similarly. Then,  $\Pr(\mathcal{E}_{\text{PUSH}}(v, u, t)) = \Pr(\mathcal{E}_{\text{PULL}}(u, v, t))$ .*

*Proof.* Consider the sample space of all possible runs of the first  $t$  rounds of the PUSH (or PULL) algorithm, where a run specifies for each vertex  $v$  and round  $i = 1, \dots, t$ , which neighbor  $v$  chooses in round  $i$ . We call these runs  $t$ -runs. (Note that a  $t$ -run specifies a choice for every  $v$  and  $i$  regardless of whether or not  $v$  knows the rumor in round  $i$ .) The probability that a given  $t$ -run occurs is the same for all  $t$ -runs and is equal to  $\prod_{v \in V} (d(v))^{-t}$ . For  $\mathcal{A} \in \{\text{PUSH}, \text{PULL}\}$ , let  $\Omega_{\mathcal{A}}(v, u, t)$  be the subset of  $t$ -runs that constitute event  $\mathcal{E}_{\mathcal{A}}(v, u, t)$ . Then,  $\Pr(\mathcal{E}_{\mathcal{A}}(v, u, t)) = |\Omega_{\mathcal{A}}(v, u, t)| \cdot \prod_{v \in V} (d(v))^{-t}$ . Thus, to prove the lemma it suffices to show that  $|\Omega_{\text{PUSH}}(v, u, t)| = |\Omega_{\text{PULL}}(u, v, t)|$ .

For a  $t$ -run  $\omega$  let  $\omega'$  be the “inverse”  $t$ -run consisting of the same sequence of rounds as  $\omega$  but executed in reverse order, i.e., if vertex  $v$  chooses vertex  $u$  in round  $i$  in  $\omega$ , then  $v$  chooses  $u$  in round  $t-i$  in  $\omega'$ . We prove next that  $\omega \in \Omega_{\text{PUSH}}(v, u, t)$  if and only if  $\omega' \in \Omega_{\text{PULL}}(u, v, t)$ . From this and the fact that distinct  $t$ -runs have distinct inverse runs, it follows that  $|\Omega_{\text{PUSH}}(v, u, t)| = |\Omega_{\text{PULL}}(u, v, t)|$ . So, it remains to show that  $\omega \in \Omega_{\text{PUSH}}(v, u, t)$  if and only if  $\omega' \in \Omega_{\text{PULL}}(u, v, t)$ .

A *PUSH-path* for a  $t$ -run is a list of vertices  $u_0, u_1, \dots, u_t$  such that for any two consecutive vertices  $u_{i-1}, u_i$ , either (i)  $u_{i-1} = u_i$ , or (ii)  $u_{i-1}$  chooses  $u_i$  in round  $i$ . A *PULL-path* is defined similarly except that condition (ii) now states that  $u_i$  chooses  $u_{i-1}$  in round  $i$ . For a given  $t$ -run,  $u$  learns a rumor started at  $v$  in at most  $t$  rounds of the PUSH (PULL) algorithm if and only if there exists a PUSH-path (respectively, PULL-path) from  $v$  to  $u$ .<sup>3</sup> Also,  $v = u_0, u_1, \dots, u_t = u$  is a PUSH-path for  $\omega$  if and only if  $u_t, u_{t-1}, \dots, u_0$  is a PULL-path for  $\omega'$ . These two observations yield that  $\omega \in \Omega_{\text{PUSH}}(v, u, t)$  if and only if  $\omega' \in \Omega_{\text{PULL}}(u, v, t)$ .  $\square$

Suppose now that for any vertex  $u$ ,  $t$  rounds of the PULL algorithm spread to all vertices a rumor started at  $u$  with probability at least  $1 - q$ . Then, by Lemma 4.1, for any vertex  $v$ ,  $t$  rounds of the PUSH algorithm spread to a given vertex  $u$  a rumor started at  $v$  with the same probability, at least  $1 - q$ ; and, by the union bound, *all* vertices  $u$  are informed in at most  $t$  rounds with probability at least  $1 - (n - 1)q$ , if  $q \leq 1/(n - 1)$ . Thus, if the broadcast time of the PULL algorithm is  $O(\phi^{-1} \log n)$  rounds w.h.p. for any start vertex, then the same is true for the PUSH algorithm. Hence, the conditions described in Section 3 guaranteeing a broadcast time for the PULL algorithm of  $O(\phi^{-1} \log n)$  rounds w.h.p. for any start vertex, apply to the PUSH algorithm as well; namely, Theorem 1.2(b), Corollary 3.7, and Theorem 3.9. Theorem 3.8 also holds for the PUSH algorithm for  $d = \delta$ . (For otherwise, by using the same reasoning as above with the roles of the PUSH and the PULL algorithms switched, we could contradict Theorem 3.8.)

<sup>3</sup>Note that if  $u$  learns the rumor earlier, say in  $t' < t$  rounds, then there is such a path with the last  $t - t'$  vertices being  $u$ .

## 5 PUSH-PULL Algorithm

In this last section we prove Theorem 1.1, which gives a bound of  $O(\phi^{-1} \log n)$  rounds w.h.p. on the broadcast time of the PUSH-PULL algorithm, and we also argue that this bound is tight.

*Proof of Theorem 1.1.* Let  $v$  be an arbitrary vertex and  $v_{\max}$  be a vertex of degree equal to the maximum degree  $\Delta$ . By Corollary 3.4, we have that: (A) W.h.p.  $O(\phi^{-1} \log n)$  rounds of the PULL algorithm suffice to spread a rumor from  $v_{\max}$  to all other vertices (and thus to  $v$ ). Combining this with Lemma 4.1 yields: (B) W.h.p.  $O(\phi^{-1} \log n)$  rounds of the PUSH algorithm suffice to spread a rumor from  $v$  to  $v_{\max}$ . The theorem now follows easily: Statement (B) implies (a fortiori) that w.h.p.  $O(\phi^{-1} \log n)$  rounds of the PUSH-PULL algorithm suffice to spread to  $v_{\max}$  a rumor started at  $v$ ; and once  $v_{\max}$  is informed, Statement (A) implies that w.h.p.  $O(\phi^{-1} \log n)$  additional rounds suffice to inform all the remaining vertices.  $\square$

The following result is known from [6].

**Lemma 5.1.** *For any  $\phi \geq 1/n^{1-\epsilon}$ , for a fixed  $\epsilon > 0$ , there exists an infinite sequence of graphs  $G_1, G_2, \dots$  such that  $G_n$  has  $\Theta(n)$  vertices, conductance  $\Theta(\phi)$ , and diameter  $\Omega(\phi^{-1} \log n)$ .*

From this lemma it is immediate that rumor spreading requires at least  $\Omega(\phi^{-1} \log n)$  rounds for some graphs, if  $\phi \geq 1/n^{1-\epsilon}$ . Thus, the bound of Theorem 1.1 is asymptotically tight for  $\phi \geq 1/n^{1-\epsilon}$ . Next we show that this is in fact true for all  $\phi = \Omega(1/n)$ .

**Lemma 5.2.** *For any  $\phi$  with  $2/(n-1) \leq \phi \leq 1/2$ , there exists an infinite sequence of graphs  $G_1, G_2, \dots$  such that  $G_n$  has  $n$  vertices and conductance  $\Theta(\phi)$ , and  $\Omega(\phi^{-1} \log n)$  rounds of the PUSH-PULL algorithm are required to inform all vertices w.h.p., for any start vertex.*

*Proof.* Consider the  $n$ -vertex graph obtained by taking two stars, one with  $\lceil \phi^{-1} \rceil$  vertices and another with  $n - \lceil \phi^{-1} \rceil$  vertices, and connecting their centers with an edge. Since  $\phi \geq 2/(n-1)$  and thus  $\lceil \phi^{-1} \rceil \leq n/2$ , the resulting graph has conductance  $\Theta(\phi)$ . We now show that for any start vertex and any constant  $c > 0$ , at least  $c \ln n / 3\phi$  rounds are required to inform all vertices with probability  $1 - n^{-c}$ . Let  $v$  and  $v'$  be the centers of the two stars, where  $v$  is the center of the star containing the start vertex. Let  $j$  be the round when  $v$  gets informed. (If  $v$  is the start vertex then  $j = 0$ .) The probability that  $v'$  is not informed at the end of round  $j + i$  (which happens if neither  $v$  chooses  $v'$  nor  $v'$  chooses  $v$  in any of the rounds  $j + 1, \dots, j + i$ ) is

$$(1 - 1/\lceil \phi^{-1} \rceil)^i (1 - 1/(n - \lceil \phi^{-1} \rceil))^i \geq (1 - 1/\lceil \phi^{-1} \rceil)^{2i} \geq (1 - \phi)^{2i} \geq e^{-3i\phi},$$

where the first inequality holds because  $\lceil \phi^{-1} \rceil \leq n/2 \leq n - \lceil \phi^{-1} \rceil$ , and the last inequality is obtained using the fact that  $1 - x \geq e^{-3x/2}$  for  $0 \leq x \leq 1/2$ . Observe that  $e^{-3i\phi} > n^{-c}$  for  $i < c \ln n / 3\phi$ . Thus, at least  $c \ln n / 3\phi$  rounds are required to inform all vertices with probability  $1 - n^{-c}$ .  $\square$

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