Analyse et Conception Formelle

Lesson 4

–

Proofs with a proof assistant

Prove logic formulas ... to prove programs

```
fun nth:: "nat => 'a list => 'a"
where
"nth 0 (x#_)=x" |
"nth x (y#ys)= (nth (x - 1) ys)"

fun index:: "'a => 'a list => nat"
where
"index x (y#ys)= (if x=y then 1 else 1+(index x ys))"

lemma nth_index: "nth (index e l) l= e"
```

How to prove the lemma nth_index? (Recall that everything is logic!)

What we are going to prove is thus a formula of the form:

$\boxed{\text{Theory of lists}} \wedge \boxed{\text{Equations for nth}} \wedge \boxed{\text{Equations for index}} \longrightarrow$ nth_index

Outline

1. Finding counterexamples
   - nitpick
   - quickcheck
2. Proving true formulas
   - Proof by cases: apply (case_tac x)
   - Proof by induction: apply (induct x)
   - Combination of decision procedures: apply auto and apply simp
   - Solving theorems in the Cloud: sledgehammer

Acknowledgements: some material is borrowed from T. Nipkow's lectures and from Concrete Semantics by Nipkow and Klein, Springer Verlag, 2016.

More details (in french) about those proof techniques can be found in:

- http://people.irisa.fr/Thomas.Genet/ACF/TPs/pc.thy
- The video https://youtu.be/qwlQIS46TLA

Finding counterexamples

Why? because «90% of the theorems we write are false!»

- Because this is not what we want to prove!
- Because the formula is imprecise
- Because the function is false
- Because there are typos...

Before starting a proof, always first search for a counterexample!

Isabelle/HOL offers two counterexample finders:

- nitpick: uses finite model enumeration
  - \+ Works on any logic formula, any type and any function
  - \- Rapidly exhausted on large programs and properties
- quickcheck: uses random testing, exhaustive testing and narrowing
  - \- Does not covers all formula and all types
  - \+ Scales well even on large programs and complex properties

Nitpick

To build an interpretation I such that $I \not\models \phi$ (or $I \models \neg\phi$) .... `nitpick`

`nitpick` principle: build an interpretation $I \models \neg\phi$ on a finite domain D

- Choose a cardinality k
- Enumerate all possible domains D_τ of size k for all types τ in $\neg\phi$
- Build all possible interpretations of functions in $\neg\phi$ on all D_τ
- Check if one interpretation satisfy $\neg\phi$ (this is a counterexample for ϕ)
- If not, there is no counterexample on a domain of size k for ϕ

`nitpick` algorithm:

- Search for a counterexample to ϕ with cardinalities 1 upto n
- Stops when I such that $I \models \neg\phi$ is found (counterex. to ϕ), **or**
- Stops when maximal cardinality n is reached (10 by default), **or**
- Stops after 30 seconds (default timeout)

Nitpick (II)

Exercise 1

By hand, iteratively check if there is a counterexample of cardinality 1, 2, 3 *for the formula* ϕ, *where* ϕ *is* `length la <= 1`.

Remark 1

- *The types occurring in* ϕ *are* `'a` *and* `'a list`
- **One** *possible domain* $D_{'a}$ *of cardinality* 1*:* $\{a_1\}$
- **One** *possible domain* $D_{'a\ list}$ *of cardinality* 1*:* $\{[\,]\}$ $\cancel{\{[a_1]\}}$
  Domains have to be **subterm-closed**, *thus* $\{[a_1]\}$ *is not valid*
- **One** *possible domain* $D_{'a}$ *of cardinality* 2*:* $\{a_1, a_2\}$
- **Two** *possible domains* $D_{'a\ list}$ *of cardinality* 2*:* $\{[\,], [a_1]\}$ *and* $\{[\,], [a_2]\}$
- **One** *possible domain* $D_{'a}$ *of cardinality* 3*:* $\{a_1, a_2, a_3\}$
- **Twelve** *possible domains* $D_{'a\ list}$ *of cardinality* 3*:* $\{[\,], [a_1], [a_1, a_1]\}$, $\{[\,], [a_1], [a_2]\}$, $\{[\,], [a_1], [a_3, a_1]\}$, ... $\cancel{\{[\,], [a_1], [a_3, a_2]\}}$

Nitpick (III)

`nitpick` options:

- `timeout=t`, set the timeout to `t` seconds (`timeout=none` possible)
- `show_all`, displays the domains and interpretations for the counterex.
- `expect=s`, specifies the expected outcome where `s` can be `none` (no counterexample) or `genuine` (a counterexample exists)
- `card=i-j`, specifies the cardinalities to explore

For instance:

```
nitpick [timeout=120, show_all, card=3-5]
```

Exercise 2

- *Explain the counterexample found for* `rev l = l`
- *Is there a counterexample to the lemma* `nth_index`*?*
- *Correct the lemma and definitions of* `index` *and* `nth`
- *Is the lemma* `append_commut` *true? really?*

Quickcheck

To build an interpretation I such that $I \not\models \phi$ (or $I \models \neg\phi$) .... `quickcheck`

`quickcheck` principle: test ϕ with automatically generated values of size k

Either with a generator

- Random: values are generated randomly (QuickCheck)
- Exhaustive: (almost) all values of size k are generated
- Narrowing: like exhaustive but taking advantage of symbolic values

No exhautiveness guarantee!! with any of them

`quickcheck` algorithm:

- Export Haskell code for functions and lemmas
- Generate test values of size 1 upto n and, test ϕ using Haskell code
- Stops when a counterexample is found, **or**
- Stops when max. size of test values has been reached (default 5), **or**
- Stops after 30 seconds (default timeout)

Quickcheck (II)

`quickcheck` options:

- `timeout=t`, set the timeout to `t` seconds
- `expect=s`, specifies the expected outcome where `s` can be `no_counterexample`, `counterexample` or `no_expectation`
- `tester=tool`, specifies generator to use where `tool` can be `random`, `exhaustive` or `narrowing`
- `size=i`, specifies the maximal size of testing values

For instance:

```
quickcheck [tester=narrowing,size=6]
```

Exercise 3

- *Using* `quickcheck`*, find the counterexample for* `length_slice`*?*

Remark 2

Quickcheck first generates values and then does the tests. As a result, it may not run the tests if you choose bad values for `size` *and* `timeout`*.*

What to do next?

When no counterexample is found what can we do?

- Increase the timeout and size values for `nitpick` and `quickcheck`?
- ... go for a proof!

Any proof is faster than an infinite time `nitpick` or `quickcheck`

Any proof is more reliable than an infinite time `nitpick` or `quickcheck`

(They make approximations or assumptions on infinite types)

The five proof tools that we will focus on:

1. `apply case_tac`
2. `apply induct`
3. `apply auto`
4. `apply simp`
5. `sledgehammer`

How do proofs look like?

A formula of the form $A_1 \wedge \ldots \wedge A_n$ is represented by the proof goal:

```
goal (n subgoals):
1. A1
...
n. An
```

Where each subgoal to prove is either a formula of the form

$\bigwedge x_1 \ldots x_n.\ B$	meaning	prove B, or
$\bigwedge x_1 \ldots x_n.\ B \Longrightarrow C$	meaning	prove $B \longrightarrow C$, or
$\bigwedge x_1 \ldots x_n.\ B_1 \Longrightarrow \ldots B_n \Longrightarrow C$	meaning	prove $B_1 \wedge \ldots \wedge B_n \longrightarrow C$

and $\bigwedge x_1 \ldots x_n$ means that those variables are local to this subgoal.

Example 1 (Proof goal)

```
goal (2 subgoals):
 1. member [] e ⟹ nth (index e []) [] = e
 2. ⋀a l. e ≠ a ⟹ member (a # l) e ⟹
           ¬ member l e ⟹ nth (index e l) l = e
```


Proof by cases

... possible when the proof can be split into a finite number of cases

Proof by cases on a formula F

Do a proof by cases on a formula `F` . . . . `apply (case_tac "F")`
Splits the current goal in two: one with assumption `F` and one with `¬ F`

Example 2 (Proof by case on a formula)

With `apply (case_tac "F::bool")`

```
goal (1 subgoal):          becomes     goal (2 subgoals):
 1. A ⟹ B                               1. F ⟹ A ⟹ B
                                         2. ¬ F ⟹ A ⟹ B
```

Exercise 4

Prove that for any natural number x*, if* $x < 4$ *then* $x * x < 10$.

Proof by cases (II)

Proof by cases on a variable x of an enumerated type of size n

Do a proof by cases on a variable x .... `apply (case_tac "x")`
Splits the current goal into n goals, one for each case of x.

Example 3 (Proof by case on a variable of an enumerated type)

In Course 3, we defined `datatype color= Black | White | Grey`
With `apply (case_tac "x")`

```
goal (1 subgoal):
 1. P (x::color)
```

becomes

```
goal (3 subgoals):
 1. x = Black ⟹ P x
 2. x = White ⟹ P x
 3. x = Grey ⟹ P x
```

Exercise 5

On the `color` *enumerated type or course 3, show that for all color x if the* `notBlack x` *is true then x is either white or grey.*

Proof by induction

«Properties on recursive functions need proofs by induction»

Recall the basic induction principle on naturals:

$$\boxed{P(0) \wedge \forall x \in \mathbb{N}.\,(P(x) \longrightarrow P(x+1)) \quad\longrightarrow\quad \forall x \in \mathbb{N}.\,P(x)}$$

All recursive datatype have a similar induction principle, *e.g.* `'a lists`:

$$P([\,]) \wedge \forall e \in \texttt{'a}.\ \forall l \in \texttt{'a list}.(P(l) \longrightarrow P(e\#l)) \quad\longrightarrow\quad \forall l \in \texttt{'a list}.P(l)$$

Etc...

Example 4

`datatype 'a binTree= Leaf | Node 'a "'a binTree" "'a binTree"`

$$\boxed{\begin{array}{l} P(\texttt{Leaf}) \wedge \forall e \in \texttt{'a}.\ \forall t1\ t2 \in \texttt{'a binTree}. \\ (P(t1) \wedge P(t2) \longrightarrow P(Node\ e\ t1\ t2)) \quad\longrightarrow\quad \forall t \in \texttt{'a binTree}.P(t) \end{array}}$$

Proof by induction (II)

$$P([\,]) \wedge \forall e \in \texttt{'a}.\ \forall l \in \texttt{'a list}.(P(l) \longrightarrow P(e\#l)) \quad\longrightarrow\quad \forall l \in \texttt{'a list}.P(l)$$

Example 5 (Proof by induction on lists)

Recall the definition of the function append:

```
   (1) append [] l    =  l
 (2) append (x#xs) l  =  x#(append xs l)
```

To prove $\boxed{\forall l \in \texttt{'a list}.\,(append\ l\,[\,]) = l}$ by induction on l, we prove:

1. $append\,[\,]\,[\,] = [\,]$, proven by the first equation of append
2. $\forall e \in \texttt{'a}.\ \forall l \in \texttt{'a list}.$
   $(append\ l\,[\,]) = l \quad\longrightarrow\quad (append\,(e\#l)\,[\,]) = (e\#l)$
   using the second equation of append, it becomes
   $(append\ l\,[\,]) = l \quad\longrightarrow\quad e\#(append\ l\,[\,]) = (e\#l)$
   using the (induction) hypothesis, it becomes
   $(append\ l\,[\,]) = l \quad\longrightarrow\quad e\#l = (e\#l)$

Proof by induction: `apply (induct x)`

To apply induction principle on variable x .... `apply (induct x)`

Conditions on the variable chosen for induction (induction variable):

- The variable x has to be of an inductive type (`nat`, datatypes, ...)
  Otherwise `apply (induct x)` fails
- The terms built by induction cases should easily be reducible!

Example 6 (Choice of the induction variable)

```
   (1) append [ ] l   =  l
 (2) append (x#xs) l  =  x#(append xs l)
```

To prove $\boxed{\forall l_1\ l_2 \in \texttt{'a list}.\,(length\,(append\ l_1\ l_2)) \geq (length\ l_2)}$

An induction proof on l_1, instead of l_2, is more likely to succeed:

- an induction on l_1 will require to prove:
  $(length\,(append\,(e\#l_1)\ l_2) \geq (length\ l_2)$
- an induction on l_2 will require to prove:
  $(length\,(append\ l_1\,(e\#l_2)) \geq (length\,(e\#l_2))$

Proof by induction: apply (induct x) (II)

Exercise 6

Recall the datatype of binary trees we defined in lecture 3. Define and prove the following properties:

1. *If* `member x t`*, then there is at least one node in the tree* `t`*.*
2. *Relate the fact that* `x` *is a sub-tree of* `y` *and their number of nodes.*

Exercise 7

Recall the functions `sumList`*,* `sumNat` *and* `makeList` *of lecture 3. Try to state and prove the following properties:*

1. *Relate the length of list produced by* `makeList i` *and* `i`
2. *Relate the value of* `sumNat i` *and* `i`
3. *Give and try to prove the property relating those three functions*

Proof by induction: generalize the goals

By defaut `apply induct` may produce too weak induction hypothesis

Example 7

When doing an `apply (induct x)` on the goal `P (x::nat) (y::nat)`

```
goal (2 subgoals):
 1. P 0 y
 2. ⋀x. P x y ⟹ P (Suc x) y
```

In the subgoals, y is fixed/constant!

Example 8

With `apply (induct x arbitrary:y)` on the same goal

```
goal (2 subgoals):
 1. ⋀y. P 0 y
 2. ⋀x y. P x y ⟹ P (Suc x) y
```

The subgoals range over any y

Exercise 8

Prove the `sym` *lemma on the* `leq` *function.*

Proof by induction: : induction principles

Recall the basic induction principle on naturals:

$$\boxed{P(0) \wedge \forall x \in \mathbb{N}.\, (P(x) \longrightarrow P(x+1)) \quad \longrightarrow \quad \forall x \in \mathbb{N}.\, P(x)}$$

In fact, there are infinitely many other induction principles

- $P(0) \wedge P(1) \wedge \forall x \in \mathbb{N}.\, ((x > 0 \wedge P(x)) \longrightarrow P(x+1)) \quad \longrightarrow \quad \forall x \in \mathbb{N}.\, P(x)$
- ...
- Strong induction on naturals
  $\forall x, y \in \mathbb{N}.\, ((y < x \wedge P(y)) \longrightarrow P(x)) \quad \longrightarrow \quad \forall x \in \mathbb{N}.\, P(x)$
- Well-founded induction on any type having a well-founded order $<<$
  $\forall x, y.\, ((y << x \wedge P(y)) \longrightarrow P(x)) \quad \longrightarrow \quad \forall x.\, P(x)$

Proof by induction: : induction principles (II)

Apply an induction principle adapted to the function call `(f x y z)`
.... `apply (induct x y z rule:f.induct)`

Apply strong induction on variable `x` of type `nat`

.... `apply (induct x rule:nat_less_induct)`

Apply well-founded induction on a variable `x`
.... `apply (induct x rule:wf_induct)`

Exercise 9

Prove the lemma on function `div2`*.*

Combination of decision procedures `auto` and `simp`

Automatically solve or simplify **all subgoals** .... `apply auto`

`apply auto` does the following:

- Rewrites using equations (function definitions, etc)
- Applies a bit of arithmetic, logic reasoning and set reasoning
- **On all subgoals**
- Solves them all or stops when stuck and shows the remaining subgoals

Automatically simplify **the first subgoal** .... `apply simp`

`apply simp` does the following:

- Rewrites using equations (function definitions, etc)
- Applies a bit of arithmetic
- **on the first subgoal**
- Solves it or stops when stuck and shows the simplified subgoal

Combination of decision procedures `auto` and `simp` (II)

Want to know what those tactics do?

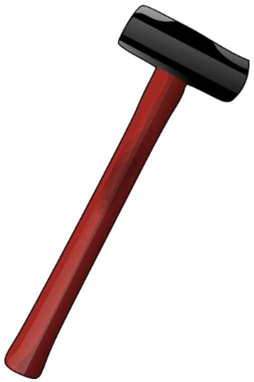
- Add the command `using [[simp_trace=true]]` in the proof script
- Look in the `output` buffer

Example 9

Switch on tracing and try to prove the lemma:

```
lemma "(index (1::nat) [3,4,1,3]) = 2"
using [[simp_trace=true]]
apply auto
```


Sledgehammer



«Sledgehammers are often used in destruction work...»

Sledgehammer

«Solve theorems in the Cloud»

Architecture:

Isabelle/HOL	Formula to prove + relevant definitions and lemmas $\longrightarrow$ $\longleftarrow$ Proof (click on it)	External **ATPs**[1] Local or in the Cloud

Prove the first subgoal using state-of-the-art[2] ATPs ...... `sledgehammer`

- Call to local or distant ATPs: SPASS, E, Vampire, CVC4, Z3, etc.
- Succeeds or stops on timeout (can be extended, *e.g.* `[timeout=120]`)
- Provers can be explicitely selected (*e.g.* `[provers= z3 spass]`
- A proof consists of applications of lemmas or definition using the Isabelle/HOL tactics: `metis`, `smt`, `simp`, `fast`, etc.

[1] Automatic Theorem Provers

[2] See `http://www.cs.miami.edu/~tptp/CASC/`.

Sledgehammer (II)

Remark 3

By default, sledgehammer does not use all available provers. But, you can remedy this by defining, once for all, the set of provers to be used:

```
sledgehammer_params [provers=cvc4 spass z3 e remote_vampire]
```

Exercise 10

Finish the proof of the property relating `nth` *and* `index`

Exercise 11

Recall the functions `sumList`, `sumNat` *and* `makeList` *of lecture 3. Try to state and prove the following properties:*

1. *Prove that there is no repeated occurrence of elements in the list produced by* `makeList`
2. *Finish the proof of the property relating those three functions*

Hints for building proofs in Isabelle/HOL

When stuck in the proof of `prop1`, add relevant intermediate lemmas:

1. In the file, define a lemma **before** the property `prop1`
2. **Name** the lemma (say `lem1`) (to be used by sledgehammer)
3. Try to find a counterexample to `lem1`
4. If no counterexample is found, close the proof of `lem1` by `sorry`
5. Go back to the proof of `prop1` and check that `lem1` helps
6. If it helps then prove `lem1`. If not try to guess another lemma

To build correct theories, do not confuse `oops` and `sorry`:

- Always close an unprovable property by `oops`
- Always close an unfinished proof of a provable property by `sorry`

Example 10 (Everything is provable using contradictory lemmas)

We can prove that $1 + 1 = 0$ using a false lemma.