Reduced Ordered Binary Decision Diagrams

Lecture #13 of Advanced Model Checking

Joost-Pieter Katoen

Lehrstuhl 2: Software Modeling & Verification

E-mail: `katoen@cs.rwth-aachen.de`

June 5, 2012

Switching functions

- Let *Var* $= \{z_1, \ldots, z_m\}$ be a finite set of Boolean variables
- An evaluation is a function $\eta : \mathit{Var} \rightarrow \{0, 1\}$
  - let *Eval*$(z_1, \ldots, z_m)$ denote the set of evaluations for $z_1, \ldots, z_m$
  - shorthand $[z_1 = b_1, \ldots, z_m = b_m]$ for $\eta(z_1) = b_1, \ldots, \eta(z_m) = b_m$
- $f : \mathit{Eval}(\mathit{Var}) \rightarrow \{0, 1\}$ is a *switching function* for *Var* $= \{z_1, \ldots, z_m\}$
- Logical operations and quantification are defined by:

$$\begin{array}{rcl} f_1(\cdot) \wedge f_2(\cdot) & = & \min\{ f_1(\cdot), f_2(\cdot) \} \\ f_1(\cdot) \vee f_2(\cdot) & = & \max\{ f_1(\cdot), f_2(\cdot) \} \\ \exists z.\, f(\cdot) & = & f(\cdot)|_{z=0} \vee f(\cdot)|_{z=1}, \text{ and} \\ \forall z.\, f(\cdot) & = & f(\cdot)|_{z=0} \wedge f(\cdot)|_{z=1} \end{array}$$

Ordered Binary Decision Diagram

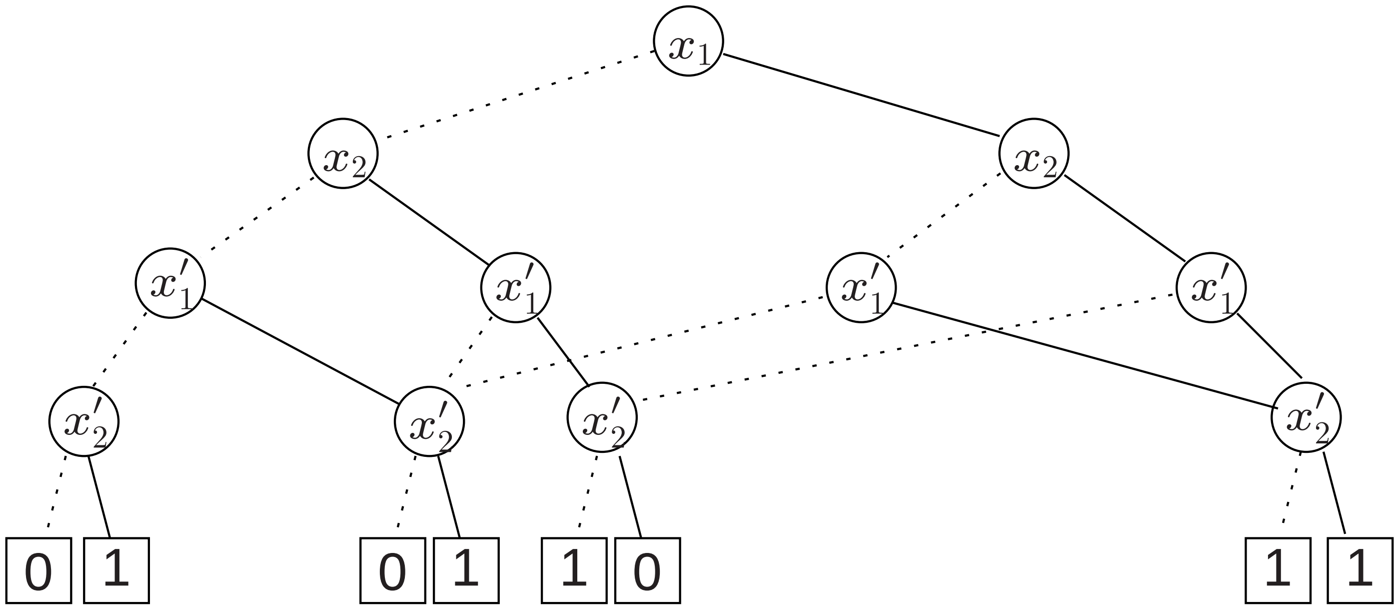
Let $\wp$ be a variable ordering for *Var* where $z_1 <_\wp \ldots <_\wp z_m$

An $\wp$-OBDD is a tuple $\mathfrak{B} = (V, V_I, V_T, \mathit{succ}_0, \mathit{succ}_1, \mathit{var}, \mathit{val}, v_0)$ with

- a finite set V of nodes, partitioned into V_I (inner) and V_T (terminals)
  - and a distinguished root $v_0 \in V$
- successor functions $\mathit{succ}_0, \mathit{succ}_1 : V_I \to V$
  - such that each node $v \in V \setminus \{v_0\}$ has at least one predecessor
- labeling functions $\mathit{var} : V_I \to \mathit{Var}$ and $\mathit{val} : V_T \to \{0, 1\}$ satisfying

$$v \in V_I \;\wedge\; w \in \{\, \mathit{succ}_0(v), \mathit{succ}_1(v) \,\} \cap V_I \;\Rightarrow\; \mathit{var}(v) <_\wp \mathit{var}(w)$$

Transition relation as an OBDD



An example OBDD representing $f_{\rightarrow}$ for our example using $x_1 < x_2 < x_1' < x_2'$

Symbolic composition operators

Consistent co-factors in OBDDs

- Let f be a switching function for *Var*
- Let $\wp = (z_1, \ldots, z_m)$ a variable ordering for *Var*, i.e., $z_1 <_\wp \ldots <_\wp z_m$
- Switching function g is a $\wp$-*consistent cofactor* of f if

$$g = f|_{z_1=b_1,\ldots,z_i=b_i} \quad \text{for some } i \in \{0, 1, \ldots, m\}$$

- Then it holds that:
  1. for each node v of an $\wp$-OBDD $\mathfrak{B}$, f_v is a $\wp$-consistent cofactor of $f_\mathfrak{B}$
  2. for each $\wp$-consistent cofactor g of $f_\mathfrak{B}$ there is a node $v \in \mathfrak{B}$ with $f_v = g$

Reduced OBDDs

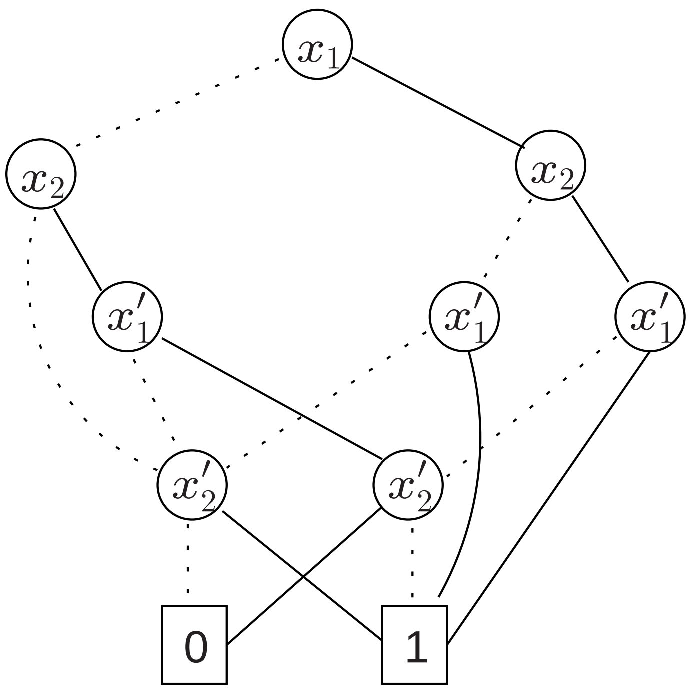
A $\wp$-OBDD $\mathfrak{B}$ is *reduced* if for every pair (v, w) of nodes in $\mathfrak{B}$:

$v \neq w$ implies $f_v \neq f_w$

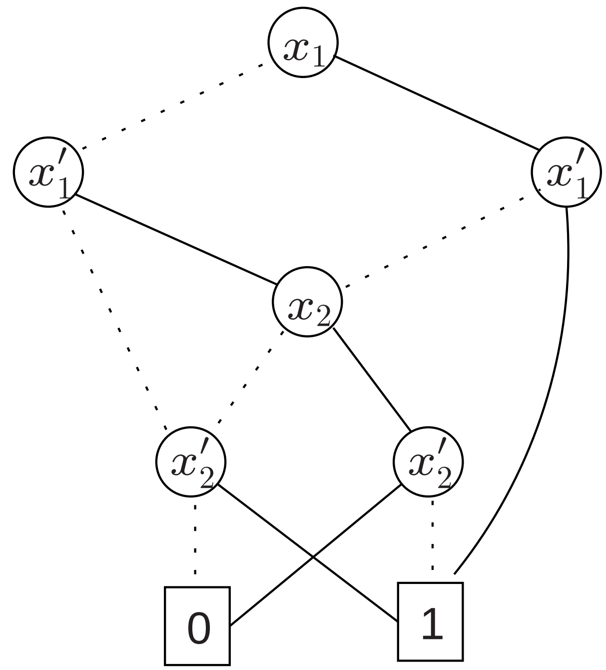
(A reduced $\wp$-OBDD is abbreviated as $\wp$-ROBDD)

$\Rightarrow$ $\wp$-ROBDDs any $\wp$-consistent cofactor is represented by exactly one node

Transition relation as an ROBDD



(a) ordering $x_1 < x_2 < x_1' < x_2'$



(b) ordering $x_1 <' x_1' <' x_2 <' x_2'$

Universality and canonicity theorem

[Fortune, Hopcroft & Schmidt, 1978]

Let *Var* be a finite set of Boolean variables and $\wp$ a variable ordering for *Var*. Then:

(a) For each switching function f for *Var* there exists a $\wp$-ROBDD $\mathfrak{B}$ with $f_{\mathfrak{B}} = f$

(b) Any $\wp$-ROBDDs $\mathfrak{B}$ and $\mathfrak{C}$ with $f_{\mathfrak{B}} = f_{\mathfrak{C}}$ are isomorphic

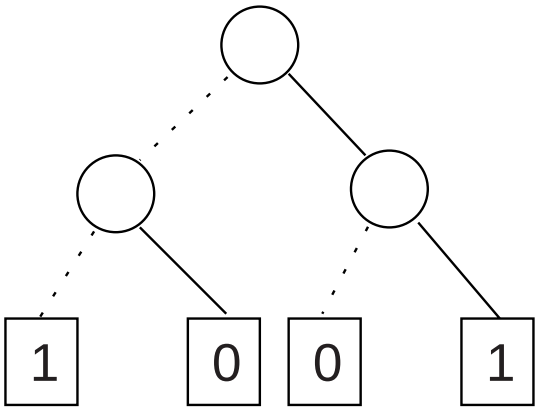
Any $\wp$-OBDD $\mathfrak{B}$ for f is reduced iff *size*$(\mathfrak{B}) \leqslant$ *size*$(\mathfrak{C})$ for each $\wp$-OBDD $\mathfrak{C}$ for f

Reducing OBDDs

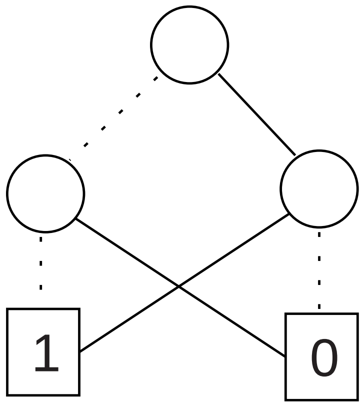
- Generate an OBDD (or BDT) for a switching function, then reduce
  - by means of a recursive descent over the OBDD
- Elimination of duplicate leafs
  - for a duplicate 0-leaf (or 1-leaf), redirect all incoming edges to just one of them
- Elimination of "don't care" (non-leaf) vertices
  - if $\mathit{succ}_0(v) = \mathit{succ}_1(v) = w$, delete v and redirect all its incoming edges to w
- Elimination of isomorphic subtrees
  - if $v \neq w$ are roots of isomorphic subtrees, remove w and redirect all incoming edges to w to v

note that the first reduction is a special case of the latter

How to reduce an OBDD?

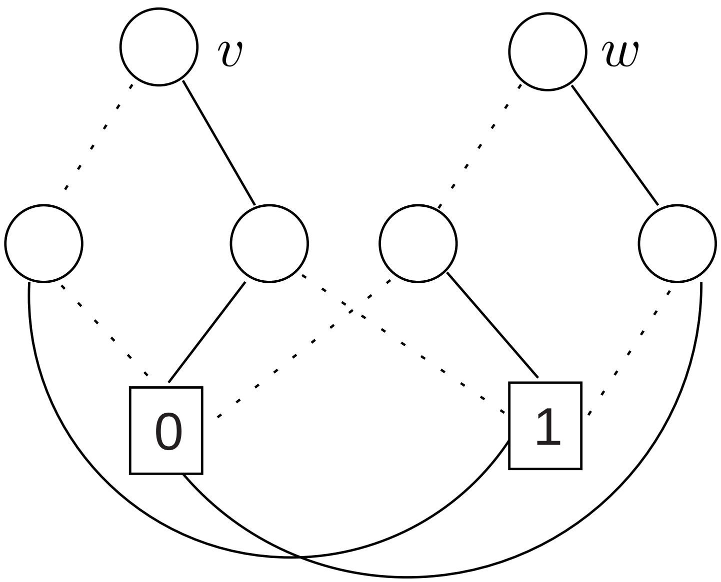


becomes

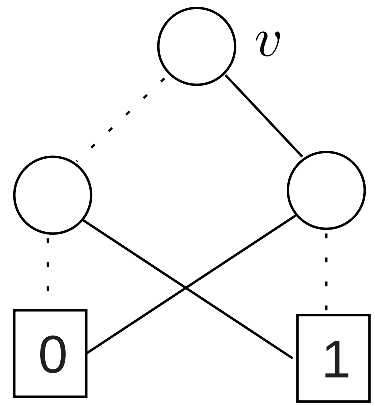


elimination of duplicated leaves

How to reduce an OBDD?

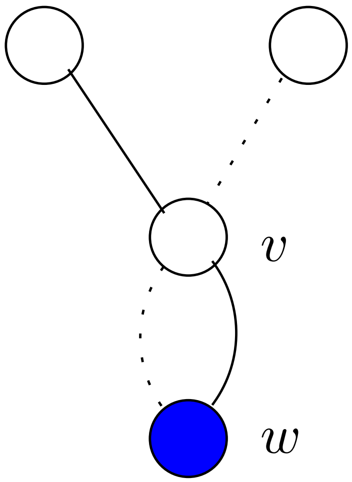


becomes

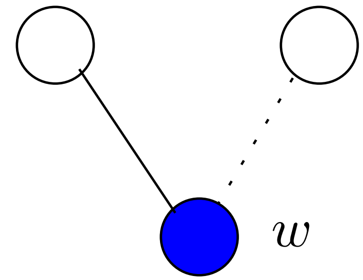


isomorphism rule

How to reduce an OBDD?



becomes



elimination rule

Soundness and completeness

if $\mathfrak{C}$ arises from a $\wp$-OBDD $\mathfrak{B}$ by applying the elimination or isomorphism rule, then:

$\mathfrak{C}$ is a $\wp$-OBDD with $f_{\mathfrak{B}} = f_{\mathfrak{C}}$

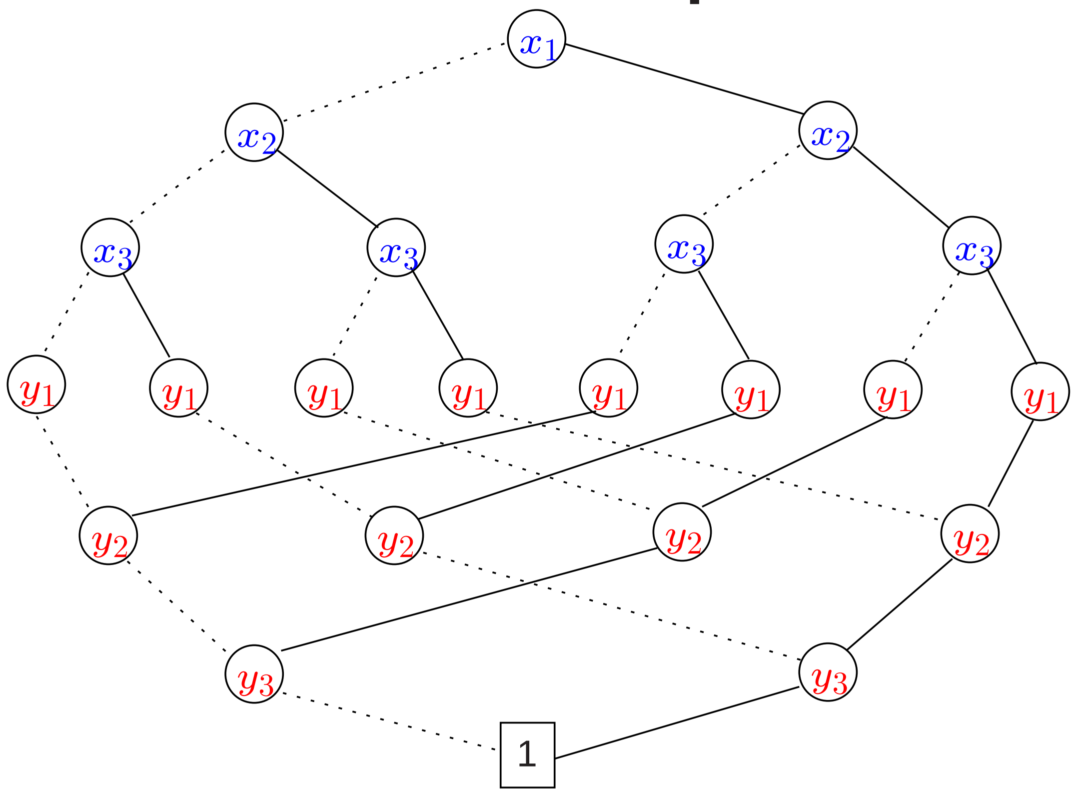
$\wp$-OBDD $\mathfrak{B}$ is reduced if and only if no reduction rule is applicable to $\mathfrak{B}$

Proof

Variable ordering

- ROBDDs are canonical for a fixed variable ordering
  - the size of the ROBDD crucially depends on the variable ordering
  - $\#$ nodes in ROBDD $\mathfrak{B}$ = $\#$ of $\wp$-consistent co-factors of f
- Some switching functions have linear and exponential ROBDDs
  - e.g., the addition function, or the stable function
- Some switching functions only have polynomial ROBDDs
  - this holds, e.g., for symmetric functions (see next)
  - examples $f(\ldots) = x_1 \oplus \ldots \oplus x_n$, or $f(\ldots) = 1$ iff $\geqslant k$ variables x_i are true
- Some switching functions only have exponential ROBDDs
  - this holds, e.g., for the middle bit of the multiplication function

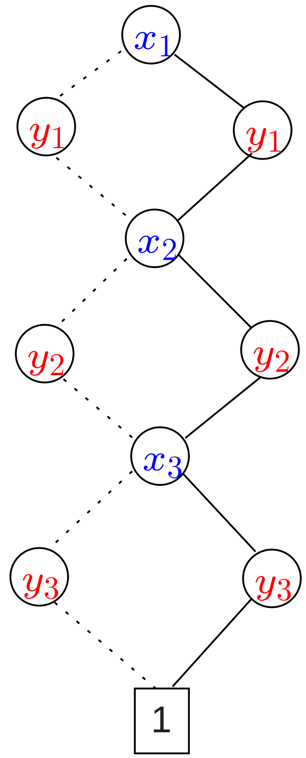
The function stable with exponential ROBDD



The ROBDD of $f_{stab}(\overline{x}, \overline{y}) = (x_1 \leftrightarrow y_1) \;\wedge\; \ldots \;\wedge\; (x_n \leftrightarrow y_n)$

has $3{\cdot}2^n - 1$ vertices under ordering $x_1 < \ldots < x_n < y_1 < \ldots < y_n$

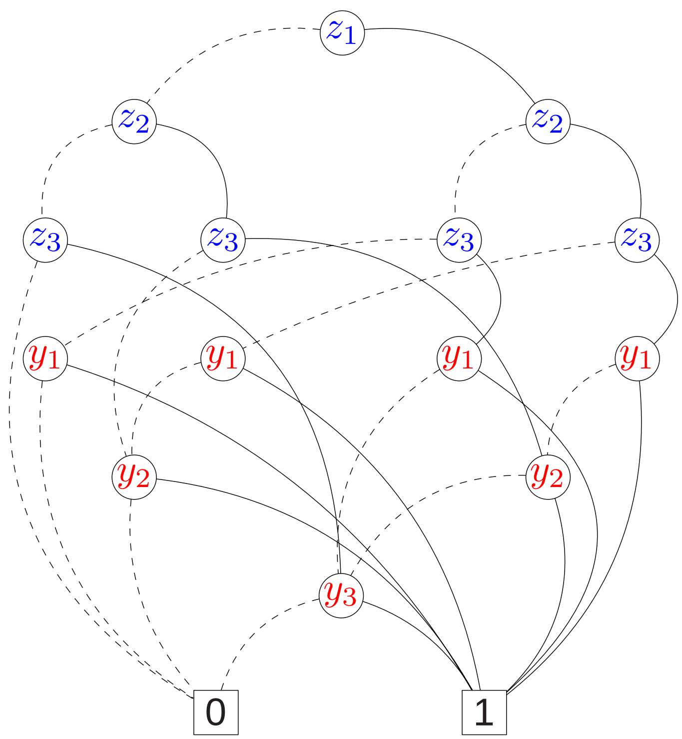
The function stable with linear ROBDD



The ROBDD of $f_{stab}(\overline{x}, \overline{y}) = (x_1 \leftrightarrow y_1) \;\wedge\; \ldots \;\wedge\; (x_n \leftrightarrow y_n)$

has $3 \cdot n + 2$ vertices under ordering $x_1 < y_1 < \ldots < x_n < y_n$

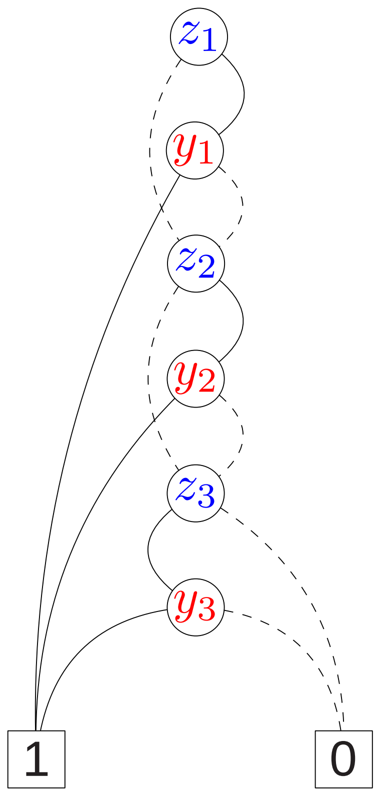
Another function with an exponential ROBDD



ROBDD for $f_3(\overline{z}, \overline{y}) = (z_1 \wedge y_1) \vee (z_2 \wedge y_2) \vee (z_3 \wedge y_3)$

for the variable ordering $z_1 < z_2 < z_3 < y_1 < y_2 < y_3$

And an optimal linear ROBDD



- ROBDD for $f_3(\cdot) = (z_1 \wedge y_1) \vee (z_2 \wedge y_2) \vee (z_3 \wedge y_3)$
- for ordering $z_1 < y_1 < z_2 < y_2 < z_3 < y_3$
- as all variables are essential for f, this ROBDD is optimal
- that is, for no variable ordering a smaller ROBDD exists

Symmetric functions

$f \in \textit{Eval}(z_1, \ldots, z_m)$ is symmetric if and only if

$$f([z_1 = b_1, \ldots, z_m = b_m]) \;=\; f([z_1 = b_{i_1}, \ldots, z_m = b_{i_m}])$$

for each permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$

E.g.: $z_1 \vee z_2 \vee \ldots \vee z_m$, $z_1 \wedge z_2 \wedge \ldots \wedge z_m$, the parity function, and the majority function

> If f is a symmetric function with m essential variables, then
> for each variable ordering $\wp$ the $\wp$-ROBDD has size $\mathcal{O}(m^2)$

The even parity function

$f_{even}(x_1, \ldots, x_n) = 1$ iff the number of variables x_i with value 1 is even

truth table or propositional formula for f_{even} has exponential size

but an ROBDD of linear size is possible

The multiplication function

- Consider two n-bit integers
  - let $b_{n-1}b_{n-2}\ldots b_0$ and $c_{n-1}c_{n-2}\ldots c_0$
  - where b_{n-1} is the most significant bit, and b_0 the least significant bit
- Multiplication yields a $2n$-bit integer
  - the ROBDD $\mathfrak{B}_{f_{n-1}}$ has at least 1.09^n vertices
  - where f_{n-1} denotes the $(n-1)$-st output bit of the multiplication

Optimal variable ordering

- The size of ROBDDs is dependent on the variable ordering

- Is it possible to determine $\wp$ such that the ROBDD has minimal size?
  - to check whether a variable ordering is optimal is NP-hard
  - polynomial reduction from the 3SAT problem [Bollig & Wegener, 1996]

- There are many switching functions with large ROBDDs
  - for almost all switching functions the minimal size is in $\Omega(\frac{2^n}{n})$

- How to deal with this problem in practice?
  - guess a variable ordering in advance
  - rearrange the variable ordering during the ROBDD manipulations
  - not necessary to test all $n!$ orderings, best known algorithm in $\mathcal{O}(3^n \cdot n^2)$

Variable swapping

Sifting algorithm

[Rudell, 1993]

Dynamic variable ordering using variable swapping:

1. Select a variable x_i in OBDD at hand
2. Successively swap x_i to determine *size*$(\mathfrak{B})$ at any position for x_i
3. Shift x_i to position for which *size*$(\mathfrak{B})$ is minimal
4. Go back to the first step until no improvement is made

- Characteristics:
  - a variable may change position several times during a single sifting iteration
  - often yields a local optimum, but works well in practice

Interleaved variable ordering

- Which variable ordering to use for transition relations?
- The interleaved variable ordering:
  - for encodings $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of state s and t respectively:

$$x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n$$

- This variable ordering yields compact ROBDDs for binary relations
  - for transition relation with $z_1 \ldots z_m$ be the encoding of action α, take:

$$\underbrace{z_1 < z_2 < \ldots < z_m}_{\text{encoding of } \alpha} < \underbrace{x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n}_{\text{interleaved order of states}}$$

Symbolic model checking

- Take a symbolic representation of a transition system (Δ and χ_B)
- Backward reachability *Pre*$^*(B) = \{ s \in S \mid s \models \exists \Diamond B \}$
- Initially: $f_0 = \chi_B$ characterizes the set $T_0 = B$
- Then, successively compute the functions $f_{j+1} = \chi_{T_{j+1}}$ for:

$$T_{j+1} \; = \; T_j \cup \{ s \in S \mid \exists s' \in S.\, s' \in \textit{Post}(s) \; \wedge \; s' \in T_j \; \}$$

- The second set i the above union is given by: $\exists \overline{x}'.\, (\underbrace{\Delta(\overline{x}, \overline{x}')}_{s' \, \in \, \textit{Post}(s)} \; \wedge \; \underbrace{f_j(\overline{x}')}_{s' \in T_j})$
  - $f_j(\overline{x}')$ arises from f_j by renaming the variables x_i into their primed copies x_i'

Symbolic computation of *Sat*$(\exists(C \, \mathsf{U} \, B))$

$f_0(\overline{x}) := \chi_B(\overline{x})$;
$j := 0$;
repeat
$\quad f_{j+1}(\overline{x}) := f_j(\overline{x}) \vee \big(\chi_C(\overline{x}) \wedge \exists \overline{x}'. (\Delta(\overline{x}, \overline{x}') \wedge f_j(\overline{x}')) \big)$;
$\quad j := j + 1$
until $f_j(\overline{x}) = f_{j-1}(\overline{x})$;
return $f_j(\overline{x})$.

Symbolic computation of *Sat*$(\exists\Box\, B)$

Compute the largest set $T \subseteq B$ with*Post*$(t) \cap T \neq \varnothing$ for all $t \in T$

Take $T_0 = B$ and $T_{j+1} = T_j \cap \{ s \in S \mid \exists s' \in S.\, s' \in \mathit{Post}(s) \wedge s' \in T_j \}$

Symbolically this amounts to:

$f_0(\overline{x}) := \chi_B(\overline{x})$;
$j := 0$;
repeat
  $f_{j+1}(\overline{x}) := f_j(\overline{x}) \wedge \exists \overline{x}'.\, (\Delta(\overline{x}, \overline{x}') \wedge f_j(\overline{x}'))$;
  $j := j + 1$
until $f_j(\overline{x}) = f_{j-1}(\overline{x})$;
return $f_j(\overline{x})$.

Symbolic model checkers mostly use ROBDDs to represent switching functions

Synthesis of ROBDDs

- Construct a $\wp$-ROBDD for f_1 *op* f_2 given $\wp$-ROBDDs for f_1 and f_2
  - where *op* is a Boolean connective such as disjunction, implication, etc.

- Idea: use a single ROBDD with (global) variable ordering $\wp$ to represent several switching functions

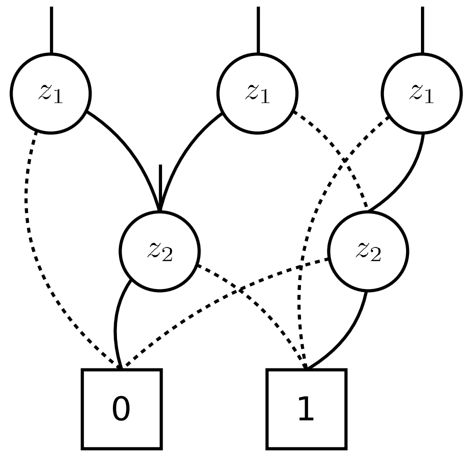
- This yields a shared OBDD, which is:

  a combination of several ROBDDs with variable ordering $\wp$
  by sharing nodes for common $\wp$-consistent cofactors

- The size of $\wp$-SOBDD $\overline{\mathfrak{B}}$ for functions $f_1, \ldots, f_k$ is at most $N_{f_1} + \ldots + N_{f_k}$ where N_f denotes the size of the $\wp$-ROBDD for f

Implementation: shared OBDDs

A shared $\wp$-OBDD is an OBDD with multiple roots



Shared OBDD representing $\underbrace{z_1 \wedge \neg z_2}_{f_1}$, $\underbrace{\neg z_2}_{f_2}$, $\underbrace{z_1 \oplus z_2}_{f_3}$ and $\underbrace{\neg z_1 \vee z_2}_{f_4}$

Main underlying idea: combine several OBDDs with same variable ordering such that common $\wp$-consistent co-factors are shared

Synthesizing shared ROBDDs

Relies on the use of two tables

- The unique table
  - keeps track of ROBDD nodes that already have been created
  - table entry $\langle \mathit{var}(v), \mathit{succ}_1(v), \mathit{succ}_0(v) \rangle$ for each inner node v
  - main operation: $\mathit{find\_or\_add}(z, v_1, v_0)$ with $v_1 \neq v_0$
    - return v if there exists a node $v = \langle z, v_1, v_0 \rangle$ in the ROBDD
    - if not, create a new z-node v with $\mathit{succ}_0(v) = v_0$ and $\mathit{succ}_1(v) = v_1$
  - implemented using hash functions (expected access time is $\mathcal{O}(1)$)

- The computed table
  - keeps track of tuples for which ITE has been executed (memoization)

  $\Rightarrow$ realizes a kind of dynamic programming

Using shared OBDDs for model checking Φ

Use a single SOBDD for:

- $\Delta(\overline{x}, \overline{x}')$ for the transition relation
- $f_a(\overline{x})$, $a \in AP$, for the satisfaction sets of the atomic propositions
- The satisfaction sets *Sat*(Ψ) for the state subformulae Ψ of Φ

In practice, often the interleaved variable order for Δ is used.

ITE normal form

The ITE (if-then-else) operator: $\mathit{ITE}(g, f_1, f_2) \;=\; (g \wedge f_1) \;\vee\; (\neg\, g \wedge f_2)$

The ITE operator and the representation of the SOBDD nodes in the unique table:

$$f_v \;=\; \mathit{ITE}\Big(\, z,\; f_{\mathit{succ}_1(v)},\; f_{\mathit{succ}_0(v)}\,\Big)$$

Then:

$$\begin{aligned}
\neg f &= \mathit{ITE}(f, 0, 1) \\
f_1 \vee f_2 &= \mathit{ITE}(f_1, 1, f_2) \\
f_1 \wedge f_2 &= \mathit{ITE}(f_1, f_2, 0) \\
f_1 \oplus f_2 &= \mathit{ITE}(f_1, \neg f_2, f_2) \;=\; \mathit{ITE}(f_1, \mathit{ITE}(f_2, 0, 1), f_2)
\end{aligned}$$

If g, f_1, f_2 are switching functions for *Var*, $z \in \mathit{Var}$ and $b \in \{0, 1\}$, then

$$\mathit{ITE}(g, f_1, f_2)|_{z=b} \;=\; \mathit{ITE}(g|_{z=b}, f_1|_{z=b}, f_2|_{z=b})$$

ITE-operator on shared OBDDs

- A node in a $\wp$-SOBDD for representing *ITE*(g, f_1, f_2) is a node w with *info*$\langle z, w_1, w_0 \rangle$ where:
  - z is the minimal (wrt. $\wp$) essential variable of *ITE*(g, f_1, f_2)
  - w_b is an SOBDD-node with $f_{w_b} =$ *ITE*$(g|_{z=b}, f_1|_{z=b}, f_2|_{z=b})$

- This suggests a recursive algorithm:
  - determine z
  - recursively compute the nodes for ITE for the cofactors of g, f_1 and f_2

ITE(u, v_1, v_2) on shared OBDDs (initial version)

if u is terminal **then**
  if *val*$(u) = 1$ **then**
    $w := v_1$ (* *ITE*$(1, f_{v_1}, f_{v_2}) = f_{v_1}$ *)
  else
    $w := v_2$ (* *ITE*$(0, f_{v_1}, f_{v_2}) = f_{v_2}$ *)
  fi
else
  $z := \min\{var(u), var(v_1), var(v_2)\}$; (* minimal essential variable *)
  $w_1 := ITE(u|_{z=1}, v_1|_{z=1}, v_2|_{z=1})$;
  $w_0 := ITE(u|_{z=0}, v_1|_{z=0}, v_2|_{z=0})$;
  if $w_0 = w_1$ **then**
    $w := w_1$; (* elimination rule *)
  else
    $w := find\_or\_add(z, w_1, w_0)$; (* isomorphism rule *)
  fi
fi
return w

ROBDD size under ITE

The size of the $\wp$-ROBDD for *ITE*(g, f_1, f_2) is bounded by $N_g \cdot N_{f_1} \cdot N_{f_2}$
where N_f denotes the size of the $\wp$-ROBDD for f

ROBDD size under ITE

The size of the $\wp$-ROBDD for *ITE*(g, f_1, f_2) is bounded by $N_g \cdot N_{f_1} \cdot N_{f_2}$
where N_f denotes the size of the $\wp$-ROBDD for f

But how to avoid multiple invocations to ITE?

$\Rightarrow$ Store triples (u, v_1, v_2) for which ITE already has been computed

Efficiency improvement by memoization

if there is an entry for (u, v_1, v_2, w) in the computed table **then**
  return node w
else
  if u is terminal **then**
    if $\mathit{val}(u) = 1$ **then** $w := v_1$ **else** $w := v_2$ **fi**
  else
    $z := \min\{\mathit{var}(u), \mathit{var}(v_1), \mathit{var}(v_2)\}$;
    $w_1 := \mathit{ITE}(u|_{z=1}, v_1|_{z=1}, v_2|_{z=1})$;
    $w_0 := \mathit{ITE}(u|_{z=0}, v_1|_{z=0}, v_2|_{z=0})$;
    if $w_0 = w_1$ **then** $w := w_1$ **else** $w := \mathit{find\_or\_add}(z, w_1, w_0)$ **fi**;
    insert (u, v_1, v_2, w) in the computed table;
    return node w
  fi
fi

The number of recursive calls for the nodes u, v_1, v_2 equals the $\wp$-ROBDD size of $\mathit{ITE}(f_u, f_{v_1}, f_{v_2})$, which is bounded by $N_u \cdot N_{v_1} \cdot N_{v_2}$

Some experimental results

- Traffic alert and collision avoidance system (TCAS) (1998)
  - 277 boolean variables, reachable state space is about 9.610^{56} states
  - $|\mathrm{B}| = 124,618$ vertices (about 7.1 MB), construction time 46.6 *sec*
  - checking $\forall\Box\,(p \rightarrow q)$ takes 290 sec and 717,000 BDD vertices
- Synchronous pipeline circuit (1992)
  - pipeline with 12 bits: reachable state space of 1.510^{29} states
  - checking safety property takes about $10^4 - 10^5$ *sec*
  - $|\mathrm{B}_{\rightarrow}|$ is linear in data path width
  - verification of 32 bits (about 10^{120} states): 1h 25m
  - using partitioned transition relations

Some other types of BDDs

- Zero-suppressed BDDs
  - like ROBDDs, but non-terminals whose 1-child is leaf 0 are omitted
- Parity BDDs
  - like ROBDDs, but non-terminals may be labeled with $\oplus$; no canonical form
- Edge-valued BDDs
- Multi-terminal BDDs (or: algebraic BDDs)
  - like ROBDDs, but terminals have values in $\mathbb{R}$, or $\mathbb{N}$, etc.
- Binary moment diagrams (BMD)
  - generalization of ROBDD to linear functions over bool, int and real
  - uses edge weights

Further reading

- R. Bryant: Graph-based algorithms for Boolean function manipulation, 1986
- R. Bryant: Symbolic boolean manipulation with OBDDs, Computing Surveys, 1992
- M. Huth and M. Ryan: Binary decision diagrams, Ch 6 of book on Logics, 1999
- H.R. Andersen: Introduction to BDDs, Tech Rep, 1994
- K. McMillan: Symbolic model checking, 1992
- Rudell: Dynamic variable reordering for OBDDs, 1993

Advanced reading: Ch. Meinel & Th. Theobald (Springer 1998)