RECALL ON REGULAR EXPRESSIONS

Fix an alphabet $\Sigma = \{A, B, \ldots\}$

Syntax of regular expressions (over $\Sigma$):

$E, E' : = \emptyset \mid \epsilon \mid A \mid E + E' \mid E \cdot E' \mid E^*$ where $A \in \Sigma$

Semantics of regular expressions

Set of finite words

$E \mapsto \mathcal{Z}(E) \subseteq \Sigma^*$

$\mathcal{Z}(\emptyset) = \emptyset$ $\quad \mathcal{Z}(\epsilon) = \{\epsilon\}$ $\quad \mathcal{Z}(A) = \{A\}$

$\mathcal{Z}(E + E') = \mathcal{Z}(E) \cup \mathcal{Z}(E')$ (union)

$\mathcal{Z}(E \cdot E') = \mathcal{Z}(E) \circ \mathcal{Z}(E')$ (concatenation)

$\mathcal{Z}(E^*) = \mathcal{Z}(E)^*$ (Kleene closure)

Exercise

Formally define $L \cdot L'$ and $L^*$, for arbitrary $L, L' \subseteq \Sigma^*$
$\omega$ - REGULAR EXPRESSIONS

$\omega$ "Omega"

Kleene star for "finite repetitions" $E^*$

$\omega$-operator for "infinite repetition" $E^\omega$

Given a language $L \subseteq \Sigma^*$ (of finite words over $\Sigma$)

define $L^\omega \overset{\Delta}{=} \{ w_0 w_1 w_2 \ldots \mid w_i \in L \text{ for all } i \in \mathbb{N} \}$

Exercise

$\{A, B\}^\omega$?

$\{AB, A\}^\omega$?

Remark

If $\epsilon \in L$, then $L \subseteq \Sigma^\omega$. 

**W-REGULAR EXPRESSIONS**

Syntax of w-regular expressions (over \( \Sigma \))

An \( w \)-regular expression \( G \) has the form

\[
G = E_1 \cdot F_1^w + E_2 \cdot F_2^w + \ldots + E_m \cdot F_m^w
\]

where \( m \geq 1 \) and \( E_1, \ldots, E_m, F_1, \ldots, F_m \) are regular expressions over \( \Sigma \), such that \( E \notin \mathcal{Z}(F_i) \) for all \( 1 \leq i \leq m \).

Semantics of \( w \)-regular expressions (Set of infinite words)

\[
G \mapsto \mathcal{Z}_w(G) \subseteq \Sigma^w
\]

The semantics of the \( w \)-regular expression \( G \) is

\[
\mathcal{Z}_w(G) = \mathcal{Z}(E_1)^w \cdot \mathcal{Z}(F_1)^w \cup \ldots \cup \mathcal{Z}(E_m)^w \cdot \mathcal{Z}(F_m)^w
\]

**DEF Two \( w \)-regular expressions \( G_1 \) and \( G_2 \) are equivalent if**

\[
\mathcal{Z}_w(G_1) = \mathcal{Z}_w(G_2)
\]
EXAMPLES OF $\omega$-REGULAR EXPRESSIONS

- The semantics of $(A^*B)^\omega$ is the language of all infinite words over $\Sigma = \{A, B\}$ containing infinitely many B's.

- The semantics of $(A^*B)^\omega + (B^*A)^\omega$ is the language of all infinite words over $\Sigma = \{A, B\}$ containing infinitely many B's or infinitely many A's. It is equivalent to the $\omega$-regular expression $\Sigma^\omega$.

Exercise

Try some expressions and determine their semantics.
A language \( L \subseteq \Sigma^w \) is \( w \)-regular if there exists an \( \omega \)-regular expression \( G \) such that \( L_\omega(G) = L \).

Let \( AP = \{ a_1, b_1, \ldots \} \) be a set of atomic propositions.

A linear-time property is a subset \( P \) of \((\Sigma^*)^w\).

It is \( w \)-regular if there exists an \( \omega \)-regular expression \( G \) over \( \Sigma^AP \) (\( = \Sigma \)) such that \( P = L_\omega(G) \).
INTERMEZZO ON \(\omega\)-REGULAR EXPRESSIONS

Semaphore-based mutual exclusion algorithm for 2 processes

Use two atomic propositions: \(AP = \{\text{crit}_1, \text{crit}_2\}\)

\(\text{crit}_1\) (resp. \(\text{crit}_2\)) hold when process \(\text{Proc}_1\) (resp. \(\text{Proc}_2\)) is in its critical section

Execution traces of this algorithm are for instance:
\[\phi \land \text{crit}_1 \land \phi \land \text{crit}_2 \land \phi \land \{\text{crit}_3\} \subseteq (\phi \{\text{crit}_1, \text{crit}_2\}\omega \phi \{\text{crit}_3\})\]

Exercise for others...
EXAMPLES OF \( \omega \)-REGULAR PROPERTIES

Fix \( AP = \{a, b\} \), and consider alphabet \( \mathcal{L}^A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \)

- The expression \((\emptyset + \{a\} + \{a, b\})\) denote words where letter \{b\} never occurs. It corresponds to the invariant \( a^* b^* \)

\( \emptyset \) means \( a^\omega b^\omega \)
\( \{a\} \) means \( a a^\omega b^\omega \)
\( \{b\} \) means \( a^\omega b b^\omega \)
\( \{a, b\} \) means \( a a^\omega b b^\omega \)

We may write \((a a^\omega b^\omega)^\omega\) instead of \((\emptyset + \{a\} + \{a, b\})^\omega\)

- "Infinitely often a" \((1a^*)^\omega\)
- "Always a" \(a^\omega\)
- "Eventually a" \(true^* a^* true^*\)
- "From some moment on a" \(true^* a^\omega\)
BÜCHI AUTOMATA

Introduced by Julius Richard Büchi (1924-1984), a Swiss mathematician

"On a decision method in restricted second order arithmetic"

[Please have a look at Wikipedia]

DEF A non-deterministic Büchi automaton (NBA) is a structure $A = (Q, \Sigma, \delta, Q_0, F)$ where

- $Q$ is a finite set of states
- $\Sigma$ is an alphabet
- $\delta : Q \times \Sigma \to 2^Q$ is the transition relation
- $Q_0 \subseteq Q$ is a set of initial states
- $F \subseteq Q$ is a set of accepting states

DEF Let $|A|$ be the size of $A$, i.e., $|Q| + |\Sigma| + |F|$
RUNS OF NBA

DEF

A run of an NBA \( M = (Q, \Sigma, \delta, Q_0, F) \) over an input word \( A_0 A_1 A_2 \ldots \in \Sigma^* \) is an infinite sequence \( q_0, q_1, q_2, \ldots \) of states such that \( q_0 \in Q_0 \) and \( q_i \xrightarrow{A_i} q_{i+1} \) for every \( i \in \mathbb{N} \).

A run \( q_0 q_1 q_2 \ldots \) is accepting if there are infinitely many \( q_i \)'s in \( F \).

DEF

The accepted language of \( M \) is

\[ L_\omega(M) = \{ \omega \in \Sigma^* \mid \text{there is an accepting run on } \omega \text{ in } M \} \]
GRAPHICAL REPRESENTATION OF NBA

What is the accepted language?

"Symbolic" versions (i.e., with $2 = 2^{|A|}$)

For language $(a^*b^*c^*)_x$, true
GRAPHICAL REPRESENTATION OF NBA

EXERCISE

1. Draw a NBA for property
   - Always and

- Infinitely often a and always and
A Kleene Theorem for \(\omega\)-Regular Languages
From NBA to \(\omega\)-Regular Expressions

**Theorem**

For every NBA \(\mathcal{A}\), there exists an \(\omega\)-regular expression \(G\) such that

\[
L_{\omega}(G) = L_{\omega}(\mathcal{A})
\]

**Proof**

Let \(\mathcal{A} = (Q, \Sigma, \delta, q_0, F)\) be an NBA.

For any \(q, p \in Q\), define \(\mathcal{A}_{q,p} = (Q, \Sigma, \delta, \{q\}, \{p\})\), a finite state automaton where \(q\) is its initial state and \(p\) is its final state.

Let \(L_{q,p} = L(\mathcal{A}_{q,p}) \subseteq \Sigma^\ast\).

**Claim**

\[
L_{\omega}(\mathcal{A}) = \bigcup_{q \in Q_0} \bigcup_{p \in F} L_{q,p} \cdot (L_{p,p} \setminus \{\varepsilon\})^\omega
\]

**Exercise** Prove the claim and conclude the proof.
EXAMPLE

\[
\begin{align*}
    \delta & : \\
    1 & \rightarrow 2 \\
    2 & \rightarrow 1 \\
    2 & \rightarrow 3 \\
    3 & \rightarrow 3 \\
    A & \rightarrow A \\
    A & \rightarrow B \\
    C & \rightarrow C \\
    \end{align*}
\]

\[
\begin{align*}
    L_{12} &= A L_{22} \\
    L_{22} &= A L_{32} + B L_{12} + \varepsilon \\
    L_{32} &= A L_{12} \\
\end{align*}
\]

\[
\begin{align*}
    Z_w(\delta) &= L_{12}(L_{22} \cdot \{ \varepsilon \})^w + L_{22}(L_{22} \cdot \{ \varepsilon \})^w \\
    &= A(A^3 + BA)^w + (A^3 + BA)^w \\
    &= (A + \varepsilon)(A^3 + BA)^w
\end{align*}
\]

EXERCISE:

Solve this algebraic system for regular languages.

You should find:

\[
\begin{align*}
    L_{12} &= (A^3 + BA)^* \\
    L_{22} \cdot \{ \varepsilon \} &= (A^3 + BA)^+ \\
    L_{32} &= A(A^3 + BA)^* \\
\end{align*}
\]
A Kleene Theorem for \( \omega \)-Regular Languages

From \( \omega \)-Regular Expressions to NBA

**Theorem**

For every \( \omega \)-regular expression

\[
G = E_1 \cdot F^\omega_1 + \ldots + E_m \cdot F^\omega_m
\]

there exists an NBA \( \mathcal{A}_G \) such that \( \mathcal{L}_\omega(G) = \mathcal{L}_\omega(\mathcal{A}_G) \)

**Proof**

Let \( \mathcal{A}_i \) be a finite-state automaton for \( E_i \).

\( \mathcal{A}_i = \quad \text{NFA} \)

We establish the following construction (for each \( 1 \leq i \leq m \))

1. We build an NBA \( \mathcal{A}_{i \cdot F}^\omega \) for \( F^\omega_i \)
2. We build an NBA \( \mathcal{C}_i \) for \( E_i \cdot F^\omega_i \)

Then we build the NBA \( \mathcal{A}_G \) corresponding to \( \bigcup_{1 \leq i \leq m} \mathcal{L}_\omega(\mathcal{C}_i) \)
AN NBA FOR THE \( \omega \)-CLOSURE OF A REGULAR LANGUAGE

**Lemma**

For every (non-deterministic) finite automaton \( \mathcal{A} \) such that \( \mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A_\omega}) \), we can effectively build an NBA \( \mathcal{A_\omega} \) such that

\[
\mathcal{L}(\mathcal{A_\omega}) = \mathcal{L}(\mathcal{A})^\omega
\]

Moreover, \( |\mathcal{A_\omega}| = O(|\mathcal{A}|) \)

**Proof** Without loss of generality, we can assume that \( \mathcal{A} \) has a unique initial state \( q_0 \).

**Exercise**

Define \( \mathcal{A_\omega} = (Q, \Sigma, \delta', \{q_0\}, \{q_0\}) \) by

\[
\delta'(q, A) = \begin{cases} 
\delta(q, A) & \text{if } \delta(q, A) \cap F = \emptyset \\
\delta(q, A) \cup \{q_0\} & \text{otherwise}
\end{cases}
\]

**Claim** \( \mathcal{L}(\mathcal{A_\omega}) = \mathcal{L}(\mathcal{A})^\omega \)
PROOF OF THE CLAIM \( Z^\omega (x^\omega) = Z^\omega (t^\omega) \)
Now it satisfies the hypothesis.

Let it be given by:

Remark: $q_b$ becomes useless.
CONCATENATION OF AN NFA AND AN NBA

LEMMA

For an NFA \(\mathcal{M}\) and an NBA \(\mathcal{M}'\) (both over alphabet \(\Sigma\)), there exists an NBA \(\mathcal{M}''\) such that

\[ L_\omega (\mathcal{M}) = L(\mathcal{M}) \cdot L_\omega (\mathcal{M}') \]

Moreover \(|\mathcal{M}''| = O(|\mathcal{M}| + |\mathcal{M}'|)\).

(PROOF) Let \(\mathcal{M} = (Q, \Sigma, \delta, Q_0, F)\) and \(\mathcal{M}' = (Q', \Sigma, \delta', Q'_0, F')\) with \(Q \cap Q' = \emptyset\).

Define \(\mathcal{M}'' = (Q'', \Sigma, \delta'', Q''_0, F'')\) where

\[ Q''_0 = \begin{cases} Q_0 & \text{if } Q_0 \cap F = \emptyset \\ Q_0 \cup Q'_0 & \text{otherwise} \end{cases} \]

\[ F'' = F' \]

\[ \delta''(q, a) = \begin{cases} \delta(q, a) & \text{if } q \in Q \text{ and } \delta(q, a) \cap F = \emptyset \\ \delta(q, a) \cup Q'_0 & \text{if } q \in Q \text{ and } \delta(q, a) \cap F \neq \emptyset \end{cases} \]

\[ \delta'(q, a) = \delta'(q, a) \]

EXERCISE: Check that \(\mathcal{M}''\) fulfills the lemma.
UNION OF NBA's

**LEMMA**

For NBA $ct_1$ and $ct_2$ (both over alphabet $\Sigma$) there exists an NBA $ct_0$ such that

$$L_\omega (ct_0) = L_\omega (ct_1) \cup L_\omega (ct_2)$$

Moreover, $|ct_0| = O(|ct_1| + |ct_2|)$

**PROOF**

Assume $Q_1 \cap Q_2 = \emptyset$

Define

$$ct_0 = (Q_1 \cup Q_2, \Sigma, \delta, Q_{10} \cup Q_{20}, F_1 \cup F_2)$$

where $\delta(q, A) = \delta_i(q, A)$ if $q \in Q_i$ for $i = 1, 2$

**EXERCISE** Check that $ct_0$ satisfies the lemma.
NON-EMPTINESS OF NBA

**Lemma**

Let $M = (Q, \Sigma, \delta, Q_0, F)$ be an NBA.

The following statements are equivalent:

(a) $L_M(M) \neq \emptyset$

(b) There exists a reachable accepting state $q$ that belong to a cycle in $M$.

Formally:

$$\exists q_0 \in Q_0, \exists q \in F, \exists w \in \Sigma^*, \exists v \in \Sigma^+, q \in \delta^*(q_0, w) \cap \delta^*(q, v)$$

\[ \begin{array}{c}
\xrightarrow{w} q_0 \\
\xrightarrow{v} q
\end{array} \]

The word $wv^\omega$ is ultimately periodic.
Proof of the Lemma

(a) ⇒ (b)

Assume \( L_{2,\omega}(\mathcal{A}) \neq \emptyset \), and let \( \pi = q_0 q_1 q_2 \ldots \) be an accepting run of \( \mathcal{A} \).

Necessarily, there exists \( q \in F \) such that \( q = q_i \) for infinitely many \( i \)'s.

Let \( 0 \leq i < j \) be such that \( q_i = q_j = q \), and consider run

\[ \pi' = q_0 \ldots q_{i-1} (q_i \ldots q_{j-1}) w \]

\( \pi' \) is accepting and it is easy to pick two words \( w, v \in \Sigma^* \) such that

\[ q_i \in \delta^*(q_0, w) \cap \delta^*(q_i, v) \]

Hence (b) holds.
PROOF OF THE LEMMA

(b) \implies (a)

Let \( q_0, q_1, w, v \) be as in statement (b).

Then there a run

\[ \pi = q_0 \ldots q \ldots q \ldots q \ldots \text{ over } w v w \]

that visits \( q \in F \) infinitely often.

\( \pi \) is therefore accepting

which entails

\[ z_w (c_0) \neq \emptyset \]
NON-EMPTINESS PB FOR NBA

Input: a NBA \( \Delta \)

Output: "Yes" if \( L(\Delta) \neq \emptyset \)

"No" otherwise.

THEO

The non-emptiness for NBA can be solved in linear time.

[It is actually \textit{NL}-complete.]

(\textit{PROOF}) We use ultimately periodic word Lemma:

Search in the automaton graph if there exists a Strongly Connected Component which can be reached from \( q_0 \).