Secure Communication in Networks: Mechanism Design, Games and Cryptography

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References

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Mechanism design

A *social environment* $\mathscr{E}$ is:

- A set of players $N := \{1, \ldots, n\}$;
- A finite set of feasible alternatives A;
- A finite set Θ_i of types of player i;
- A belief $P_i(\cdot|\theta_i)$ over Θ_{-i};
- A utility function $u_i : A \times \Theta \longrightarrow \mathbb{R}$ for each player i.

We assume $P_i(\theta_{-i}|\theta_i) > 0$ for all $(\theta_i, \theta_{-i}) \in \Theta$ and for all $i \in N$.

- The environment has a *worst outcome* if there exists $\underline{a} \in A$ such that for each player i, each type profile θ and each $a \in A \setminus \{\underline{a}\}$, $u_i(\underline{a}, \theta) < u_i(a, \theta)$.

Social choice functions

A social choice function is a mapping $f : \Theta \rightarrow A$.

Such a function is incentive compatible if $\forall i, \theta_i, \theta_i'$,

$$\sum_{\theta_{-i}} u_i(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) P_i(\theta_{-i}|\theta_i) \geq \sum_{\theta_{-i}} u_i(f(\theta_i', \theta_{-i}), \theta_i, \theta_{-i}) P_i(\theta_{-i}|\theta_i)$$

This is an equilibrium outcome of a game of direct communication between the players and the designer.
Examples:
-Dictatorial s.c.f.
-2nd price auction.

Communication networks

- A communication network is a directed graph with $n+1$ vertices representing the n players and the designer (player 0). There is a directed edge ij from player i to player j, if i can send a message to j securely and privately.
- $C(i) = \{j \in N \cup \{0\} : ij \in \mathcal{N}\} = \{\textit{successors of } i\}$
- $D(i) = \{j \in N \cup \{0\} : ji \in \mathcal{N}\} = \{\textit{predecessors of } i\}$

We assume that the network is strongly 1-connected (for each $i \in N$, there exists a directed path from i to 0) and *acyclic*. Each player speaks only once and the designer sends no message.

Mechanisms on communication networks

Given a network $\mathcal{N}$, a mechanism is a pair $\langle (M_{ij})_{ij\in\mathcal{N}}, g\rangle$ where for each edge ij, M_{ij} is the set of messages that player i can send to player j, and $g : \times_{i\in D(0)} M_{i0} \to A$ is the allocation rule. The associated game unfolds as follows.

- Each player i reads the messages he receives from players in $D(i)$. Then he send messages to players in $C(i)$ (he may send different messages to different players).
- The designer reads the messages he receives from players in $D(0)$, and selects an outcome accordingly.

Implementation on a network

Let $\mathbb{P}_{\sigma,\theta}$ be the probability distributions over messages induced by the strategy σ at state θ.

Definition

The mechanism $\langle (M_{ij})_{ij\in\mathcal{N}}, g\rangle$ implements f on $\mathcal{N}$ if there exists a Bayesian-Nash equilibrium σ^* such that:

$$g((m^*_{i0})_{i\in D(0)}) = f(\theta)$$

for all $\theta \in \Theta$ and for all $(m^*_{i0})_{i\in D(0)}$ in the support of $\mathbb{P}_{\sigma^*,\theta}$.

Let $F_{\mathcal{N}}(\mathcal{E})$ be the set of functions partially implementable on $\mathcal{N}$ when the environment is $\mathcal{E}$.

From the revelation principle: $F_{\mathcal{N}}(\mathcal{E}) \subseteq IC$ and $F_{\mathcal{N}^\star}(\mathcal{E}) = IC$ (all players directly connected to 0).

The goal of this project is to give conditions on $\mathcal{N}$ for which $F_{\mathcal{N}}(\mathcal{E}) = IC$, $\forall \mathcal{E}$.

Reformulation: Secret information transmission

A necessary condition for implementation of all IC function is the following.

Definition

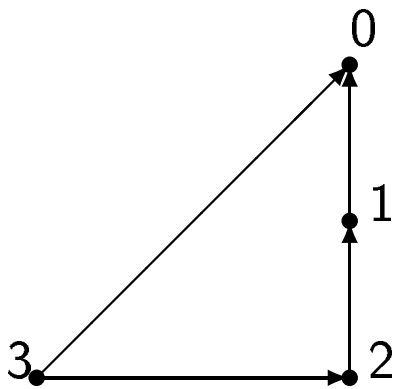
Secret information transmission is possible on the network if there exists a communication protocol such that:
-The designer correctly learns all types,
-No player gets information beyond his own type. I.e. the messages received by j are probabilistically independent from θ_i.

Example: a contribution game (provision of a public good).

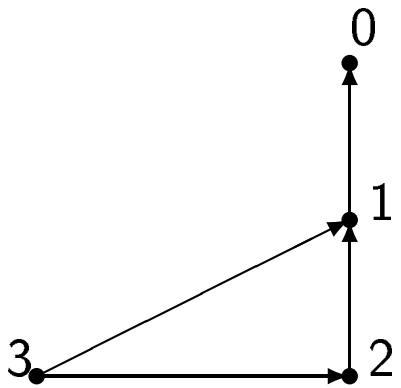
Definition

The communication network $\mathcal{N}$ is *weakly 2-connected* if for each player $i \in N \setminus D(0)$, there exist two disjoint paths (not necessarily directed) π_i^1 and π_i^2 from player i to the designer.

Example 1: A network weakly 2-connected.



Example 2: A network not weakly 2-connected: player 1 "controls" all information.



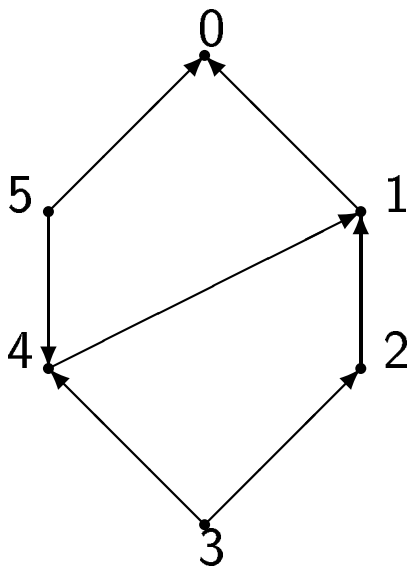
Worst outcome

Theorem (Renou-T09)

-In all environments $\mathscr{E}$ with a worst outcome, $F_{\mathscr{N}}(\mathscr{E}) = IC$ if and only if $\mathscr{N}$ is weakly 2-connected.
-Secret information transmission is possible if and only if $\mathscr{N}$ is weakly 2-connected.

An example

Consider the following network:



- All directed paths from the unique predecessor of 2 (player 3) to the designer, go through player 1.
- Secret information transmission is possible and all IC functions are implementable (under worst outcome).

Probabilistic coding

Assume w.l.o.g. $\Theta_i \subseteq F_2^n \setminus \{0\}$.

- Let X, Y be independent r.v.'s variables in F_2^n. If X is uniformly distributed, so is $X + Y$.
- Let $X_1, \ldots, X_K, \theta$ be independent r.v.'s such that all X_k's are uniformly distributed. Let

$$Y_j \in \text{vect}\{X_1, \ldots, X_K, \theta\}, j = 1, \ldots, J$$

If $\theta \notin \text{vect}\{Y_1, \ldots, Y_J\}$, then $(Y_1, \ldots, Y_J)$ is independent from θ.

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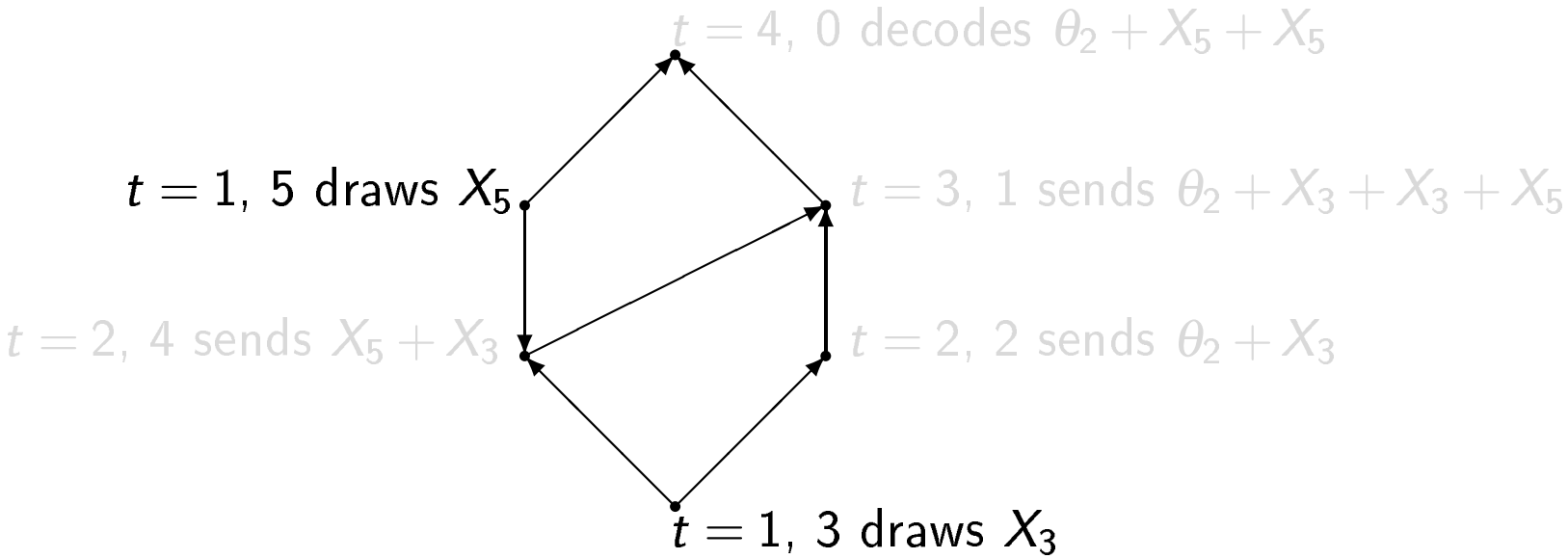

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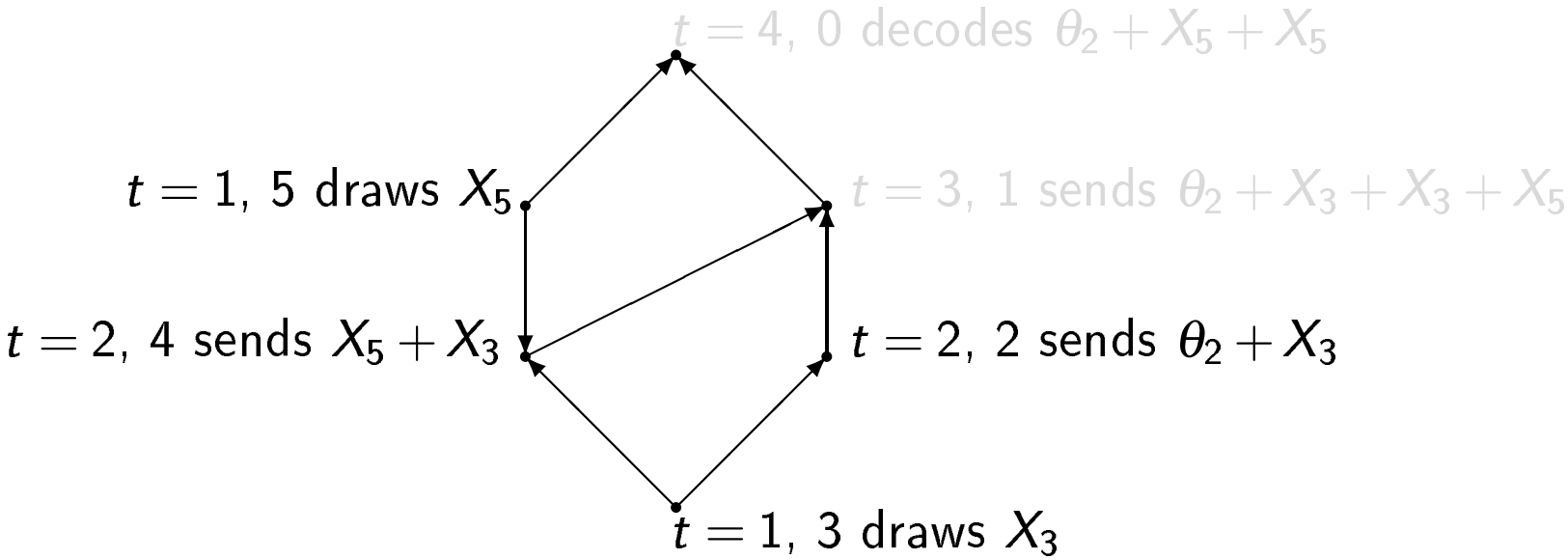

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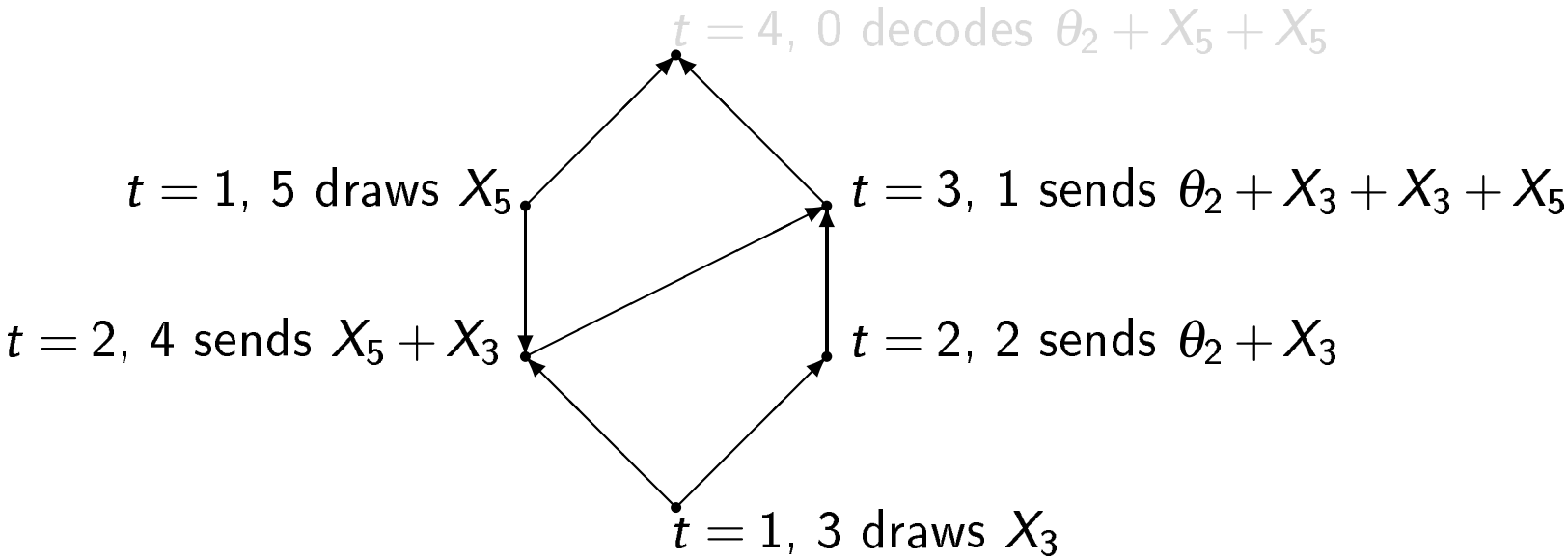

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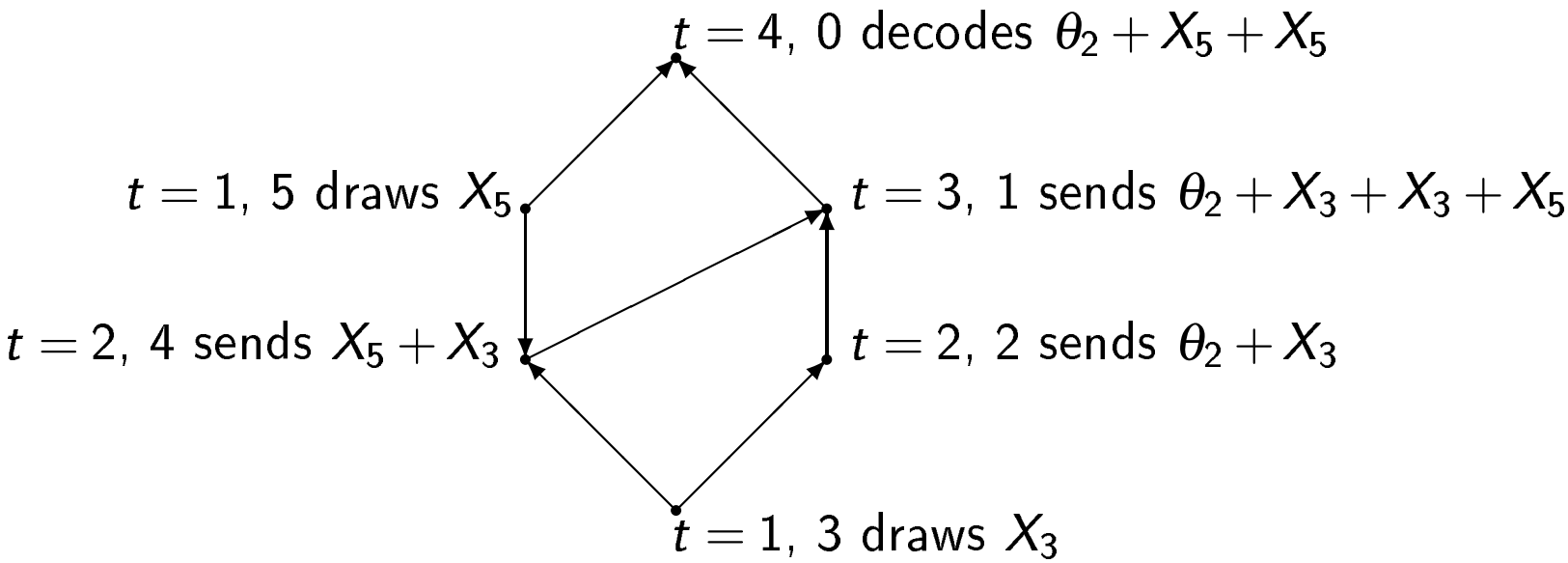


Detering deviations

This allows secret information transmission. To ensure implementation under worst outcome, the designer must *detect* deviations.

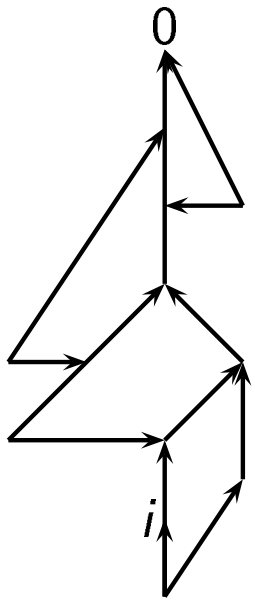
- Authentication scheme (check-vectors in Rabin and Ben-or).
  Assume strong 2-connectedness.
  - Sender draws keys A, B, C, D, X and sets $Y = X + \theta$, $U = AX + B$, $V = CY + D$.
  - Sender sends (X, U, C, D) on one path and (Y, V, A, B) on the other.
  - Receiver checks $U = AX + B$ and $V = CY + D$. If he accepts, he decodes $\theta = X + Y$. Otherwise, he punishes.
  The probability to tamper with the message AND to pass the test is small.

Detering deviations

- Without strong 2-connectedness (as in example), run in parallel a large number independent copies of the previous protocol and let the Sender (player 2): -select one of these protocols at random, -input his type in the selected protocol, -input 0 in all others. Receiver accepts if exactly one protocol has a non-zero output and punishes otherwise.

The general proof

We show that a directed graph which is acyclic, strongly 1-connected and weakly 2-connected contains subgraphs of the type:



Secret information transmission: general case

Fix a sender i and the receiver 0. Let k be an integer.

Definition

k-Secret information transmission from i to 0 is possible on the network if there exists a communication protocol such that:
- 0 correctly learns θ_i,
- For every $S \subseteq N \setminus i$, $|S| \leq k$, the joint messages received by S are probabilistically independent from θ_i.

Information transmission against an "honest but curious" adversary.

Secret information transmission: general case

Assume that the graph is strongly 1-connected. The graph is $k+1-C(i,0)$ if there are $k+1$ disjoint undirected paths from i to 0.

Theorem (Renault-Renou-T10)

k-Secret information transmission from i to 0 is possible IFF $i \in D(0)$ or the graph is $k+1-C(i,0)$.

The condition is clearly necessary.

Secret information transmission: general case

Secret sharing

Let θ be a random variable in F_2^n.
For each n, there exists $Y_1, \ldots, Y_n$ such that:

- $\theta = \sum_{i=1}^{n} Y_i$,
- For each i, $(Y_j)_{j \neq i}$ is independent from θ.

Draw $Y_1, \ldots, Y_{n-1}$ i.i.d. uniform and set $Y_n := \theta + \sum_{i=1}^{n-1} Y_i$.

More generally, k-secret sharing between n-players (Shamir) is such that no coalition of size k can decode and any coalition of size $k+1$ can. Efficient algorithm draws a random polynomial f of degree k in F_p and informs player i of $f(i)$ (p is a prime $p > n$).

Secret information transmission: general case

Sufficiency: Assume that the digraph is acyclic and consider the following protocol.

- Player i computes $s_i = \sum_{k \in C(i)} m_{ki}(+\theta_i)$.
- Share s_i in n_i pieces $Y_1, \ldots, Y_{n_i}$ and sends the ℓ-th piece to the ℓ-th successor (i has n_i successors).

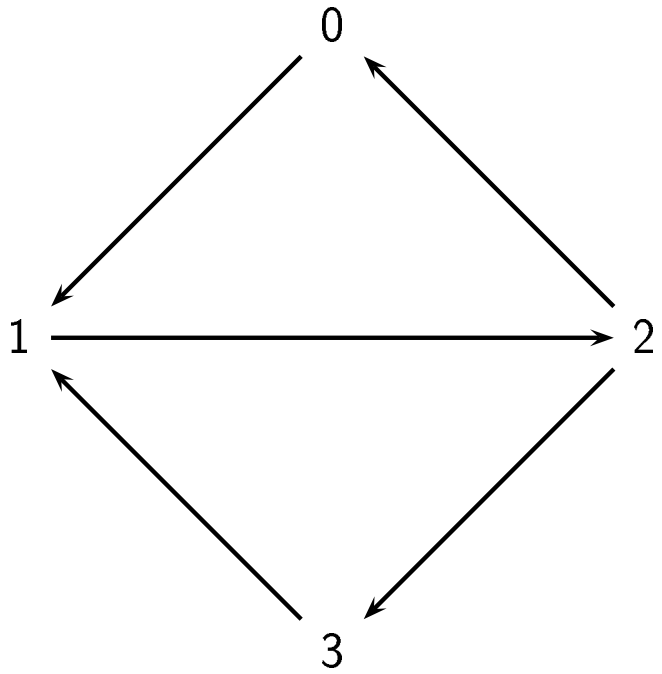
Claim 1.

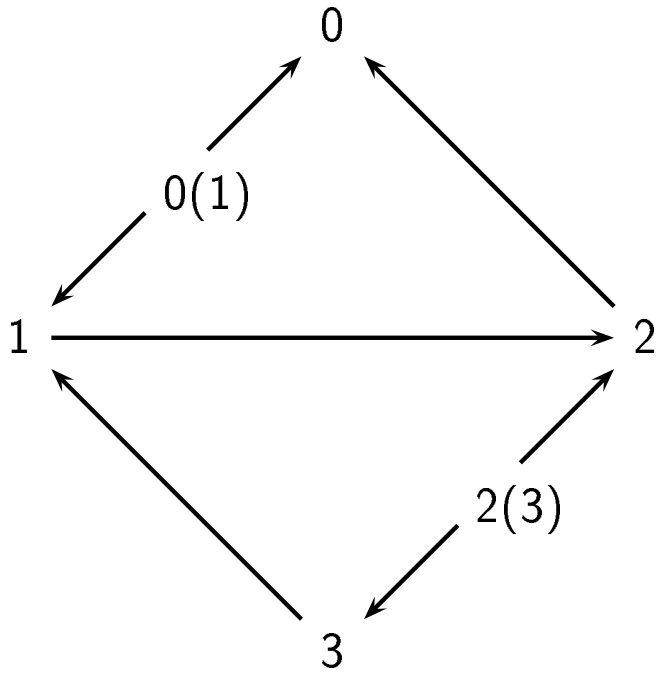
$$s_0 := \sum_{k \in D(0)} m_{k0} = \theta_i$$

Claim 2. If $|S| \leq k$, the joint messages received by S are independent from θ_i.

Dispensing with acyclicity

If the graph is strongly 1-connected a weakly 2-connected, all results go through. We associate an acyclic graph as follows:





Common independent beliefs and private values

Theorem

In all environments $\mathscr{E}$ with common independent beliefs and private values, $F_{\mathscr{N}}(\mathscr{E}) = IC$ if and only if the network $\mathscr{N}$ is weakly 2-connected.

Consider the following network.

We show how to implement the dictatorial s.c.f. of player 2.

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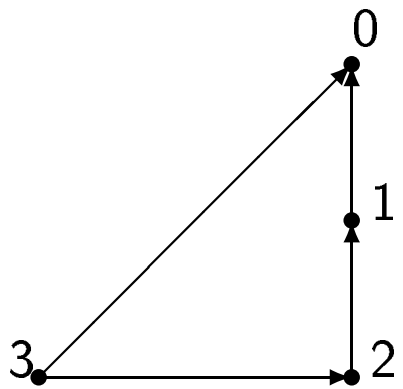

We show how to implement the dictatorial s.c.f. of player 2.

Assume that $\Theta_2 = \{a, b\}$ and $P(\theta_2 = a) = p$. The protocol (i.e., mechanism and strategies) is the following:

- Player 3 draws an encoding key y uniformly on $[0,1]$, and sends it to player 2 and the designer.
- Player 2 of type a draws x uniformly on $[0,p)$ and sends $z = x + y \bmod_{01}$ to player 1.
- Player 2 of type b draws x' uniformly on $(p,1]$ and sends $z = x' + y \bmod_{01}$ to player 1.
- Players 1 forwards his message to the designer.
- The designer computes $\hat{x} = (z - y) \bmod_{01}$ and decodes $\theta_2' = a\mathbf{1}_{\{\hat{x} \in [0,p)\}} + b\mathbf{1}_{\{\hat{x} \in [p,1]\}}$.

$\theta_2' =^d \theta_2$ under unilateral deviation of player 1 or 3.

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Pure equilibria

Let $F^p_{\mathcal{N}}(\mathcal{E})$ be the set of functions partially implementable by *pure equilibria* on $\mathcal{N}$ when the environment is $\mathcal{E}$.

Theorem

$F^p_{\mathcal{N}}(\mathcal{E}) = IC^p$ *for all* $\mathcal{E}$ *if and only if* $D(0) = N$.

If $\exists i \notin D(0)$, we construct an incentive compatible social choice function and a utility profile such that:

- Each player $j \neq i$ has an incentive to lie about his own type, given the type of player i,
- the social choice depends on each player's type.

E.g. a contribution game.

Then, at a pure equilibrium, some player $j \neq i$ learns (something about) the type of player i.

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Secure communication

Implementation of all IC functions in all environments is more demanding.

Definition

SECURE information transmission is possible on the network if there exists a communication protocol such that:
-The designer correctly learns all types, even under unilateral deviation (RELIABLE),
-No player gets information beyond his own type, even under unilateral deviaton (SECRET).

This is a necessary and sufficient condition for implementation of all IC functions in all environments.
Strong 3-connectedness is sufficient with continuous mechanisms. Strong 4-connectedness is sufficient with finite mechanisms.

Secure communication in 2-way networks

Dolev et al. characterize k-secure communication in 2-way networks, i.e. undirected graphs. Fix a sender and a receiver.

Theorem (DDWY93)

k-secure communication between the sender and the receiver is possible IFF the (undirected) graph is $k+1$-connected between sender and receiver.

The protocol for $k = 1$

Assume that there are 3 disjoint path from sender to receiver, through player 1, player 2 and player 3 respectively.

- Sender draws X, X_1, X_2, X_3, X', X'_1, X'_2, X'_3 and sends:
  $(X_1 + X, X_2, X_3)$, $(X'_1 + X', X'_2, X'_3)$ to player 1
  $(X_1, X_2 + X, X_3)$, $(X'_1, X'_2 + X', X'_3)$ to player 2
  $(X_1, X_2, X_3 + X)$, $(X'_1, X'_2, X'_3 + X')$ to player 3.
  The message to player 1 *(resp. 2, 3) conveys no information on* (X, X'). *On another hand, the pooled messages of any two players reveal both* X *and* X'.

The protocol for $k = 1$, ctd.

Denote the messages received by receiver:
$(y_1, x_2(1), x_3(1))$, $(y_1', x_2'(1), x_3'(1))$ from player 1
$(x_1(2), y_2, x_3(2))$, $(x_1'(2), y_2', x_3'(2))$ from player 2
$(x_1(3), x_2(3), y_3)$, $(x_1'(3), x_2'(3), y_3')$ from player 3.

- If $x_1(2) = x_1(3)(= X_1)$, $x_2(1) = x_2(3)(= X_2)$ and $x_3(1) = x_3(2)(= X_3)$, then sender computes $y_1 + X_1$, $y_2 + X_2$ and $y_3 + X_3$. Under unilateral deviation, at least two of these values coincide $(= X)$. Then receiver broadcasts OK and sender broadcasts back $X + \theta$.
- Otherwise there exists i such that $x_i(j) \neq x_j(k)$ (with $\{i, j, k\} = \{1, 2, 3\}$). In this case j contradicts k.

The protocol for $k = 1$, ctd.

- If j contradicts the two other players, then sender recognizes j as deviating. Then $x_i(k) = X_k$ and $x_i(k) + y_k = X$. Then receiver broadcasts OK and sender broadcasts back $X + \theta$.
- If j contradicts only one player k then receiver does not know who of j and k is deviating. He broadcasts $(x_i(j), x_i(k))$. Sender compares $x_i(j)$ and $x_i(k)$ with X_i. If $x_i(j) \neq X_i$ (and $x_i(k) = X_i$), sender broadcasts $\{j\}$ and $X' + \theta$. Receiver computes $X' = y'_i + x_k(i)$ and finds out θ.