# Continuous Control of Lagrangian Data

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**Abstract.** This paper addresses the challenging problem of globally controlling several (and possibly independent) moving agents that together form a whole, generally called swarm, which may display interesting properties. Applications are numerous and can be related either to robotics or computer animation. Assuming the agents are driven by their own dynamics (such act as Newtonian particles), controlling this swarm is known as the particle swarm control problem. In this paper, the theory of an original approach to solve this issue, where we rely on a centralized control rather than focusing on designing individual and simple rules for the agents, is presented. To that end, we propose a framework to control several particles with constraints either expressed on a per-particle basis, or expressed as a function of their environment. We refer to these two categories as respectively *Lagrangian* or *Eulerian* constraints.

## 1 Introduction

Planning trajectories is the task that consists in estimating the best path of a set of particles in order to reach a desired configuration/scenario. This issue has many applications such as robot swarms [1], simulation of schools and herds of animals [2], human crowd simulation [3] or even mobile networks with switching topology [4]. Many competitive approaches have been proposed in the literature to solve this tricky problem. Most of the work on particle swarm control focuses on individual control laws, usually based on potential fields (see e.g. [5, 6]), to define the control policy. The latter induces an emerging and global behavior of the swarm also known as *the swarm intelligence*, which has raised several important stability issues [7, 8]. A good review of those problems and the common associated recipes, along with stability results, can be found in [6].

In this paper, unlike most of the previous studies, particles are not examined individually but we rather prefer to define the problem as a *centralized control*. We indeed assume that each particle evolves under a continuous low and particles are therefore dependent each others. Hence, the swarm control operates globally to enforce consistency at the scene level.

This approach enables flexibility since the commands can then either be expressed

a) at a particle level: position of particles, spatial relations between particles, ... at a given time $t$. This is referenced as *Lagrangian constraints*, or

b) at a domain specific level: spatially-continuous density of particles, velocity, ... at a given time $t$ . This is identified as *Eulerian constraints*.

It is obvious that such properties in a unified framework enable a rich range of applications.

The general idea to deal with this problem is introduced in this paper. Roughly speaking, the technique consists in rewriting a completely continuous control theory (based on the variational assimilation framework) in order to allow situations where the system state, the observations and the dynamical model do not belong to the same space of representation. The mapping between the different spaces involved (eulerian and lagrangian) is performed thanks to operators inspired from the graph theory [9].

The paper is organized as follows. The section 2 presents the basic concepts of optimal control theory based on variational data assimilation. In section 3, we introduce the key points enabling a representation of lagrangian data and their relationships with the eulerian space. Due to space limitations, only some video screenshots are drawn but videos can be seen on supplementary files.

## 2 Variational data assimilation

In this section we present the main principles of variational data assimilation. We refer the reader to [10] for complete methodological aspects.

The general problem consists in recovering, from an initial condition, a system's state $\mathbf{X}$ partially observed and driven by approximately known dynamics $\mathbb{M}$. This can be formalized as finding $\mathbf{X}(t)$ at time $t \in [t_0, t_f]$, that satisfies the system:

$$\frac{\partial \mathbf{X}}{\partial t} + \mathbb{M}(\mathbf{X}) = \boldsymbol{\epsilon}_{\mathbb{M}} \quad ; \quad \mathbf{X}(t_0) = \mathbf{X}_0 + \boldsymbol{\epsilon}_0 \quad ; \quad \mathbb{H}(\mathbf{X}) = \mathbf{Y} + \boldsymbol{\epsilon}_{\mathbb{H}} \tag{1}$$

where $\mathbb{M}$ is the non-linear operator relative to the dynamics, $\mathbf{X}_0$ is the initial vector at time $t_0$ and $(\boldsymbol{\epsilon}_{\mathbb{M}}, \boldsymbol{\epsilon}_0)$ are (unknown) additive control variables relative to noise on the dynamics and the initial condition respectively. In addition, noisy measurements $\mathbf{Y}$ of the unknown state are available through the non-linear operator $\mathbb{H}$ up to $\boldsymbol{\epsilon}_{\mathbb{H}}$. These observations can either be issued from *real data* (in this context, we try to estimate some hidden parameters $\mathbf{X}$) but they can also be defined *by the user*. This latter situation is more related to a pure control process. To estimate the system's state, a common methodology relies on the minimization of the cost function that takes into account the three above equations. Unfortunately, for practical reasons due to the complexity of the dynamical systems and to the dimension of the system state, the direct minimization of such a cost function is in practice unfeasible. A known way to cope with this difficulty is to write *an adjoint formulation* of the problem. It can indeed be shown (see [10] for details) that recovering $\mathbf{X}$ can be easily done using a direct and a backward model integration, which considerably simplify the complexity.

This framework is then an appealing solution for large system states and complex dynamical models, as the ones we are likely to deal with a high number of swarm particles submitted by continuous physical laws. However with swarm particles, the system state is Lagrangian (i.e. defined for particles) whereas the command can either be defined at a Lagrangian level (configuration of particles) or in an Eulerian context (any continuous data like the density). It is therefore of primary interest to design some mathematical tools to switch from Lagrangian/Eulerian spaces. The general idea is presented in the next section.

## 3 Graph-based data representation

We consider the swarm as a connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ composed of a set of $N$ particles, seen as vertices. The population $\mathcal{V} \in \mathbb{V}^N$ has interconnections, seen as edges $\mathcal{E} \in \mathbb{E}^Z$, where $Z$ is the number of edges connecting two vertices of $\mathcal{V}$. One can easily understand that we are not interested in connecting every particle to all others, and therefore we aim at having $Z \ll N^2$. We obtain the definition of the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ defining in a row particles and their interactions. For example, the position of particles, also called configuration, can be written as:

$$\mathbf{y}(\mathcal{V}) = [y(p_1), ..., y(p_i), ..., y(p_N)]^T = [y_1, ..., y_i, ..., y_N]^T, \tag{2}$$

where $y(p_i) = y_i$ are the spatial locations of the particles. In order to write graph quantity expressions, we introduce the diagonalisation operator $\underline{.}$ which creates a pure diagonal matrix made of the vector values. For instance, $\underline{\mathbf{f}}\mathbf{y}$ gives us a new graph quantity being the product of $\mathbf{f}$ and $\mathbf{y}$. The connectivity between the different vertices $\mathcal{V}$ is defined by the adjacency matrix $\boldsymbol{A} \in \mathbb{Z}^{N\times N}$ which captures the structure of the graph $\mathcal{G}$, i.e. $\boldsymbol{A}_{ij} = 1$ if there exists an edge between $p_i$ and $p_j$ and $\boldsymbol{A}_{ij} = 0$ elsewhere. Depending on applications, this adjacency matrix can also be balanced by weighted $w(f(p_i), f(p_j)) = w_{ij}^{\mathbf{f}} \in \mathbb{W}$ to promote some specific relations, which when combined to the adjacency matrix leads to $\boldsymbol{A}_{w^{\mathbf{f}}} = [w_{ij}^{\mathbf{f}}] \circ \boldsymbol{A}$, $\circ$ being the Hadamard, or entrywise, product. The aim being to deal, on the one hand, with a system state defined on such a graph and on the other hand, with a continuous dynamic model and eventually eulerian observations, we have to define a Lagrangian to Eulerian operator (able to extract continuous quantity from a graph) as well as an Eulerian to Lagrangian projector (that evaluates a continuous quantity on a graph).

**Lagrangian to Eulerian projection**. This operation can be viewed as the transcription of a graph of $N$ particles on an other graph composed of $M$ vertices, $M$ being the dimension of the Eulerian grid (i.e. the number of pixels). With the help of an adjacency matrix, the operation consisting in deriving an eulerian value $\boldsymbol{\rho}_{\mathcal{C}}$ on a grid $\mathcal{C}$ from a graph $\mathbf{q}$ reads $\boldsymbol{\rho}_{\mathcal{C}} = \mathcal{A}_{\boldsymbol{G}^{\mathbf{y}}}\mathbf{q}$. In this case, $\mathcal{A}_{\boldsymbol{G}^{\mathbf{y}}} \in \mathbb{W}^{M\times N}$ is an Eulerian adjacency matrix such as $\boldsymbol{G}_{i,j}^{\mathbf{y}} = f(\mathbf{y}(i) - \mathbf{y}(j))$ is a kernel function, such as a gaussian one for instance.

**Eulerian to Lagrangian projection**. To evaluate an Eulerian quantity on a graph, several possibility are available. An efficient and simple method, used in this paper, is the bilinear interpolation.

The control of a Lagrangian system state defined on a graph submitted to a

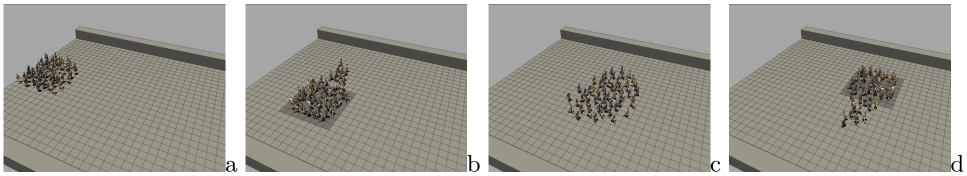


**Fig. 1. Example**: a basic swarm model is used in this example. The swarm is given the goal to reach the opposite part of the space. Meanwhile, this evolution model is controlled to fulfill two density constraints represented as a grey zone in figure (b) and (d), thanks to our methodology

continuous dynamical model can then been managed by mixing the continuous theory of section 2 with the above operators able to switch from Eulerian/Lagrangian spaces. An example of a control with an Eulerian command is shown in figure 1 where a swarm, whose agents are represented by pedestrians, is controlled to reach two given density observations.

## 4 Conclusion

In this paper we have proposed an original framework to deal with the control of dynamic particle swarm using the variational assimilation theory with a system state defined on graphs. This theoretical framework is able to manage command/observations both issued from Lagrangian or Eulerian quantities.

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