ELEMENT OF INFORMATION THEORY

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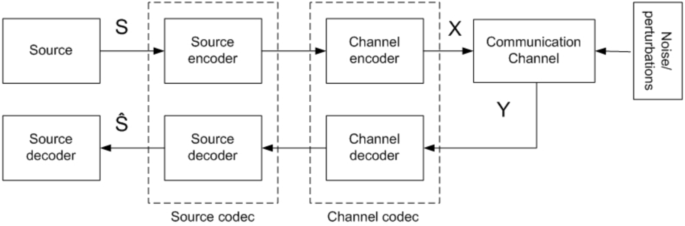
ELEMENTS OF INFORMATION THEORY

Information Theory

Goal and Framework of the communication system

- To transmit an information at the minimum rate for a given quality;
- Seminal work of Claude Shannon (1948)[Shannon,48].



Ultimate goal

The source and channel codec must be designed to ensure a good transmission of a message given a minimum bit rate or a minimum level of quality.

Goal and framework of the communication system

Three major research axis:

1. Measure: Amount of information carried by a message.
2. Compression:
   - Lossy vs lossless coding...
   - Mastering the distortion $d(S, \widehat{S})$
3. Transmission:
   - Channel and noise modelling
   - Channel capacity

Some definitions

Definition

Source of information: something that produces messages!

Definition

Message: a stream of symbols taking theirs values in a predefined alphabet.

Example

Source: a camera
Message: a picture
Symbols: RGB coefficients
alphabet$= (0, ..., 255)$

Example

Source: book
Message: a text
Symbols: letters
alphabet$= (a, ..., z)$

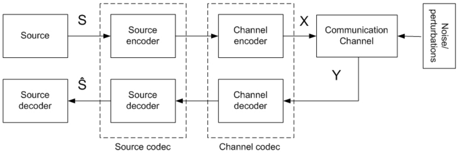
Some definitions

Definition

Source Encoder: the goal is to transform S in a binary signal X of size as small as possible (eliminate the redundancy).
Channel Encoding: the goal is to add some redundancy in order to be sure to transmit the binary signal X without errors.

Definition

$$\text{Compression Rate: } \sigma = \frac{\text{Nb bits of input}}{\text{Nb bits of output}}$$



Information Theory

Random variables and probability distribution

The transmitted messages are considered as a random variable with a finite alphabet.

Definition (Alphabet)

An alphabet $\mathcal{A}$ is a set of data $\{a_1, ..., a_N\}$ that we might wish to encode.

Definition (Random Variable)

A discrete random variable X is defined by an alphabet $\mathcal{A} = \{x_1, ..., x_N\}$ and a probability distribution $\{p_1, ..., p_N\}$, i.e. $p_i = P(X = x_i)$.

Remark: a symbol is the outcome of a random variable.

Properties

- $0 \leq p_i \leq 1$;
- $\sum_{i=1}^{N} p_i = 1$ also noted $\sum_{x \in \mathcal{A}} p(x)$.
- $p_i = P(X = x_i)$ is equivalent to $P_X(x_i)$ and $P_X(i)$.

Joint probability

Definition (Joint probability)

Let X and Y be discrete random variables defined by alphabets $\{x_1, ..., x_N\}$ and $\{y_1, ..., y_M\}$, respectively.
A and B are the events $X = x_i$ and $Y = y_j$,
$P(X = x_i, Y = y_j)$ is the joint probability also called $P(A, B)$ or p_{ij}.

Properties of the joint probability density function (pdf)

- $\sum_{i=1}^{N} \sum_{j=1}^{M} P(X = x_i, Y = y_j) = 1$,
- If $A \cap B = \oslash$, $P(A, B) = P(X = x_i, Y = y_j) = 0$,
- Marginal probability distribution of X and Y:
  - $P(A) = P(X = x_i) = \sum_{j=1}^{M} P(X = x_i, Y = y_j)$ is the probability of the event A,
  - $P(B) = P(Y = y_j) = \sum_{i=1}^{N} P(X = x_i, Y = y_j)$ is the probability of the event B.

Joint probability

Example

Let X and Y be discrete random variables defined by alphabets $\{x_1, x_2\}$ and $\{y_1, y_2, y_3, y_4\}$, respectively.
The sets of events of (X, Y) can be represented in a joint probability matrix:

X, Y	y_1	y_2	y_3	y_4
x_1	(x_1, y_1)	(x_1, y_2)	(x_1, y_3)	(x_1, y_4)
x_2	(x_2, y_1)	(x_2, y_2)	(x_2, y_3)	(x_2, y_4)

Joint probability

Example

Let X and Y be discrete random variables defined by alphabets $\{R, NR\}$ and $\{S, Su, A, W\}$, respectively.

X, Y	S	Su	A	W
R	0.15	0.05	0.15	0.20
NR	0.10	0.20	0.10	0.05

Joint probability

Example

Let X and Y be discrete random variables defined by alphabets $\{R, NR\}$ and $\{S, Su, A, W\}$, respectively.

X, Y	S	Su	A	W
R	0.15	0.05	0.15	0.20
NR	0.10	0.20	0.10	0.05

Questions:

- Does (X, Y) define a pdf? => Yes, $\sum_{i=1}^{2}\sum_{j=1}^{4} P(X = x_i, Y = y_j) = 1$;
- Is it possible to define the marginal pdf of X? => Yes, $X = \{R, NR\}$,
  $P(X = R) = \sum_{j=1}^{4} P(X = R, Y = y_j) = 0.55$,
  $P(X = NR) = \sum_{j=1}^{4} P(X = NR, Y = y_j) = 0.45$.

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Conditional probability and Bayes rule

Notation: the conditional probability of $X = x_i$ knowing that $Y = y_j$ is written as $P(X = x_i | Y = y_j)$.

Definition (Bayes rule)

$$P(X = x_i | Y = y_j) = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)}$$

Properties

- $\sum_{k=1}^{N} P(X = x_k | Y = y_j) = 1$;
- $P(X = x_i | Y = y_j) \neq P(Y = y_j | X = x_i)$.

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Conditional probability and Bayes rule

Example

Let X and Y be discrete random variables defined by alphabets $\{R, NR\}$ and $\{S, Su, A, W\}$, respectively.

X, Y	S	Su	A	W
R	0.15	0.05	0.15	0.20
NR	0.10	0.20	0.10	0.05

Question:
What is the conditional probability distribution of $P(X = x_i | Y = S)$?

Conditional probability and Bayes rule

Example

Let X and Y be discrete random variables defined by alphabets $\{R, NR\}$ and $\{S, Su, A, W\}$, respectively.

X, Y	S	Su	A	W
R	0.15	0.05	0.15	0.20
NR	0.10	0.20	0.10	0.05

Question:
What is the conditional probability distribution of $P(X = x_i|Y = S)$?
$P(Y = S) = \sum_{i=1}^{2} P(Y = y_i) = 0.25$, and from Bayes
$P(X = R|Y = S) = \frac{0.15}{0.25}$ and $P(X = NR|Y = S) = \frac{0.10}{0.25}$

Statistical independence of two random variables

Definition (Independence)

Two discrete random variables are independent if the joint pdf is egal to the product of the marginal pdfs:

$$P(X = x_i, Y = y_j) = P(X = x_i) \times P(Y = y_j) \; \forall \; i \text{ and } j.$$

Remark: If X and Y independent, $P(X = x_i | Y = y_j) = P(X = x_i)$ (From Bayes).

While independence of a set of random variables implies independence of any subset, the converse is not true. In particular, random variables can be pairwise independent but not independent as a set.

Information Theory

Self-Information

Let X be a discrete random variable defined by the alphabet $\{x_1, ..., x_N\}$ and the probability density $\{p(X = x_1), ..., p(X = x_N)\}$.

How to measure the amount of information provided by an event A, $X = x_i$?

Definition (Self-Information proposed by Shannon)

$$I(A) \stackrel{def}{=} -log_2 p(A) \Leftrightarrow I(X = x_i) \stackrel{def}{=} -log_2 p(X = x_i) \text{ Unit: bit/symbol.}$$

Properties

- $I(A) \geq 0$;
- $I(A) = 0$ if $p(A) = 1$;
- if $p(A) < p(B)$ then $I(A) > I(B)$;
- $p(A) \to 0$, $I(A) \to +\infty$.

Shannon entropy

Definition (Shannon Entropy)

The entropy of a discrete random variable X defined by the alphabet $\{x_1, ..., x_N\}$ and the probability density $\{p(X = x_1), ..., p(X = x_N)\}$ is given by:

$$H(X) = E\{I(X)\} = -\sum_{i=1}^{N} p(X = x_i) log_2(p(X = x_i)), \text{ unit: bit/symbol.}$$

The entropy $H(X)$ is a measure of the amount of *uncertainty, a measure of surprise* associated with the value of X.

Entropy gives the average number of bits per symbol to represent X

Properties

- $H(X) \geq 0$;
- $H(X) \leq log_2 N$ (equality for a uniform probability distribution).

Shannon entropy

Example

- **Example 1: The value of $p(0)$ is highly predictable, the entropy (amount of *uncertainty*) is zero.**

	p(0)	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	p(7)	Entropy (bits/symbol)
Example1	1.0	0	0	0	0	0	0	0	0.00

Shannon entropy

Example

- Example 1: The value of $p(0)$ is highly predictable, the entropy (amount of *uncertainty*) is zero;
- Example 2: This is a probability distribution of Bernoulli $\{p, 1-p\}$.

	p(0)	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	p(7)	Entropy (bits/symbol)
Example1	1.0	0	0	0	0	0	0	0	0.00
Example2	0	0	0.5	0.5	0	0	0	0	1.00

Shannon entropy

Example

- Example 1: The value of $p(0)$ is highly predictable, the entropy (amount of *uncertainty*) is zero;
- Example 2: This is a probability distribution of Bernoulli $\{p, 1-p\}$;
- Example 3: Uniform probability distribution ($P(X = x_i) = \frac{1}{M}$, with $M = 4$, $i \in \{2, \ldots, 5\}$).

	p(0)	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	p(7)	Entropy (bits/symbol)
Example1	1.0	0	0	0	0	0	0	0	0.00
Example2	0	0	0.5	0.5	0	0	0	0	1.00
Example3	0	0	0.25	0.25	0.25	0.25	0	0	2.00

Shannon entropy

Example

- Example 1: The value of $p(0)$ is highly predictable, the entropy (amount of *uncertainty*) is zero;
- Example 2: This is a probability distribution of Bernoulli $\{p, 1-p\}$;
- Example 3: Uniform probability distribution ($P(X = x_i) = \frac{1}{M}$, with $M = 4$, $i \in \{2, \ldots, 5\}$);
- Example 4: –;

	p(0)	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	p(7)	Entropy (bits/symbol)
Example1	1.0	0	0	0	0	0	0	0	0.00
Example2	0	0	0.5	0.5	0	0	0	0	1.00
Example3	0	0	0.25	0.25	0.25	0.25	0	0	2.00
Example4	0.06	0.23	0.30	0.15	0.08	0.06	0.06	0.06	2.68

Shannon entropy

Example

- Example 1: The value of $p(0)$ is highly predictable, the entropy (amount of *uncertainty*) is zero;
- Example 2: This is a probability distribution of Bernoulli $\{p, 1-p\}$;
- Example 3: Uniform probability distribution ($P(X = x_i) = \frac{1}{M}$, with $M = 4$, $i \in \{2, \ldots, 5\}$);
- Example 4: –;
- Example 5: Uniform probability distribution ($P(X = x_i) = \frac{1}{N}$, with $N = 8$, $i \in \{0, \ldots, 7\}$)

	p(0)	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	p(7)	Entropy (bits/symbol)
Example1	1.0	0	0	0	0	0	0	0	0.00
Example2	0	0	0.5	0.5	0	0	0	0	1.00
Example3	0	0	0.25	0.25	0.25	0.25	0	0	2.00
Example4	0.06	0.23	0.30	0.15	0.08	0.06	0.06	0.06	2.68
Example5	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	3.00

Joint information/joint entropy

Definition (Joint information)

Let X and Y be discrete random variables defined by alphabet $\{x_1, ..., x_N\}$ and $\{y_1, ..., y_M\}$, respectively.

The joint information of two events $(X = x_i)$ and $(Y = y_i)$ is defined by

$$I(X = x_i, Y = y_j) = -log_2(p(X = x_i, Y = y_j)).$$

Definition (Joint entropy)

The joint entropy of the two discret random variables X and Y is given by:

$$H(X, Y) = E\{I(X = x_i, Y = y_j)\}.$$

$$H(X, Y) = -\sum_{i=1}^{N} \sum_{j=1}^{M} p(X = x_i, Y = y_j) log_2(p(X = x_i, Y = y_j))$$

Remark: $H(X, Y) \leq H(X) + H(Y)$ (Equality if X and Y independant).

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AConditional information/Conditional entropy

Definition (Conditional information)

The conditional information ... $I(X = x_i|Y = y_j) = -log_2(p(X = x_i|Y = y_j))$

Definition (Conditional entropy)

The conditional entropy of Y given the random variable X:

$$H(Y|X) = \sum_{i=1}^{N} p(X = x_i) H(Y|X = x_i)$$

$$H(Y|X) = \sum_{i=1}^{N} \sum_{j=1}^{M} p(X = x_i, Y = y_j) log_2 \frac{1}{p(Y = y_j|X = x_i)}$$

The conditional entropy $H(Y|X)$ is the amount of uncertainty remaining about Y after X is known.

Remarks:

- **We always have $H(Y|X) \leq H(Y)$ (The knowledge reduces the uncertainty);**
- **Entropy chain rule: $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$ (From Bayes);**
- **Sub-Additivity: $H(X,Y) \leq H(X) + H(Y)$**

Mutual information

Definition (Mutual information)

- The Mutual information of two events $X = x_i$ and $Y = y_j$ is defined as $I(X = x_i; Y = y_j) = -log_2 \frac{p(X=x_i)p(Y=y_j)}{p(X=x_i, Y=y_j)}$
- The Mutual information of two random variables X and Y is defined as $I(X;Y) = -\sum_{i=1}^{N} \sum_{j=1}^{M} p(X = x_i, Y = y_j) log_2 \frac{p(X=x_i)p(Y=y_j)}{p(X=x_i, Y=y_j)}$

The mutual information $I(X;Y)$ measures the information that X and Y share...

Be careful to the notation, $\boldsymbol{I(X,Y) \neq I(X;Y)}$

Properties

- Symmetry: $I(X = x_i; Y = y_j) = I(Y = y_j; X = x_i)$;
- $I(X;Y) \geq 0$; zero if and only if X and Y are independent variables;
- $H(X|X) = 0 \Rightarrow H(X) = I(X;X) \Rightarrow I(X;X) \geq I(X;Y)$.

Mutual information

Mutual information and dependency

The mutual information can be expressed as :

$$I(X;Y) = D(p(X = x_i, Y = y_j)||p(X = x_i)p(Y = y_j))$$

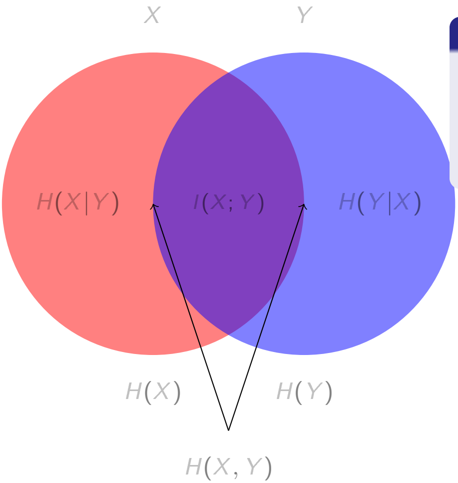
where,

- $p(X = x_i, Y = y_j)$ joint pdf of X and Y;
- $p(X = x_i)$ and $p(Y = y_i)$ marginal probability distribution of X and Y, respectively;
- $D(.||.)$ the divergence of Kullback-Leibler.

Remarks regarding the KL-divergence:

- $D(p||q) = -\sum_{i=1}^{N} p(X = x_i) log_2 \frac{p(X=x_i)}{p(Q=q_i)}$, Q random variable $\{q_1, ..., q_N\}$;
- This is a measure of divergence between two pdfs, not a distance;
- $D(p||q) = 0$, if and only if the two pdfs are strictly the same.

Venn's diagram



We retrieve:

$$H(X,Y) = H(X) + H(Y|X)$$
$$H(X,Y) = H(Y) + H(X|Y)$$
$$H(X,Y) = H(X|Y) + H(Y|X) + I(X;Y)$$
$$I(X;Y) = H(X) - H(X|Y)$$
$$I(X;Y) = H(Y) - H(Y|X)$$

Information Theory

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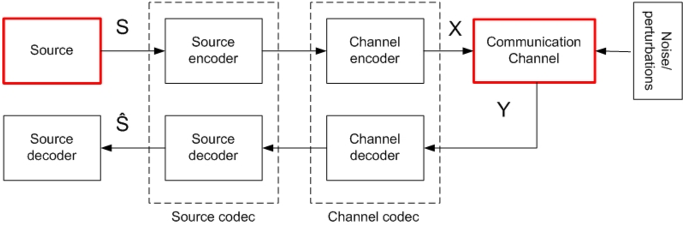
Introduction

Remind of the goal

- To transmit an information at the minimum rate for a given quality;
- Seminal work of Claude Shannon (1948)[Shannon,48].



Parameters of a discrete source

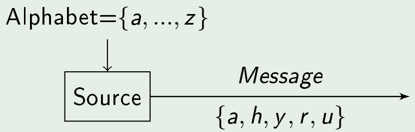
Definition (Alphabet)

An alphabet $\mathcal{A}$ is a set of data $\{a_1, ..., a_N\}$ that we might wish to encode.

Definition (discrete source)

A source is defined as a discrete random variable S defined by the alphabet $\{s_1, ..., s_N\}$ and the probability density $\{p(S = s_1), ..., p(S = s_N)\}$.

Example (Text)



Discrete memoryless source

Definition (Discrete memoryless source)

A discrete source S is memoryless if the symbols of the source alphabet are independent and identically distributed:

$$p(S = s_1, ..., S = s_N) = \prod_{i=1}^{N} p(S = s_i)$$

Remarks:

- Entropie: $H(S) = -\sum_{i=1}^{N} p(S = s_i) log_2 p(S = s_i)$ bit;
- Particular case of a uniform source: $H(S) = log_2 N$.

Extension of a discrete memoryless source

Rather than considering individuals symbols, more useful to deal with blocks of symbols.
Let S be a discrete source with an alphabet of size N. The output of the source is grouped into blocks of K symbols. The new source, called S^K, is defined by an alphabet of size N^K.

Definition (Discrete memoryless source, K^{th} extension of a source S)

If the source S^K is the K^{th} extension of a source S, the entropy per extended symbols of S^K is K times the entropy per individual symbol of S:

$$H(S^K) = K \times H(S)$$

Remark:
the probability of a symbol $s_i^K = (s_{i_1}, ..., s_{i_K})$ from the source S^K is given by $p(s_i^K) = \prod_{j=1}^{K} p(s_{i_j})$.

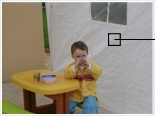
Discrete source with memory (Markov source)

Discrete memoryless source

This is not realistic!

Successive symbols are not completely independent of one another...

- in a picture: a pel (S_0) depends statistically on the *previous* pels.



200	210	207	205	200	202
201	205	199	199	200	201
202	203	203	201	200	204
200	210	207	205	200	202

–	–	–	–	–	–
–	–	s_5	s_4	s_3	s_2
–	–	s_1	s_0	–	–
–	–	–	–	–	–

This dependence is expressed by the conditionnal probability $p(S_0|S_1, S_2, S_3, S_4, S_5)$.
$p(S_0|S_1, S_2, S_3, S_4, S_5) \neq p(S_0)$

- in the langage (french): $p(S_k = u) \leq p(S_k = e)$,
$p(S_k = u|S_{k-1} = q) >> p(S_k = e|S_{k-1} = q)$;

Discrete source with memory (Markov source)

Definition (Discrete source with memory)

A discrete source with memory of order N (N^{th} order Markov) is defined as:

$$p(S_k|S_{k-1}, S_{k-2}, ..., S_{k-N})$$

The entropy is given by:

$$H(S) = H(S_k|S_{k-1}, S_{k-2}, ..., S_{k-N})$$

Example (One dimensional Markov model)

The pel value S_0 depends statistically only on the pel value S_1.

$\xrightarrow{Q}$

85	85	170	0	255
85	85	85	170	255

$$H(X) = 1.9 \ bit/symb, \ H(Y) = 1.29, \ H(X,Y) = 2.15, \ H(X|Y) = 0.85$$

Information Theory

Source code

Definition (Source code)

A source code C for a random variable X is a mapping from $x \in \mathcal{X}$ to $\{0,1\}^*$. Let c_i denotes the code word for x_i and l_i denote the length of c_i.

$\{0,1\}^*$ is the set of all finite binary string.

Definition (Prefix code)

A code is called a prefix code (instantaneous code) if no code word is a prefix of another code word

Not required to wait for the whole message to be able to decode it.

Kraft inequality

Definition (Kraft inequality)

A code C is instantaneous if it satisfies the following inequality:

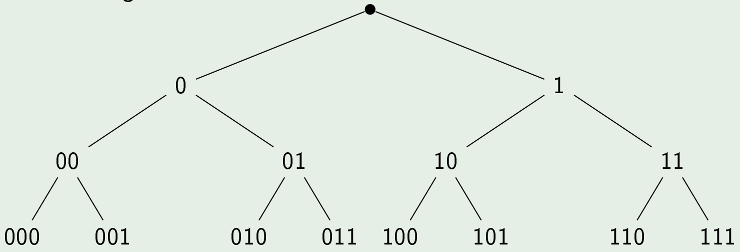
$$\sum_{i=1}^{N} 2^{-l_i} \leq 1$$

with, l_i the length of code word length i

Kraft inequality

Example (Illustration of the Kraft inequality using a coding tree)

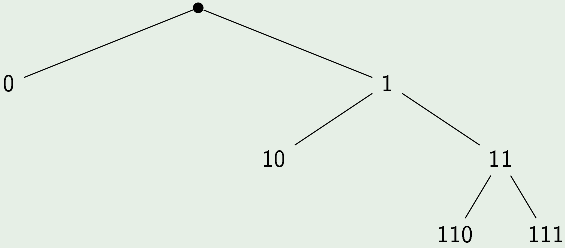
The following tree contains all three-bit codes:



Kraft inequality

Example (Illustration of the Kraft inequality using a coding tree)

The following tree contains a prefix code. We decide to use the code word 0 and 10.



The remaining leaves constitute a prefix code:

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = 1$$

Higher bound of entropy

Let S a discret source defined by the alphabet $\{s_1, ..., s_N\}$ and the probability density $\{p(S = s_1), ..., p(S = s_N)\}$.

Definition (Higher bound of entropy)

$$H(S) \leq log_2 N$$

Interpretation

- the entropy is limited by the size of the alphabet;
- a source with a uniform pdf provides the highest entropy.

Source coding theorem

Let S a discrete source defined by the alphabet $\{s_1, ..., s_N\}$ and the probability density $\{p(S = s_1), ..., p(S = s_N)\}$. Each symbol s_i is coded with a length l_i bits:

Definition (Source coding theorem or First Shannon's theorem)

$$H(S) \leq \overline{l_C} \text{ with } \overline{l_C} = \sum_{i=1}^{N} p_i l_i$$

The entropy of the source gives the limit of the lossless compression. We can not encode the source with less than $H(S)$ bit per symbol. The entropy of the source is the lower-bound.

Warning....

$\{l_i\}_{i=1,...,N}$ must satisfy Kraft's inequality.

Remarks:

- $\overline{l_C} = H(S)$, when $l_i = -log_2 p(X = x_i)$.

Source coding theorem

Definition (Source coding theorem (bis))

Whatever the source S, there exist an instantaneous code C, such that

$$H(S) \leq \overline{l_C} < H(S) + 1$$

The upper bound is equal to $H(S) + 1$, simply because the Shannon information gives a fractionnal value.

Source coding theorem

Example

Let X a random variable with the following probability density. The optimal code lengths are given by the self-information:

X	x_1	x_2	x_3	x_4	x_5
$P(X = x_i)$	0.25	0.25	0.2	0.15	0.15
$I(X = x_i)$	2.0	2.0	2.3	2.7	2.7

The entropy $H(X)$ is equal to 2.2855 bits. The source coding theorem gives:

$$2.2855 \leq \bar{l} < 3.2855$$

Rabbani-Jones extension

Symbols can be coded in blocks of source samples instead of one at a time (block coding). In this case, further bit-rate reductions are possible.

Definition (Rabbani-Jones extension)

Let S be an ergodic source with an entropy $H(S)$. Consider encoding blocks of N source symbols at a time into binary codewords.
For any $\delta > 0$, it is possible to construct a code that the average number of bits per original source symbol $\overline{l_C}$ satisfies:

$$H(S) \leq \overline{l_C} < H(S) + \delta$$

Remarks:

- Any source can be **losslessly** encoded with a code very close to the source entropy in bits;
- There is a high benefit to increase the value N;
- But, the number of symbols in the alphabet becomes very high. Example: block of 2×2 pixels (coded on 8 bits) leads to 256^4 values per block...

Channel coding theorem

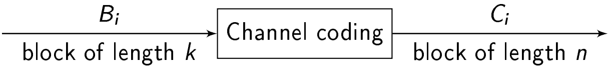
Let a discrete memoryless channel of capacity C. The channel coding transform the messages $\{b_1, ..., b_k\}$ into binary codes having a length n.

Definition (Transmission rate)

The transmission rate R is given by:

$$R \stackrel{def}{=} \frac{k}{n}$$

R is the amount of information stemming from the symbols b_i per transmitted bits.



Channel coding theorem

Example (repetition coder)

The coder is simply a device repeating r times a particular bit. Below, example for $r = 2$. $R = \frac{1}{3}$.

$$\xrightarrow[|011|]{B_t} \boxed{\text{Channel coding}} \xrightarrow[|000111111|]{C_t}$$

This is our basic scheme to communicate with others! We repeat the information...

Definition (Channel coding theorem or second Shannon's theorem)

- $\forall R < C$, $\forall p_e > 0$: it is possible to transmit information nearly without error at any rate below the channel capacity;
- if $R \geq C$, all codes will have a probability of error greater than a certain positive minimal level, and this level increases with the rate.

Shannon's theorem
Source/Channel theorem

Let a noisy channel having a capacity C and a source S having an entropy H.

Definition (Source/Channel theorem)

- if $H < C$ it is possible to transmit information nearly without error. Shannon showed that it was possible to do that by making a source coding followed by a channel coding;
- if $H \geq C$, the transmission cannot be done with an arbitrarily small probability.

Information Theory

YOU MUST KNOW

Let X a random variable defined by $\mathcal{X} = \{x_1, ..., x_N\}$ and the probabilities $\{p_{x_1}, ..., p_{x_N}\}$.
Let Y a random variable defined by $\mathcal{Y} = \{y_1, ..., y_N\}$ and the probabilities $\{p_{y_1}, ..., p_{y_N}\}$.

- $\sum_{i=1}^{N} p_{x_i} = 1$
- Independence: $p(X = x_1, ..., X = x_N) = \prod_{i=1}^{N} p(X = x_i)$
- Bayes rule: $p(X = x_i | Y = y_j) = \frac{p(X=x_i, Y=y_j)}{p(Y=y_j)}$
- Self information: $I(X = x_i) = -log_2 p(X = x_i)$
- Mutual information:
  $I(X; Y) = -\sum_{i=1}^{N} \sum_{j=1}^{M} p(X = x_i, Y = y_j) log_2 \frac{p(X=x_i)p(Y=y_j)}{p(X=x_i, Y=y_j)}$
- Entropy: $H(X) = -\sum_{i=1}^{N} p(X = x_i) log_2 p(X = x_i)$
- Conditional entropy of Y given X: $H(Y|X) = \sum_{i=1}^{N} p(X = x_i) H(Y|X = x_i)$
- Higher Bound of entropy: $H(X) \leq log_2 N$
- Limit of the lossless compression : $H(X) \leq \bar{l}_C$, $\bar{l}_C = \sum_{i=1}^{N} p_{x_i} l_i$

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Suggestion for further reading...

[Shannon,48] C.E. Shannon. A Mathematical Theory of Communication. Bell System Technical Journal, 27, 1948.

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