

# Quantitative verification of weighted Kripke structures

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**Abstract.** Extending formal verification techniques to handle quantitative aspects, both for the models and for the properties to be checked, has become a central research topic over the last twenty years. Following several recent works, we study model checking for (one-dimensional) weighted Kripke structures with positive and negative weights, and temporal logics constraining the total and/or average weight. We prove decidability when only accumulated weight is constrained, while allowing average-weight constraints alone already is undecidable.

## 1 Introduction

**Quantitative verification.** Model checking [CGP00] has been developed for almost 40 years as a formal method for verifying correctness flushing out bugs of computerized systems: this technique first consists in representing the system under study as a mathematical model (a finite-state transition system, in the most basic setting), expressing the correctness property in some logical formalism, and running an algorithm that exhaustively explores the set of behaviours of the model for proving or disproving the property. Model checking has been successfully applied on various real-life case studies.

Though model checking has primarily concentrated on pure qualitative analysis via the development of various temporal logics [Pnu77,CE82,QS82], rich models and logics have also been developed in order to take into account quantitative aspects of reactive systems and of their correctness properties.

In particular, there has been quite a lot of efforts invested in the study of *weighted* discrete transition systems, like weighted Kripke structures [CC95] or counter automata (or VASS) [EN94]. In these models, the weight gives some quantitative information on the system, which might be timing information and constraints, or energy consumption, or value of a discrete variable, etc. This weight can either be some information that we observe on the system (like in weighted Kripke structures) or its value can constrain the further behaviour of the system (like in counter automata).

On such models, there are many interesting verification questions that can be asked. First one can be interested in qualitative structural and logical properties of the system, that can for instance be expressed using some logical formalism.

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Then one can be more interested in quantitative properties of the system, like (among others) mean-payoff constraints [ZP96] (the limit-average of the weight along an execution satisfies some constraint), or energy constraints [BFL<sup>+</sup>08] (all along the execution, the weight satisfies some constraint). More interestingly, one might be interested in properties that might mix qualitative logical properties and quantitative constraints. For instance, in a robot-planning system, one would like to verify that an autonomous robot can always go back to its home base without running out-of-energy.

In this setting, weights in weighted transition systems have been most often restricted to range over the *nonnegative integers* (mostly for representing timing information), and temporal logics have been augmented either with constrained modalities [Koy90,EMSS92] or with explicit variables [AH94], and efficient algorithms have been developed and implemented [BLN03,JLSØ13]. Models with both positive and negative weights have also been studied, but mostly qualitative behaviours alone have been analyzed (this is the case in counter automata—see [HKOW09,GHOW10,Haa12] for recent references), or quantitative constraints have been analyzed (in weighted Kripke structures or games [CDHR10,CRR12]). Mixing qualitative logical properties and quantitative constraints has only poorly been addressed so far, and many works only consider specifications given as a conjunction of a qualitative logical property and a quantitative constraint: this is for instance the case of optimal reachability, mean-payoff parity games [BMOU11], energy parity games [CD12], mean-payoff LTL synthesis [BBFR13].

In that direction, the most relevant and advanced propositions are those of [DG09] and of [BCHK11]. In [DG09], LTL is (roughly) extended with Presburger constraints over weights and interpreted over one-counter automata (and an extension thereof). The satisfiability and model checking problems are addressed, and it is shown that only a restriction to a single weight leads to decidability. In [BCHK11], CTL and LTL are extended with (prefix) accumulative values over finitely many variables: these logics embed numerical assertions such as  $\text{Sum}(x) \sim c$  (*e.g.* to compare the accumulated amount of some resource  $x$  against some value  $c$ ) and  $\text{Avg}(x) \sim c$  (*e.g.* to constrain the average consumption of some resource  $x$ ). In this context, the authors of [BCHK11] show undecidability of the logics in general, and propose several fragments for which model checking is decidable: (i) LTL with limit-average, where (roughly) a property can be rewritten as a conjunction of a qualitative logical property and a quantitative constraint; and (ii) CTL restricted to **EF** and **EX** modalities, with all kinds of numerical assertions, but where the qualitative logical part of the formula is rather poor. Notice that this decidability result is rather surprising, since reachability in two-counter machines is undecidable. The difference is that here counters can go negative, and taking the Parikh image of a path is enough for checking properties expressed in the **EF**-fragment. Designing logical languages that can express intricate properties mixing qualitative logical features and quantitative constraints, and for which model checking remains decidable, seems therefore to be a real challenge!

**Our contribution.** We investigate further the temporal logics with prefix accumulation that has been proposed in [BCHK11], and we study the impact of restricting the logic to a single weight. The logics we consider are therefore based on CTL and LTL, and they extend the standard logics with two kinds of numerical assertions:  $\text{Sum} \sim c$  expresses that the accumulated weight satisfies the constraint  $\sim c$ , and  $\text{Avg} \sim c$  expresses that the current average of the weight satisfies the constraint  $\sim c$ . The extension of CTL is called WCTL and the extension of LTL is called WLTL. Those two logics will be interpreted on weighted Kripke structures, and we will be interested in the model checking problem.

We prove in this paper that when using only Sum constraints, model checking for both CTL and LTL extensions is decidable. On the contrary, allowing Avg constraints leads to undecidability for both branching and linear time. This undecidability result for our logic  $\text{WLTL}_{\text{Avg}}$  is to be compared with the decidability of LTL with limit-average modalities of [BCHK11]: limit-average constraints can be made disjoint from the logical property expressed by the formula of LTL, whereas average modalities are really mixed with the logical property. Finally, we define a *flat* fragment of WLTL allowing both Sum and Avg constraints, but restricting the way they can be nested in the formula; we prove that this fragment has decidable model checking.

Only drafts of proofs are given in the main part of the paper. More detailed proofs can be found in the technical appendix.

## 2 Definitions

### 2.1 Weighted Kripke structures

**Definition 1.** Let  $\text{AP}$  be a finite set of atomic propositions. A weighted Kripke structure over  $\text{AP}$  is a tuple  $\mathcal{K} = \langle S, R, \ell \rangle$  where  $S$  is a finite set of states,  $R \subseteq S \times W \times S$  (where  $W \subseteq \mathbb{Z}$  is the set of weights<sup>1</sup> of  $\mathcal{K}$ , which we assume are given in binary notation) is a weighted transition relation (which we assume total, meaning that for all  $s \in S$ , there exists  $w \in \mathbb{Z}$  and  $s' \in S$  s.t.  $(s, w, s') \in R$ ), and  $\ell: S \rightarrow 2^{\text{AP}}$  is a function labelling the states with atomic propositions.

A weighted Kripke structure with zero-tests is a weighted Kripke structure with extra zero-test transitions, that is,  $R \subseteq S \times (W \cup \{= 0\}) \times S$ .

A weighted Kripke structure (with zero-tests) is unitary if its set of weights  $W$  is included in  $\{-1, 0, +1\}$ .

Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure,  $s_0 \in S$  be a state of  $\mathcal{K}$ , and  $w_0 \in \mathbb{Z}$ . A run in  $\mathcal{K}$  from  $(s_0, w_0)$  is a (finite or infinite) sequence  $\pi = (q_i, w_i)_{i \in I}$  such that  $q_0 = s_0$ ,  $I$  is an interval of  $\mathbb{N}$  containing 0, and for all  $i \in I \setminus \{0\}$ ,  $(q_{i-1}, w_i - w_{i-1}, q_i) \in R$ . When  $I$  is finite, we write  $|\pi|$  for the length of  $\pi$  (the cardinal of  $I$ ), and we write  $\text{last}(\pi)$  (resp.  $\text{last}_q(\pi)$ ,  $\text{last}_w(\pi)$ ) for the configuration  $(q_{\max(I)}, w_{\max(I)})$  (resp. the state  $q_{\max(I)}$ , the weight  $w_{\max(I)}$ ). Given a path

<sup>1</sup> Rational weights would easily be handled, after scaling the values by the least common multiple of their denominators.

$\pi = (q_i, w_i)_{i \in I}$  and  $k \in I$ , the prefix of  $\pi$  up to  $k$  is the path  $\pi_{\leq k} = (q_i, w_i)_{i \in [0, k]}$ ; the suffix of  $\pi$  from  $k$  is the path  $\pi_{\geq k} = (q_{k+i}, w_{k+i})_{i \in \mathbb{N} \cap (I - k)}$ . When  $\mathcal{K}$  has zero-tests, a run in  $\mathcal{K}$  is a sequence  $\pi = (q_i, w_i)_{i \in I}$  where for all  $i \in I \setminus \{0\}$ , either  $(q_{i-1}, w_i - w_{i-1}, q_i) \in R$ , or  $w_i = w_{i-1} = 0$  and  $(q, = 0, q') \in R$ .

Weighted Kripke structures (with or without zero-tests) are related to counter automata [Min61]. A *one-counter automaton*<sup>2</sup> is a weighted Kripke structure with zero-tests in which runs are restricted to only visit configurations with nonnegative weight: the state space is  $S \times \mathbb{N}$ , and it is not possible to take a transition that would make the counter (or weight) negative. It is also usual to have branching zero-tests in one-counter automata, instead of our simple “guarded transitions” (zero-tests transitions). However, branching tests in one-counter automata can easily be implemented in weighted Kripke structures with zero-tests (for instance, to test whether the value of the counter is positive, we put in sequence a decrementation of  $-1$  followed by an incrementation of  $+1$ ).

The execution tree of a weighted Kripke structure  $\mathcal{K}$  from some configuration  $(s_0, w_0)$  is the  $(S \times \mathbb{Z})$ -tree

$$\mathcal{T} = \{\pi \in (S \times \mathbb{Z})^* \mid \pi \text{ is a run from } (s_0, w_0) \text{ in } \mathcal{K}\}$$

For convenience, we label each node  $\pi$  of  $\mathcal{T}$  with  $\text{last}(\pi)$ , which then relates each node of  $\mathcal{T}$  with its corresponding state of  $\mathcal{K}$ . A *branch* of  $\mathcal{T}$  is an infinite run  $\pi$  from  $(s_0, w_0)$  in  $\mathcal{K}$ : every prefix  $\pi_{\leq i}$  ( $i \in \mathbb{N}$ ) is then an element of  $\mathcal{T}$ . We write  $B(\mathcal{T})$  for the set of branches of  $\mathcal{T}$ .

## 2.2 Weighted temporal logics

We extend both branching-time temporal logic CTL [CE82, QS82] and linear-time temporal logic LTL [Pnu77] with numerical constraints on weights:

**Definition 2.** A numerical assertion is built on the following grammar:

$$\alpha ::= \text{Sum} \sim c \mid \text{Avg} \sim c$$

where  $\sim$  ranges over  $\{<, \leq, =, \geq, >\}$  and  $c$  ranges over  $\mathbb{Q}$ .

Fix a set AP of atomic propositions. The syntax of WCTL over AP is given as

$$\phi ::= p \mid \alpha \mid \neg \phi \mid \phi \wedge \phi \mid \mathbf{EX} \phi \mid \mathbf{AX} \phi \mid \mathbf{E}\phi \mathbf{U} \phi \mid \mathbf{A}\phi \mathbf{U} \phi \mid w \cdot \phi$$

where  $p$  ranges over AP and  $\alpha$  ranges over numerical assertions.

The syntax of WTL over AP is given as

$$\phi ::= p \mid \alpha \mid \neg \phi \mid \phi \wedge \phi \mid \mathbf{X} \phi \mid \phi \mathbf{U} \phi \mid w \cdot \phi$$

where  $p$  ranges over AP and  $\alpha$  ranges over numerical assertions.

The operation  $w \cdot \phi$  in both logics is called the reset operation.

<sup>2</sup> Counter automata are often required to have set of weights included in  $\{-1, 0, +1\}$ , which we call *unitary* counter automata in the sequel. The counter automata we consider here correspond to *succinct* counter automata of [HKOW09, GHOW10].

In the sequel, we write  $\text{WCTL}_{\text{Sum}}$  (resp.  $\text{WCTL}_{\text{Avg}}$ ) for the fragments of WCTL using only **Sum** (resp. **Avg**) in numerical assertions. We also write  $\text{WCTL}^{\text{rf}}$ ,  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  and  $\text{WCTL}_{\text{Avg}}^{\text{rf}}$  for the respective fragments with no reset operations. Also, we write  $\text{WLTL}_{\text{Sum}}$  (resp.  $\text{WLTL}_{\text{Avg}}$ ) for the fragments of WLTL using only **Sum** (resp. **Avg**) in numerical assertions. We also write  $\text{WLTL}^{\text{rf}}$ ,  $\text{WLTL}_{\text{Sum}}^{\text{rf}}$  and  $\text{WLTL}_{\text{Avg}}^{\text{rf}}$  for the respective fragments with no reset operations.

The semantics of numerical assertions is defined on finite runs  $\pi$  of a weighted Kripke structure  $\mathcal{K}$  as follows (boolean combinations omitted):

$$\begin{aligned} \pi \models \text{Sum} \sim c & \quad \text{iff} \quad \text{last}_w(\pi) \sim c \\ \pi \models \text{Avg} \sim c & \quad \text{iff} \quad \text{last}_w(\pi) \sim c \cdot |\pi| \end{aligned}$$

Such constraints can for instance be used to express the so-called *energy constraints* [BFL<sup>+</sup>08,CDHR10], requiring that **Sum**  $\geq 0$  all along a run. We can also reinforce this condition by additionally requiring that, at the end of the run, the average energy level (over the prefix) has remained within a given range.

**Semantics of WCTL.** The semantics of WCTL is defined inductively, on the execution tree of a weighted Kripke structure. Let  $\mathcal{K}$  be a weighted Kripke structure and  $(s_0, w_0)$  be an initial configuration. Let  $\mathcal{T}$  be the execution tree of  $\mathcal{K}$  from  $(s_0, w_0)$ ; fix a branch  $\pi$  of  $\mathcal{T}$ , a position  $i \in \mathbb{N}$  along  $\pi$  (with the intended meaning that  $i$  corresponds to the node  $\pi_{\leq i}$  of  $\mathcal{T}$ ; in particular, the node at position 0 corresponds to  $(s_0, w_0)$ ). The semantics of WCTL is defined as follows (atomic propositions and boolean operators omitted):

$$\begin{aligned} \mathcal{T}, \pi, i \models \alpha & \quad \text{iff} \quad \pi_{\leq i} \models \alpha \\ \mathcal{T}, \pi, i \models \mathbf{EX} \phi & \quad \text{iff} \quad \exists \pi' \in B(\mathcal{T}). (\pi_{\leq i} = \pi'_{\leq i} \text{ and } \mathcal{T}, \pi', i+1 \models \phi) \\ \mathcal{T}, \pi, i \models \mathbf{AX} \phi & \quad \text{iff} \quad \forall \pi' \in B(\mathcal{T}). (\pi_{\leq i} = \pi'_{\leq i} \Rightarrow \mathcal{T}, \pi', i+1 \models \phi) \\ \mathcal{T}, \pi, i \models \mathbf{E}\phi_1 \mathbf{U} \phi_2 & \quad \text{iff} \quad \exists \pi' \in B(\mathcal{T}). \exists j \geq i. (\pi_{\leq i} = \pi'_{\leq i} \text{ and} \\ & \quad \mathcal{T}, \pi', j \models \phi_2 \text{ and } \forall i \leq k < j. \mathcal{T}, \pi', k \models \phi_1) \\ \mathcal{T}, \pi, i \models \mathbf{A}\phi_1 \mathbf{U} \phi_2 & \quad \text{iff} \quad \forall \pi' \in B(\mathcal{T}). \exists j \geq i. (\pi_{\leq i} = \pi'_{\leq i} \Rightarrow \\ & \quad \mathcal{T}, \pi', j \models \phi_2 \text{ and } \forall i \leq k < j. \mathcal{T}, \pi', k \models \phi_1) \\ \mathcal{T}, \pi, i \models w \cdot \phi & \quad \text{iff} \quad \mathcal{T}', \pi', 0 \models \phi \text{ where } \mathcal{T}' \text{ is the execution tree of } \mathcal{K} \\ & \quad \text{from } (\text{last}_q(\pi_{\leq i}), 0) \text{ and } \pi' \in B(\mathcal{T}') \end{aligned}$$

Notice that the value of  $\mathcal{T}', \pi', 0 \models \phi$  in the semantics of  $w \cdot \phi$  does not depend on the choice of the branch  $\pi'$ , so the semantics is well-defined. We can generalize that remark:

**Lemma 3.** *Pick  $\pi, \pi' \in B(\mathcal{T})$ .*

– *If  $\pi_{\leq i} = \pi'_{\leq i}$  for some position  $i \in \mathbb{N}$ , then for every formula  $\phi \in \text{WCTL}$ ,*

$$\mathcal{T}, \pi, i \models \phi \quad \text{iff} \quad \mathcal{T}, \pi', i \models \phi.$$

– If  $\text{last}(\pi_{\leq i}) = \text{last}(\pi'_{\leq j})$  for some  $i, j \in \mathbb{N}$ , then for all  $\phi \in \text{WCTL}_{\text{Sum}}$ ,

$$\mathcal{T}, \pi, i \models \phi \quad \text{iff} \quad \mathcal{T}, \pi', j \models \phi.$$

The value of  $\mathcal{T}, \pi, 0 \models \phi$  does not depend on the choice of the branch  $\pi \in B(\mathcal{T})$ ; we then define the truth value of  $\mathcal{K}, (s_0, w_0) \models \phi$  as that of  $\mathcal{T}, \pi, 0 \models \phi$ , where  $\mathcal{T}$  is the execution tree of  $\mathcal{K}$  from  $(s_0, w_0)$  and  $\pi$  is any branch of  $\mathcal{T}$ .

**Definition 4 (WCTL model-checking problem).** *Let  $\phi$  be a WCTL formula, let  $\mathcal{K}$  be a weighted Kripke structure and  $s_0$  be an initial state. The model-checking problem with fixed initial credit asks, for a given  $w_0 \in \mathbb{Z}$ , whether  $\mathcal{K}, (s_0, w_0) \models \phi$ . The model-checking problem with unknown initial credit asks whether there exists  $w_0 \in \mathbb{Z}$  such that  $\mathcal{K}, (s_0, w_0) \models \phi$ .*

**Semantics of WLTL.** The semantics of WLTL is defined inductively over infinite runs of a weighted Kripke structure. Let  $\mathcal{K}$  be a weighted Kripke structure,  $\pi$  be an infinite run in  $\mathcal{K}$  from some configuration  $(s_0, w_0)$ , and  $i \in \mathbb{N}$  be a position along  $\pi$ . The semantics of WLTL is defined as follows (simple cases omitted):

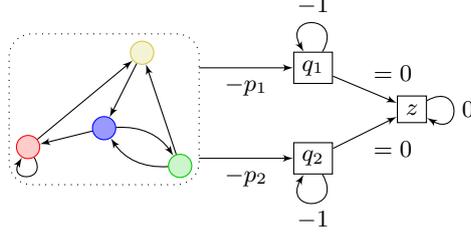
$$\begin{aligned} \pi, i \models \alpha & \quad \text{iff} \quad \pi_{\leq i} \models \alpha \\ \pi, i \models \mathbf{X} \phi & \quad \text{iff} \quad \pi, i + 1 \models \phi \\ \pi, i \models \phi_1 \mathbf{U} \phi_2 & \quad \text{iff} \quad \exists j \geq i. (\pi, j \models \phi_2 \text{ and } \forall i \leq k < j. \pi, k \models \phi_1) \\ \pi, i \models w \cdot \phi & \quad \text{iff} \quad \pi', 0 \models \phi \text{ where } \pi' \text{ is the run of } \mathcal{K} \text{ from } (\text{last}_q(\pi_{\leq i}), 0) \\ & \quad \text{that follows the same transitions as } \pi_{\geq i}. \end{aligned}$$

As for WCTL, the reset operator imposes that further numerical assertions will count from the current position only, this is why the position is reset to 0.

We write  $\mathcal{K}, (s_0, w_0) \models \phi$  whenever there exists an infinite run  $\pi$  from  $(s_0, w_0)$  in  $\mathcal{K}$  such that  $\pi, 0 \models \phi$ . Note that the choice of an existential semantics is arbitrary and harmless, given that the logic is closed under negation.

**Definition 5 (WLTL model-checking problem).** *Let  $\phi$  be a WLTL formula, let  $\mathcal{K}$  be a weighted Kripke structure and  $s_0$  be an initial state. The model-checking problem with fixed initial credit asks, for a given  $w_0 \in \mathbb{Z}$ , whether  $\mathcal{K}, (s_0, w_0) \models \phi$ . The model-checking problem with unknown initial credit asks whether there exists  $w_0 \in \mathbb{Z}$  such that  $\mathcal{K}, (s_0, w_0) \models \phi$ .*

*Remark 6.* Note that the reset operator which is used in both logics is a powerful operator, which can be used to express multiple numerical constraints on various portions of a run. This is a rather standard operator in temporal logics, which is for instance in the core of linear-time timed temporal logic TPTL [AH94]. Note nevertheless that the logics WCTL and WLTL above only allow for one weight variable in a given formula. We will see that the reset operator does not impact much on the model checking of WCTL, but has a strong impact on WLTL model checking.



**Fig. 1.** From  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  to CTL (there is one transition from each state in the dotted box (which is the original weighted Kripke structure) to each state  $q_i$ ).

### 3 Algorithm for model checking $\text{WCTL}_{\text{Sum}}$

In this section we prove the decidability of the model-checking of  $\text{WCTL}_{\text{Sum}}$  over weighted Kripke structures by reducing it to the model-checking of CTL over one-counter automata. We proceed by first removing numerical assertions from the formulas (which requires to modify also the Kripke structure), and then by building a one-counter automaton and a CTL formula. We then apply the results of [Haa12,Ser06], and carefully analyze the complexity of the algorithm.

We first focus on logic  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$ , and will explain at the end of the section how we can extend the result to  $\text{WCTL}_{\text{Sum}}$ .

#### 3.1 Moving quantitative constraints into the model

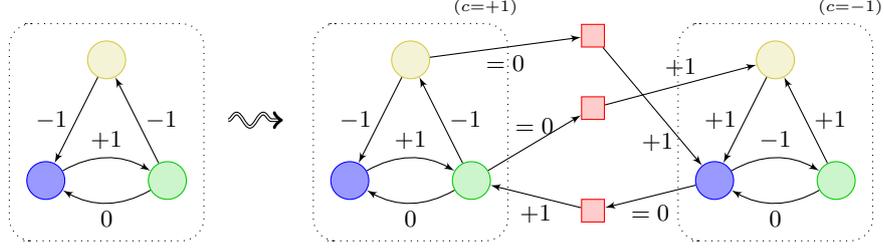
We prove that model checking  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  is logspace-reducible to model checking CTL on structures allowing zero-tests. This is achieved by adding “tests modules” in the model, and replacing Sum constraints with a CTL condition in the corresponding test modules.

Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure, and  $\phi$  be a  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  formula involving integer constants  $\mathcal{P} = \{p_1, \dots, p_k\}$ . We define a new weighted Kripke structure *with zero tests*  $\mathcal{K}'_{\mathcal{P}} = \langle S', R', \ell' \rangle$  as follows:

- $S' = S \cup \{q_i \mid 1 \leq i \leq k\} \cup \{z\}$ ,
- $R' = R \cup \{(s, -p_i, q_i), (q_i, -1, q_i), (q_i, = 0, z), (z, 0, z) \mid s \in S, 1 \leq i \leq k\}$ ,
- $\ell' : \begin{cases} s \in S \mapsto \ell(s) \\ s = q_i \mapsto a_i \\ s = z \mapsto \text{zero} \end{cases}$

The  $a_i$  are fresh atomic propositions. The construction is depicted on Fig. 1 for two constants  $p_1$  and  $p_2$ . The intuition is as follows: whenever the formula requires comparing the current weight with  $p_i$  in a state  $s$  of  $\mathcal{K}$ , the new formula will query the existence of a transition to  $q_i$ , and check that the value of the weight when reaching  $q_i$  is nonnegative (by testing whether state  $z$  is reachable).

Now, after an easy transformation of  $\phi$  into  $\hat{\phi}$  in order not to evaluate subformulas in newly-added states, we get:



**Fig. 2.** From weighted Kripke structures to one-counter automata.

**Proposition 7.** Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure. Let  $\phi$  be a  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  formula with integer constants in  $\mathcal{P}$ , and  $s_0 \in S$  and  $v \in \mathbb{Z}$ . Let  $\mathcal{K}'_{\mathcal{P}}$  be the weighted Kripke structure with zero test as defined above, and  $\hat{\phi}$  be the formula obtained from  $\phi$  by the transformation above. Then

$$\mathcal{K}, (s_0, v) \models \phi \quad \text{iff} \quad \mathcal{K}'_{\mathcal{P}}, (s_0, v) \models \hat{\phi}.$$

Note that the size of  $\mathcal{K}'_{\mathcal{P}}$  is polynomial in  $\mathcal{K}$  and  $\phi$ , and so is  $\hat{\phi}$ .

### 3.2 From weighted Kripke structures to one-counter automata

We now reduce the model checking problem for CTL on weighted Kripke structures with zero tests to the same problem on unitary one-counter automata, in order to invoke the algorithms for CTL model-checking of [Haa12, Ser06]. The one-counter automaton will be made of two copies of the weighted Kripke structure: one to be used when the accumulated weight is nonnegative, and one when it is nonpositive. We will have zero-tests between both copies. Prior to this transformation, we first make the Kripke structure unitary, so that no transition will jump from positive to negative weights, or vice-versa, without hitting zero.

We now come to the transformation of this unitary weighted Kripke structure into a unitary one-counter automaton  $\mathcal{C}$ . The natural idea is to consider two copies of each state: one is used when the accumulated weight is nonnegative, and one when the total weight is nonpositive. By considering the opposite value of the weight in the second copy, we end up with a unitary one-counter automaton. Fig. 2 illustrates this construction on a simple example.

Again rewriting the formula as  $\hat{\phi}$  to “hide” newly added states, we get:

**Proposition 8.** Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure with zero-tests,  $\phi$  be a CTL formula,  $s_0 \in S$  and  $w_0 \in \mathbb{Z}$ . Let  $\mathcal{C}$  be the one-counter automaton obtained above, and  $\hat{\phi}$  be the formula obtained from  $\phi$ . Then

$$\mathcal{K}, (s_0, w_0) \models \phi \quad \text{iff} \quad \mathcal{C}, ((s_0, \text{sign}(w_0)), |w_0|) \models \hat{\phi}$$

where  $\text{sign}(0)$  can be taken as either  $+1$  or  $-1$ .

### 3.3 $\text{WCTL}_{\text{Sum}}^{\text{rf}}$ model checking over weighted Kripke structures

We now come to the main result of this section:

**Theorem 9.** *Model checking  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  over weighted Kripke structures with fixed initial credit is EXPSPACE-complete. It is PSPACE-complete when starting from a unitary weighted Kripke structure.*

*Proof.* The algorithms are obtained by applying the previous transformations from  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  model checking over weighted Kripke structures to CTL model checking over unitary one-counter automata, and then relying on the PSPACE algorithm of [Haa12,Ser06] for model checking unitary one-counter automata.

Hardness is easily proved by reducing the model-checking problem of CTL over one-counter automata to that for  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  over weighted Kripke structures. The former problem was proved EXPSPACE-complete in [GHOW10], and PSPACE-complete over unitary one-counter automata in [GL10]. The reduction is rather straightforward, by reinforcing the CTL formula in order to enforce nonnegative value of the accumulated weight all along the paths.  $\square$

We can now extend the above algorithm to handle the reset operator: when a formula  $\phi$  contains  $w \cdot \psi$  as a subformula, we first evaluate  $\psi$  in all the states of the Kripke structure, assuming initial weight zero, and apply a classical labelling algorithm. In the end:

**Theorem 10.** *Model checking  $\text{WCTL}_{\text{Sum}}$  over weighted Kripke structures with fixed initial credit is EXPSPACE-complete; it is PSPACE-complete when starting from a unitary weighted Kripke structure.*

### 3.4 Model checking $\text{WCTL}_{\text{Sum}}$ with unknown initial credit

The model checking of  $\text{WCTL}_{\text{Sum}}$  with unknown initial credit can be reduced to the fixed initial credit case by adding an initial module which allows to set the weight to any value. We can state the following result:

**Theorem 11.** *Model checking  $\text{WCTL}_{\text{Sum}}$  over weighted Kripke structures with unknown initial credit is EXPSPACE-complete; it is PSPACE-complete when starting from a unitary weighted Kripke structure.*

This result has to be compared with the lower-bound problems in weighted timed automata which is PSPACE-complete with unknown initial credit, but becomes undecidable with fixed initial credit [BLM14].

## 4 Model checking $\text{WCTL}_{\text{Avg}}$ is undecidable

In this section, we prove that constraining the average of the weight value leads to undecidability:

**Theorem 12.** *Model checking  $\text{WCTL}_{\text{Avg}}^{\text{rf}}$  (and therefore  $\text{WCTL}_{\text{Avg}}$ ) over weighted Kripke structures is undecidable.*

*Proof.* We encode the halting problem for (deterministic) two-counter machines into our model-checking problem with fixed initial value. The values  $c_1$  and  $c_2$  of the counters at a given position along an execution is encoded by the length of the path being of the form  $2^{c_1} \cdot 3^{c_2} \cdot 5^a$ , where  $a$  is a nonnegative integer. Decrementing counter  $c_1$  will then amount to multiplying the length of the path by  $5/2$ , which will be achieved by taking a self-loop on a state until the total average value of the weight reaches a given value.

We illustrate the construction on an example. Figure 3 depicts a *module* from  $(q, A_k)$  to  $(q', A_{k'})$ . This module will be used to modify (increment or decrement) the counters. The  $\text{WCTL}_{\text{Avg}}^{\text{rf}}$  formula will enforce that the average weight value of a path ending in a state labelled  $A_k$  be exactly  $k$ , for all  $0 \leq k \leq 2$ . Consider such a path, of length  $n$ , ending in state  $(q, A_k)$  of Figure 3 (so that the accumulated weight is  $k \cdot n$ ). With  $j$  taking values 2, 3,  $5/2$  or  $5/3$  will implement all four instructions modifying the counters. Now, we extend this path following the depicted module, until reaching  $(q', A_{k'})$ . Write  $x - 2$  for the number of times we take the self-loop on state  $(r, k, k')$ . Then the length of the path when reaching  $(q', A_{k'})$  is  $n + x$ , and its total accumulated weight is  $k \cdot n + \frac{(jk' - k) \cdot x}{j - 1}$ . The requirement that the average be  $k'$  in  $(q', A_{k'})$  entails

$$k \cdot n + \frac{(jk' - k) \cdot x}{j - 1} = k' \cdot (n + x).$$

One easily checks that, provided  $k \neq k'$ , this implies  $x = (j - 1) \cdot n$ , so that the length of the whole path when reaching  $(q', A_{k'})$  is  $j \cdot n$  (assuming  $j > 1$ ), and the average is indeed  $k'$ .

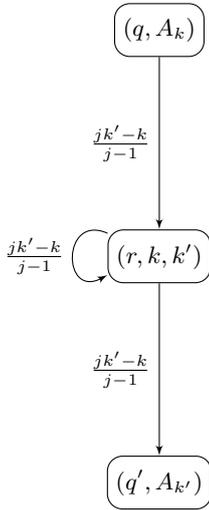


Fig. 3. Updating counters

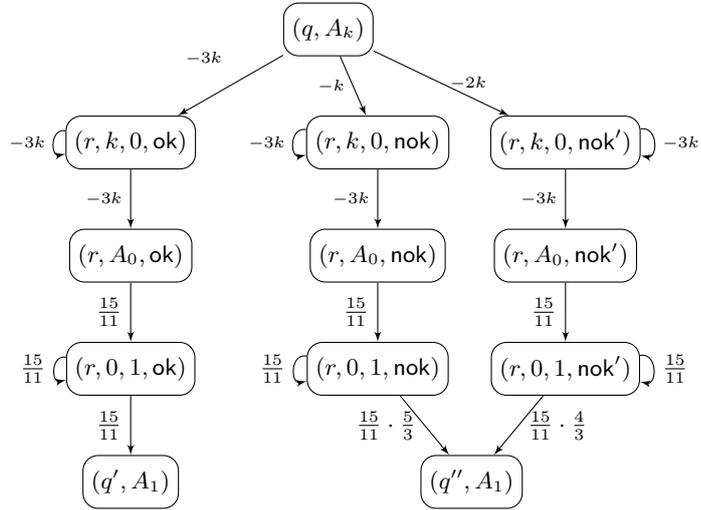


Fig. 4. Testing counters (here  $c_2$ )

Similarly, Figure 4 is a module testing whether counter  $c_2$  equals zero. Starting in state  $(q, A_k)$  with a path of length  $n$ , and total accumulated weight  $k \cdot n$ , it is rather clear that we can reach state  $(r, A_0, \text{ok})$  if, and only if,  $n$  is an integer multiple of 3 (which means that  $c_2 > 0$ ). State  $(r, A_0, \text{ok})$  is then reached by a path of total length  $4n/3$ , and accumulated weight zero. Then the path goes to  $(r, 0, 1, \text{ok})$ , and takes the loop  $x - 2$  times, and reaches  $(q', A_1)$ . The accumulated weight then is  $15x/11$ , and the length of the path is  $4n/3 + x$ . The average is 1 exactly when  $x = 11n/3$ , for which the total length of the path is  $5n$ ; this way, the counters are back to their original value when reaching  $(q', A_1)$ .

If we follow the middle branch of the module, we reach  $(r, A_0, \text{nok})$  with average zero if, and only if,  $n - 1$  is an integer multiple of 3, which implies that  $n$  is not, so that  $c_2 = 0$ . The length of the path when reaching  $(r, A_0, \text{nok})$  is  $n + 1 + (n - 1)/3$ . We then reach  $(q'', A_1)$  after looping  $x - 2$  times on  $(r, 0, 1, \text{nok})$ ; then the length of the path is  $(4n + 2)/3 + x$ , and the total weight is  $(x + 2/3) \cdot 15/11$ . Since we require the average weight to equal 1, we get  $x = (11n - 2)/3$ , which yields a total final length of  $5n$ , as expected. A similar computation can be conducted for the rightmost branch.  $\square$

*Remark 13.* Notice that this can be made to work with only nonnegative and/or integer weights, by shifting and/or multiplying all weights and average constraints by some constant.

## 5 Algorithms for model checking fragments of WTL

### 5.1 Decidability of $\text{WTL}_{\text{Sum}}^{\text{rf}}$ model checking

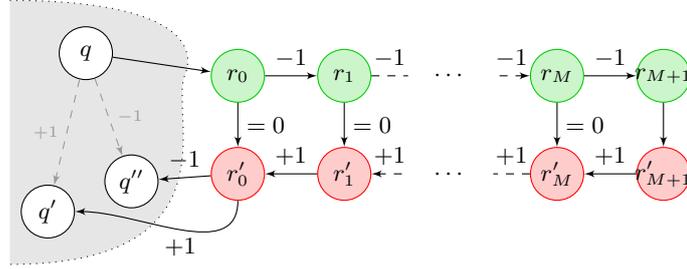
It follows from the proof of Theorem 12 that model checking  $\text{WTL}_{\text{Avg}}^{\text{rf}}$  is undecidable: the formula we built contains a single “until” modality, and can thus be interpreted as a  $\text{WTL}_{\text{Avg}}^{\text{rf}}$  formula. We believe this result is quite surprising since it was proven in [BCHK11] that the model checking of LTL extending with limit-average constraints (that is, constraints speaking on the long-run value of the average) is decidable, even for several weights.

We thus focus on WTL with only Sum constraints, and begin with proving the decidability of the case without reset operator:

**Theorem 14.** *Model checking  $\text{WTL}_{\text{Sum}}^{\text{rf}}$  over weighted Kripke structures both with fixed and unknown initial credit is PSPACE-complete.*

Our proof closely follows the ideas of [GHOW10] and [DG09]: we apply the same transformation as for WCTL, reducing our weighted Kripke structure into a unitary one-counter automaton. We then plug at each state a module (as displayed on Fig. 5, where  $M$  is the maximal absolute value of the constants involved in the formula) to encode numerical assertions into plain LTL. The LTL formula is then checked using Büchi automata.

Notice that in [DG09], the model-checking problem is proven decidable for an extension of LTL with Presburger-defined constraints (but without atomic propositions) over one-dimensional weighted Kripke structures with zero tests.



**Fig. 5.** From  $\text{WLTLSum}^f$  to LTL (notice that  $q'$  and  $q''$  also have their corresponding test modules, which we omitted to draw for the sake of readability).

How to handle atomic propositions for that problem is not addressed in [DG09]. The extension to unknown initial credit is similar to the case of  $\text{WCTL}_{\text{Sum}}$ .

## 5.2 $\text{WLTLSum}$ model checking is undecidable

In this section, we prove that contrary to the case of  $\text{WCTL}_{\text{Sum}}$ , the reset operator makes model checking undecidable. This is not so surprising, and is actually a corollary of a similar result for LTL with one register over one-counter automata [DLS10, Thm. 17].

LTL with registers extends LTL with a way of *storing* the current value of a counter (or other data, depending on the underlying model), and compare the stored value later on during the execution. For instance,  $\downarrow \phi$  stores the current value of the counter before evaluating  $\phi$ . Then  $\uparrow$  evaluates to true at positions where the value of the counter equals the value stored in the register. For instance,  $\downarrow \neg \mathbf{X} \mathbf{F} \uparrow$ , means that it must never be the case that the value of the counter equals its initial value.

The translation of LTL with *one* register into  $\text{WLTLSum}$  is then straightforward:  $\downarrow$  corresponds to setting the weight to zero, and  $\uparrow$  simply means  $\text{Sum} = 0$ . In order to encode the behaviours of a one-counter automaton as a weighted Kripke structure, we have to additionally require that the accumulated weight remains nonnegative, by adding  $\mathbf{G}(\text{Sum} \geq 0)$  as a global conjunct. The following theorem directly follows:

**Theorem 15.** *Model checking  $\text{WLTLSum}$  over weighted Kripke structures is undecidable.*

## 5.3 Model checking a *flat* fragment of WLTLSum

We conclude this part with a fragment of WLTLSum which forbids numerical assertions on the left-hand-side of an “until” formula. As we prove below, model checking our fragment is decidable, so that it offers an alternative to the fragments  $\mathbf{EF}^\Sigma$  and  $\text{LTL}^{\text{lim}}$  of [BCHK11], with a lower complexity. Also, our fragment allows us to use multiple variables, as well as average assertions. The syntactic restriction

is the price to pay for this, but we believe that the fragment remains interesting in practice when dealing with average values, as it is rarely the case that some average value has to be constrained all along an execution. We call this fragment *flat*. Notice that this adjective was already used in similar contexts, but for different restrictions [CC00].

In this section, we consider multi-dimensional weighted Kripke structures, as our algorithm will be able to handle them; they extend the 1-dimensional Kripke structures in the obvious way. FlatWTL is defined by the following syntax:

$$\text{FlatWTL} \ni \phi ::= p \mid \neg p \mid \alpha \mid \phi \vee \phi \mid \phi \wedge \phi \mid \mathbf{X} \phi \mid \mathbf{G} \psi \mid \psi \mathbf{U} \phi \mid w \cdot \phi$$

where  $\alpha$  ranges over numerical assertions,  $\psi$  ranges over LTL and, and  $w$  ranges over the set of variables. The semantics follows that of WTL.

Notice that our fragment allows both Sum and Avg constraints, as well as the reset operator. Also notice that our logic includes a restricted version of the release modality, namely  $\phi \mathbf{R} \psi$ , which can be expressed as  $\mathbf{G} \psi \vee \psi \mathbf{U} (\phi \wedge \psi)$ .

Additionally, using the techniques in [BCHK11], this allows us to transform Avg-assertions into Sum-assertions (if we have to check  $\text{Avg}(x) \leq c$ , we introduce a new variable  $x_c$  whose updates are shifted by  $-c$  compared to the updates of  $x$ , and then check  $\text{Sum}(x_c) \leq 0$ ). Hence we can assume that our formula does not contain Avg assertions.

We first prove that a path satisfying a formula in FlatWTL can be decomposed into finitely many segments, delimited with positions where some numerical assertions have to be checked.

Pick  $\phi \in \text{FlatWTL}$ , and a path  $\pi$  in a weighted Kripke structure  $\mathcal{K}$  such that  $\pi, 0 \models \phi$ . We inductively build a finite set of positions along  $\pi$  at which we may have to evaluate numerical assertions; at all other positions, only pure-LTL formulas will need to be evaluated. For this, we consider the tree of  $\phi$ , and we proceed inductively. We first decorate the root of the tree of  $\phi$  with 0 (to indicate that  $\phi$  holds true at position 0). If  $\phi$  is pure LTL, then we end the labelling. Then, from a node representing subformula  $\psi$  that has been decorated with integer  $i$ , we distinguish between the different types of nodes:

- if the node corresponds to an atomic proposition, the negation thereof, or a numerical assertion, we are done;
- if the node corresponds a reset operator (that is,  $\psi = w \cdot \psi'$ ), we label its successor node, which corresponds to subformula  $\psi'$ , with  $i$ ;
- if the node is a conjunction of subformulas (that is,  $\psi = \psi_1 \wedge \psi_2$ ), we mark all successors of this node with  $i$ . Notice that indeed all subformulas have to hold true at position  $i$  of  $\pi$ ;
- if the node is a disjunction of subformulas, then one of the disjunct has to hold true at position  $i$  of  $\pi$ . We label this successor with  $i$ ;
- if the node is a  $\mathbf{X}$ -modality, we label its successor node with  $i + 1$ ;
- if the node is a  $\mathbf{G}$ -modality, we decorate its successor node with the interval  $[i, +\infty)$  (hence the inductive labelling ends here for this branch, since the formula is flat);

- if the node is an **U**-modality, then there must be a position  $j \geq i$  along  $\pi$  at which the right-hand-side subformula of this **U**-formula holds true. We decorate the right-hand-side successor node of the present node with  $j$ , and the left-hand-side node with  $[i, j - 1]$ . The inductive labelling ends here for the left-hand-side branch since the formula is flat.

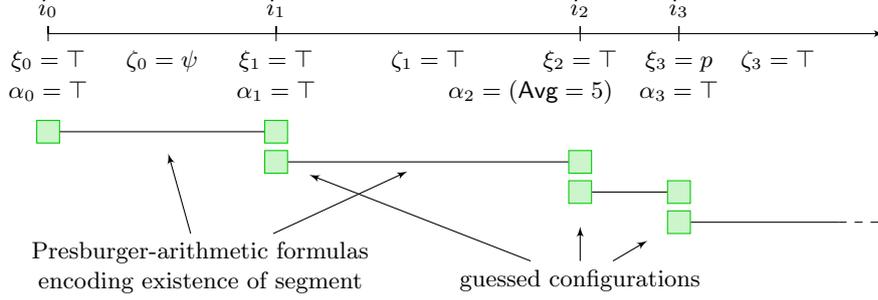
The following trivially holds: for every node that has been labelled by an integer  $i$ , if that node corresponds to subformula  $\psi$ , then  $\pi, i \models \psi$ . Conversely, if for some run  $\pi$  we can label consistently the tree representation of  $\phi$  with integers (or intervals on pure LTL formulas) such that the root is labelled with 0, then  $\pi, 0 \models \phi$ .

This way, we have identified sufficiently many witnessing positions where some numerical assertions may have to be checked. Let  $\mathcal{P} = \{i_0 = 0, i_1, \dots, i_k\}$  be the set of integers (named *breakpoints* hereafter) labelling the tree of  $\phi$ , assuming  $i_l < i_m$  whenever  $l < m$ . By construction, we have that  $k \leq |\phi|$ . Between any two such consecutive positions, the decorated tree gives us (conjunctions of) LTL subformulas to be checked at every intermediary position (the above labelling tells us that all positions between two checkpoints have to satisfy the same LTL subformulas of  $\phi$ ). The tree also indicates those breakpoints where we reset some of the weight variables. Note that given two labellings of the formula tree yielding the same order on breakpoints and making the same choices in the disjunctions, the very same formulas have to be verified between two breakpoints.

As a first step of our (non-deterministic) algorithm, we pick a number  $k + 1$  of (at most  $|\phi|$ ) breakpoints, and guess a labelling of the formula tree with the indices of the breakpoints (or intervals) that respects the rules defined earlier. Then we uniquely associate with each  $h \in [0, k]$  an LTL “right-hand-side” subformula  $\xi_h$  and a numerical assertion  $\alpha_h$  to be checked at breakpoint  $i_h$ , and an LTL formula  $\zeta_h$  that has to be checked at intermediary positions before the next breakpoint (with  $\zeta_k$  being enforced at all positions after the last breakpoint). As noted above, those formulas are uniquely fixed by the order of breakpoints and the labelling of the formula tree; moreover  $\xi_h$  and  $\zeta_h$  are conjunctions of pure LTL subformulas of  $\phi$  whereas  $\alpha_h$  are conjunctions of numerical assertions appearing as subformulas of  $\phi$  or negations of such subformulas. See Fig. 6 for an example of a formula tree labelled with breakpoints.

With each formula  $\xi_h$  selected above, we associate a Büchi automaton  $\mathcal{A}_{\xi'_h}$  where  $\xi'_h$  is the formula  $\mathbf{G}(b_h \Rightarrow \xi_h)$  and  $b_h$  is a fresh atomic proposition that only holds true at breakpoint  $i_h$  (note that the value of  $i_h$  is not known). Similarly, with formulas  $\zeta_h$ , we associate an automaton  $\mathcal{A}_{\zeta'_h}$  enforcing formula  $\zeta'_h = \mathbf{G}(c_h \Rightarrow \zeta_h)$ , where atomic proposition  $c_h$  only holds true between  $i_h$  and  $i_{h+1}$ .

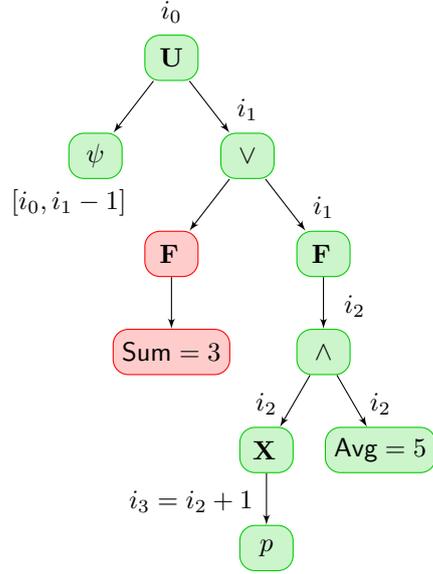
Our algorithm will check the existence of segments between two breakpoints that satisfy the required properties. Each segment corresponds to a finite path in the product  $\mathcal{L}$  of the weighted Kripke structure  $\mathcal{K}$  and all the Büchi automata built above. When working with the  $j$ -th segment, proposition  $b_j$  is set to true at the first step, and  $c_j$  holds true all along this segment. This way, the automata  $\mathcal{A}_{\xi'_j}$  and  $\mathcal{A}_{\zeta'_j}$  play their roles of checking  $\xi_j$  at the beginning of the segment, and  $\zeta_j$  at every position in the segment. The automata that have already been “activated”



**Fig. 7.** Schematics representation of our algorithm

(at previous breakpoints) keep on running, finishing their computations, while the automata corresponding to later breakpoints remain “idle”.

One configuration of  $\mathcal{L}$  (i.e. a state of  $\mathcal{K}$  and a state per Büchi automaton constructed above) can be stored using polynomial space. However, we do not have a bound on the length of the segments, which prevents us from guessing the path on-the-fly. Instead, we guess the configurations of  $\mathcal{L}$  at each breakpoint (there are at most  $|\phi|$  breakpoints). It remains to decide the existence of a path in  $\mathcal{L}$  from the configuration in one breakpoint to the configuration in the next one, and checking that the numerical assertions at each breakpoint are satisfied. Following the ideas of [BCHK11], we can encode the existence of each segment by assigning one variable with each transition of  $\mathcal{L}$ : each variable represents the number of times this transition will be taken along the segment, and one can easily write a Presburger-arithmetic formula expressing that a valuation for those variables corresponds to a path and that the numerical assertions are fulfilled. Notice that we can easily handle reset operators in those equations. In the end, our formula is in the existential fragment of Presburger arithmetic, and has size exponential, so that our procedure runs in NEXPTIME.



**Fig. 6.** Labelling of the tree of formula  $\psi \mathbf{U} [\mathbf{F} (\text{Sum} = 3) \vee \mathbf{F} (\mathbf{X} (p) \wedge \text{Avg} = 5)]$  (assuming  $\psi \in \text{LTL}$ ) with breakpoints.

**Theorem 16.** *Model checking FlatWTL over weighted Kripke structures is decidable in NEXPTIME.*

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## A Proofs of Section 3

### A.1 Moving quantitative constraints into the model

Letting  $n = (\bigvee_{1 \leq i \leq k} a_i) \vee \text{zero}$ , which we use to characterize the newly added states, we define  $\widehat{\phi}$  inductively as follows:

$$\begin{array}{ll}
\widehat{p} = p & \\
\widehat{\neg \phi} = \neg \widehat{\phi} & \widehat{\phi_1 \wedge \phi_2} = \widehat{\phi_1} \wedge \widehat{\phi_2} \\
\widehat{\mathbf{EX} \phi} = \mathbf{EX} (\neg n \wedge \widehat{\phi}) & \widehat{\mathbf{E}\phi_1 \mathbf{U} \phi_2} = \mathbf{E}(\neg n \wedge \widehat{\phi_1}) \mathbf{U} (\neg n \wedge \widehat{\phi_2}) \\
\widehat{\mathbf{AX} \phi} = \mathbf{AX} (n \vee \widehat{\phi}) & \widehat{\mathbf{A}\phi_1 \mathbf{U} \phi_2} = \mathbf{A}(\neg n \wedge \widehat{\phi_1}) \mathbf{U} (n \vee \widehat{\phi_2}) \\
\widehat{\text{Sum} \geq p_i} = \mathbf{EX} (a_i \wedge \mathbf{EF} \text{zero}) & \widehat{\text{Sum} > p_i} = \mathbf{EX} (a_i \wedge \mathbf{EX} (a_i \wedge \mathbf{EF} \text{zero})) \\
\widehat{\text{Sum} < p_i} = \neg \widehat{\text{Sum} \geq p_i} & \widehat{\text{Sum} \leq p_i} = \neg \widehat{\text{Sum} > p_i} \\
\widehat{\text{Sum} = p_i} = \mathbf{EX} (a_i \wedge \mathbf{EX} \text{zero}) & 
\end{array}$$

**Proposition 7.** *Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure. Let  $\phi$  be a  $\text{WCTL}_{\text{Sum}}^{\text{rf}}$  formula with integer constants in  $\mathcal{P}$ , and  $s_0 \in S$  and  $v \in \mathbb{Z}$ . Let  $\mathcal{K}'_{\mathcal{P}}$  be the weighted Kripke structure with zero test as defined above, and  $\widehat{\phi}$  be the formula obtained from  $\phi$  by the transformation above. Then*

$$\mathcal{K}, (s_0, v) \models \phi \quad \text{iff} \quad \mathcal{K}'_{\mathcal{P}}, (s_0, v) \models \widehat{\phi}.$$

*Proof.* Let  $\mathcal{T}$  be the execution tree of  $\mathcal{K}$  from  $(s_0, v)$  and  $\mathcal{T}'_{\mathcal{P}}$  be that of  $\mathcal{K}'_{\mathcal{P}}$  from  $(s_0, v)$ . Obviously  $\mathcal{T}$  is a subtree of  $\mathcal{T}'_{\mathcal{P}}$  ( $\mathcal{K}'_{\mathcal{P}}$  is obtained from  $\mathcal{K}$  by adding modules). We prove by induction on the structure of  $\phi$  that for every  $\pi \in B(\mathcal{T})$ , for every position  $i$ ,

$$\mathcal{T}, \pi, i \models \phi \quad \text{iff} \quad \mathcal{T}'_{\mathcal{P}}, \pi, i \models \widehat{\phi}$$

This obviously holds for atomic propositions. We now handle the case of numerical assertion  $\text{Sum} \geq p_i$ , other numerical assertions can be handled similarly. Assume that  $\mathcal{T}, \pi, i \models (\text{Sum} \geq p_i)$ : it means that  $\text{last}_w(\pi_{\leq i}) \geq p_i$ . Then, after  $\pi_{\leq i}$  we can branch and take the transition to  $q_i$ , and the weight will still be nonnegative; we then loop over  $q_i$  until the weight reaches value 0, which allows to take the transition to  $z$ . This means that  $\mathcal{T}'_{\mathcal{P}}, \pi, i \models \mathbf{E}(q_i \wedge \mathbf{EF} \text{zero})$ , which means  $\mathcal{T}'_{\mathcal{P}}, \pi, i \models \widehat{\text{Sum} \geq p_i}$ . Conversely, assume that  $\mathcal{T}'_{\mathcal{P}}, \pi, i \models \widehat{\text{Sum} \geq p_i}$ : this means that after prefix  $\pi_{\leq i}$ , one can branch and go to  $q_i$ , loop for some time on  $q_i$  and reach  $z$  when weight is 0. This means that  $\text{last}_w(\pi_{\leq i}) \geq p_i$ , hence  $\mathcal{T}, \pi, i \models (\text{Sum} \geq p_i)$ .

The case of boolean connectors is straightforward. We now deal with the other modalities, and give details for formula  $\mathbf{A}\phi_1 \mathbf{U} \phi_2$ , which is the most difficult case. The other modalities can be handled similarly. Assume that  $\mathcal{T}, \pi, i \models \mathbf{A}\phi_1 \mathbf{U} \phi_2$ . It means that for every branch  $\pi' \in B(\mathcal{T})$  such that  $\pi_{\leq i} = \pi'_{\leq i}$ , there exists  $j \geq i$  such that  $\mathcal{T}, \pi', j \models \phi_2$  and for every  $i \leq k < j$ ,  $\mathcal{T}, \pi', k \models \phi_1$ . Now let  $\pi'' \in B(\mathcal{T}'_{\mathcal{P}})$  such that  $\pi''_{\leq i} = \pi_{\leq i}$ . We distinguish two cases:

- either  $\pi'' \in B(\mathcal{T})$ , in which case we know that there is  $j \geq i$  such that  $\mathcal{T}, \pi'', j \models \phi_2$  and for every  $i \leq k < j$ ,  $\mathcal{T}, \pi'', k \models \phi_1$ . Applying the induction hypothesis, we get that  $\mathcal{T}'_{\mathcal{P}}, \pi'', j \models \widehat{\phi}_2$  and for every  $i \leq k < j$ ,  $\mathcal{T}'_{\mathcal{P}}, \pi'', k \models \widehat{\phi}_1$ .
- or there is a position  $h > i$  such that for every  $l \geq h$ ,  $\mathcal{T}'_{\mathcal{P}}, \pi'', l \models n$  and for every  $l < h$ ,  $\mathcal{T}'_{\mathcal{P}}, \pi'', l \models \neg n$ . We can find a branch  $\pi' \in B(\mathcal{T})$  such that  $\pi'_{<h} = \pi''_{<h}$ ; in particular  $\pi'_{\leq i} = \pi_{\leq i}$ . On that branch, since  $\mathcal{T}, \pi, i \models \mathbf{A}\phi_1 \mathbf{U} \phi_2$ , there is a position  $j \geq i$  that witnesses  $\phi_2$  in  $\phi_1 \mathbf{U} \phi_2$ :  $\mathcal{T}, \pi', j \models \phi_2$  and for every  $i \leq k < j$ ,  $\mathcal{T}, \pi', k \models \phi_1$ .

We apply the induction hypothesis to  $\pi' \in B(\mathcal{T})$ :  $\mathcal{T}'_{\mathcal{P}}, \pi', j \models \widehat{\phi}_2$  and for every  $i \leq k < j$ ,  $\mathcal{T}'_{\mathcal{P}}, \pi', k \models \widehat{\phi}_1$ . If  $j < h$ , then we get that  $\mathcal{T}'_{\mathcal{P}}, \pi'', j \models \widehat{\phi}_2$  and for every  $i \leq k < j$ ,  $\mathcal{T}'_{\mathcal{P}}, \pi'', k \models \widehat{\phi}_1$ . If  $j \geq h$ , then for every  $k < h$ ,  $\mathcal{T}'_{\mathcal{P}}, \pi'', k \models \widehat{\phi}_1 \wedge \neg n$  and  $\mathcal{T}'_{\mathcal{P}}, \pi'', h \models n$ .

Globally we have proven that  $\mathcal{T}'_{\mathcal{P}}, \pi, i \models \widehat{\mathbf{A}\phi_1 \mathbf{U} \phi_2}$ .

Assume now that  $\mathcal{T}'_{\mathcal{P}}, \pi, i \models \mathbf{A}(\neg n \wedge \widehat{\phi}_1) \mathbf{U} (n \vee \widehat{\phi}_2)$ . Pick a branch  $\pi \in B(\mathcal{T})$ . Then it is a branch of  $\mathcal{T}'_{\mathcal{P}}$  as well, which never visits a state labelled by  $n$ . Hence we can apply the induction hypothesis and get that  $\pi$  satisfies  $\phi_1 \mathbf{U} \phi_2$ .  $\square$

## A.2 From weighted Kripke structures to one-counter automata

We first transform the weighted Kripke structure into a unitary one. Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure with zero-tests, and  $\phi$  be a CTL formula. The transformation simply consists in splitting a transition with weight  $w \notin \{-1, 0, +1\}$  into  $|w|$  transitions of weight  $\text{sign}(w)$ .<sup>3</sup> Assuming that the integers are encoded in binary in the input, this involves an exponential blowup in the size of the model. We write  $\mathcal{K}' = \langle S', R', \ell' \rangle$  for the new unitary weighted Kripke structure with zero-tests, and we label the newly added states with  $n'$ .

It remains to inductively transform the CTL formula using the following rules:

$$\begin{array}{l}
\widehat{p} = p \\
\widehat{\neg \phi} = \neg \widehat{\phi} \qquad \widehat{\phi_1 \wedge \phi_2} = \widehat{\phi_1} \wedge \widehat{\phi_2} \\
\widehat{\mathbf{E}X \phi} = \mathbf{E}X \mathbf{E}n' \mathbf{U} (\neg n' \wedge \widehat{\phi}) \qquad \widehat{\mathbf{E}\phi_1 \mathbf{U} \phi_2} = \mathbf{E}(\neg n' \Rightarrow \widehat{\phi_1}) \mathbf{U} (\neg n' \wedge \widehat{\phi_2}) \\
\widehat{\mathbf{A}X \phi} = \mathbf{A}X \mathbf{A}n' \mathbf{U} (\neg n' \wedge \widehat{\phi}) \qquad \widehat{\mathbf{A}\phi_1 \mathbf{U} \phi_2} = \mathbf{A}(\neg n' \Rightarrow \widehat{\phi_1}) \mathbf{U} (\neg n' \wedge \widehat{\phi_2})
\end{array}$$

This correctly avoids evaluating subformulas in the newly added states, and satisfies the expected property that  $\phi$  is true in a state of  $\mathcal{K}$  if, and only if,  $\widehat{\phi}$  holds true in the corresponding state of  $\mathcal{K}'$ .

We now build a unitary one-counter automaton  $\mathcal{C} = \langle \widetilde{S}, \widetilde{R}, \widetilde{\ell} \rangle$ . Its state space is defined as follows:

$$\widetilde{S} = \{(s, c) \mid s \in S', c \in \{-1, +1\}\} \cup \{(t, c) \mid t \in R', c \in \{-1, +1\}\}$$

<sup>3</sup> If  $w < 0$  then  $\text{sign}(w) = -1$ , and if  $w > 0$  then  $\text{sign}(w) = +1$ .

Intuitively, states of the form  $(s, c)$  are the *main states*, corresponding to the states in the original Kripke structure. The copy with  $c = +1$  is used to encode the states where the accumulated weight is nonnegative, while the copy with  $c = -1$  is used when the weight is nonpositive. States of the form  $(t, c)$  are used to jump from one side (nonnegative or nonpositive) to the other one, mimicking transition  $t$ .

We now define the transition relation  $\tilde{R}$ :

$$\tilde{R} = \{((s, c), c \cdot w, (s', c)) \mid (s, w, s') \in R'\} \cup \left\{ \begin{array}{l} ((s, c), = 0, (t, c)) \\ ((t, c), +1, (s', -c)) \end{array} \mid t = (s, w, s') \text{ with } w \in \mathbb{Z} \text{ and } c \cdot w < 0 \right\}$$

with the convention that  $c \cdot w$  is a zero-test when  $w$  is a zero test.

Finally, the labelling function  $\tilde{\ell}$  is obtained from  $\ell'$  as follows:

$$\tilde{\ell}((s, c)) = \ell'(s) \cup \{c\} \qquad \tilde{\ell}((t, c)) = \{n'\}$$

where  $n'$  is the same atomic proposition already used when constructing the unitary weighted Kripke structure  $\mathcal{K}'$ . We keep evaluating formula  $\hat{\phi}$  after this transformation, and claim the following result:

**Proposition 8.** *Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure with zero-tests,  $\phi$  be a CTL formula,  $s_0 \in S$  and  $w_0 \in \mathbb{Z}$ . Let  $\mathcal{C}$  be the one-counter automaton obtained above, and  $\hat{\phi}$  be the formula obtained from  $\phi$ . Then*

$$\mathcal{K}, (s_0, w_0) \models \phi \quad \text{iff} \quad \mathcal{C}, ((s_0, \text{sign}(w_0)), |w_0|) \models \hat{\phi}$$

where  $\text{sign}(0)$  can be taken as either  $+1$  or  $-1$ .

*Proof.* The proof of the correspondence between  $\mathcal{K}$  and  $\mathcal{K}'$  has already been mentioned and is obvious: for every  $(s_0, w_0) \in S \times \mathbb{Z}$ ,

$$\mathcal{K}, (s_0, w_0) \models \phi \quad \text{iff} \quad \mathcal{K}', (s_0, w_0) \models \hat{\phi}$$

We therefore now turn to the proof of the equivalence between  $\mathcal{K}'$  and  $\mathcal{C}$ . We define a correspondence between the paths in  $\mathcal{K}'$  and those in  $\mathcal{C}$ . First, for a path  $\pi$  in  $\mathcal{K}'$ , we define  $\text{sign}(\pi)$  inductively as follows:

$$\begin{aligned} \text{sign}((s_0, w_0)) &= \text{sign}(w_0) \\ \text{sign}(\pi \cdot (s_{n+1}, w_{n+1})) &= \begin{cases} \text{sign}(w_{n+1}) & \text{if } w_{n+1} \neq 0 \\ \text{sign}(\pi) & \text{otherwise} \end{cases} \end{aligned}$$

We can now map paths in  $\mathcal{K}'$  into paths in  $\mathcal{C}$ :

$$\begin{aligned} \zeta((s_0, w_0)) &= ((s_0, \text{sign}(w_0)), |w_0|) \\ \zeta(\pi \cdot (s_{n+1}, w_{n+1})) &= \begin{cases} \zeta(\pi) \cdot ((s_{n+1}, \text{sign}(w_{n+1})), |w_{n+1}|) & \\ \quad \text{if } \text{sign}(\pi) = \text{sign}(\pi \cdot (s_{n+1}, w_{n+1})) & \\ \zeta(\pi) \cdot ((t, -\text{sign}(w_{n+1})), w_n) \cdot ((s_{n+1}, \text{sign}(w_{n+1})), |w_{n+1}|) & \\ \quad \text{otherwise} & \end{cases} \end{aligned}$$

One easily checks that  $\zeta(\pi)$  ends up in the copy indicated by  $\text{sign}(\pi)$ , so that  $\zeta(\pi)$  is indeed a valid path in  $\mathcal{C}$ .

Conversely, from a path  $\pi'$  in  $\mathcal{C}$ , we obtain a path in  $\mathcal{K}'$  by first dropping the states of the form  $(t, c)$ , and then replacing states  $((s, c), w)$  with  $(s, c \cdot w)$ . We write  $\zeta^{-1}(\pi')$  for this run.

We finish the proof by proving that for every state  $s$  and every  $w \in \mathbb{Z}$ ,

$$\mathcal{K}', (s, w) \models \widehat{\phi} \quad \text{iff} \quad \mathcal{C}, ((s, \text{sign}(w)), |w|) \models \widehat{\phi}$$

We use an induction on the structure of formula  $\widehat{\phi}$ . The result obviously holds for atomic propositions and for boolean connectives. Assume now that  $\mathcal{K}', (s, w) \models \mathbf{EX} \mathbf{E}n' \mathbf{U} (\neg n' \wedge \widehat{\phi})$  (which is  $\widehat{\mathbf{EX}} \widehat{\phi}$ ). There is an infinite run  $\pi$  from  $(s, w)$  such that the first non-initial position, say  $i$ , where  $n'$  does not hold satisfies  $\widehat{\phi}$ . Let  $(s', w') = \text{last}(\pi_{\leq i})$ . The run  $\zeta(\pi_{\leq i})$  ends in the copy indicated by  $\text{sign}(\zeta(\pi_{\leq i}))$ , and  $\text{last}(\zeta(\pi_{\leq i}))$  is the first non-initial position along  $\zeta(\pi_{\leq i})$  where  $n'$  does not hold. Also,  $\text{sign}(\zeta(\pi_{\leq i}))$  coincides with  $\text{sign}(w')$  if  $w' \neq 0$ . By induction hypothesis,  $\mathcal{C}, ((s', \text{sign}(w')), |w'|) \models \widehat{\phi}$  and therefore  $\mathcal{C}, ((s', \text{sign}(\zeta(\pi_{\leq i}))), |w'|) \models \widehat{\phi}$  (if  $w' = 0$ , the equivalence holds for  $\text{sign}(0)$  being any of  $-1$  and  $+1$ ). We conclude that  $\mathcal{C}, ((s, \text{sign}(w)), |w|) \models \mathbf{EX} \mathbf{E}n' \mathbf{U} (\neg n' \wedge \widehat{\phi})$  ( $\zeta(\pi_{\leq i})$  being a witness).

Conversely, assume that  $\mathcal{C}, ((s, \text{sign}(w)), |w|) \models \mathbf{EX} \mathbf{E}n' \mathbf{U} (\neg n' \wedge \widehat{\phi})$ , and pick a witness  $\pi$ . Then  $\zeta^{-1}(\pi)$  is a witness in  $\mathcal{K}'$  for the same formula (using the induction hypothesis).

The other modalities can be handled similarly.  $\square$

### A.3 Model checking $\text{WCTL}_{\text{Sum}}$ with unknown initial credit

**Theorem 11.** *Model checking  $\text{WCTL}_{\text{Sum}}$  over weighted Kripke structures with unknown initial credit is EXPSPACE-complete; it is PSPACE-complete when starting from a unitary weighted Kripke structure.*

*Proof.* Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure and  $\phi$  a  $\text{WCTL}_{\text{Sum}}$  formula. Fix an initial state  $s_0 \in S$  in  $\mathcal{K}$ . Define  $\mathcal{K}_{s_0} = \langle S', R', \ell' \rangle$  as follows:  $S' = S \cup \{i\}$ ,  $R' = R \cup \{(i, -1, i), (i, +1, i), (i, 0, s_0)\}$ , and  $\ell'(i) = \emptyset$  and  $\ell'(s) = \ell(s)$  if  $s \in S \setminus \{s_0\}$  and  $\ell'(s_0) = \ell(s_0) \cup \{\text{init}\}$ . Then we easily prove:

$$\mathcal{K}', (i, 0) \models \mathbf{EF} (\text{init} \wedge \phi) \quad \text{iff} \quad \exists w_0 \in \mathbb{Z}. \mathcal{K}, (s_0, w_0) \models \phi$$

This implies the EXPSPACE and PSPACE upper bounds.

Hardness follows from the fact that one can constrain the initial credit in the formula. Let  $\mathcal{K} = \langle S, R, \ell \rangle$  be a weighted Kripke structure and  $\phi$  a  $\text{WCTL}_{\text{Sum}}$  formula. Then:

$$\mathcal{K}, (s_0, 0) \models \phi \quad \text{iff} \quad \exists w_0 \in \mathbb{Z}. \mathcal{K}, (s_0, w_0) \models (\text{Sum} = 0) \wedge \phi \quad \square$$

## B Proofs of Section 4

**Theorem 12.** *Model checking  $\text{WCTL}_{\text{Avg}}^{\text{rf}}$  (and therefore  $\text{WCTL}_{\text{Avg}}$ ) over weighted Kripke structures is undecidable.*

*Proof.* We encode the halting problem for (deterministic) two-counter machines into our model-checking problem with fixed initial value. The values  $c_1$  and  $c_2$  of the counters at a given position along an execution is encoded by the length of the path being of the form  $2^{c_1} \cdot 3^{c_2} \cdot 5^a$ , where  $a$  is a nonnegative integer. Decrementing counter  $c_1$  will then amount to multiplying the length of the path by  $5/2$ , which will be achieved by taking a self-loop on a state until the total average value of the weight reaches a given value.

Before formalizing this, we take an example: Figure 3 depicts a *module* from  $(q, A_k)$  to  $(q', A_{k'})$ . This module will be used to modify (increment or decrement) the counters. The  $\text{WCTL}_{\text{Avg}}^{\text{rf}}$  formula will enforce that the average weight value of a path ending in a state labelled  $A_k$  be exactly  $k$ , for all  $0 \leq k \leq 2$ . Consider such a path, of length  $n$ , ending in state  $(q, A_k)$  of Figure 3 (so that the accumulated weight is  $k \cdot n$ ). Now, we extend this path following the depicted module, until reaching  $(q', A_{k'})$ . Write  $x - 2$  for the number of times we take the self-loop on state  $(r, k, k')$ . Then the length of the path when reaching  $(q', A_{k'})$  is  $n + x$ , and its total accumulated weight is  $k \cdot n + \frac{(jk' - k) \cdot x}{j - 1}$ . The requirement that the average be  $k'$  in  $(q', A_{k'})$  entails

$$k \cdot n + \frac{(jk' - k) \cdot x}{j - 1} = k' \cdot (n + x).$$

One easily checks that, provided  $k \neq k'$ , this implies  $x = (j - 1) \cdot n$ , so that the length of the whole path when reaching  $(q', A_{k'})$  is  $j \cdot n$  (assuming  $j > 1$ ), and the average is indeed  $k'$ . Taking values 2, 3,  $5/2$  and  $5/3$  for  $j$  will implement all four instructions modifying the counters.

Similarly, Figure 4 is a module testing whether counter  $c_2$  equals zero. Starting in state  $(q, A_k)$  with a path of length  $n$ , and total accumulated weight  $k \cdot n$ , it is rather clear that we can reach state  $(r, A_0, \text{ok})$  if, and only if,  $n$  is an integer multiple of 3 (which means that  $c_2 > 0$ ). State  $(r, A_0, \text{ok})$  is then reached by a path of total length  $4n/3$ , and accumulated weight zero. Then the path goes to  $(r, 0, 1, \text{ok})$ , and takes the loop  $x - 2$  times, and reaches  $(q', A_1)$ . The accumulated weight then is  $15x/11$ , and the length of the path is  $4n/3 + x$ . The average is 1 exactly when  $x = 11n/3$ , for which the total length of the path is  $5n$ ; this way, the counters are back to their original value when reaching  $(q', A_1)$ .

If we follow the middle branch of the module, we reach  $(r, A_0, \text{nok})$  with average zero if, and only if,  $n - 1$  is an integer multiple of 3, which implies that  $n$  is not, so that  $c_2 = 0$ . The length of the path when reaching  $(r, A_0, \text{nok})$  is  $n + 1 + (n - 1)/3$ . We then reach  $(q'', A_1)$  after looping  $x - 2$  times on  $(r, 0, 1, \text{nok})$ ; then the length of the path is  $(4n + 2)/3 + x$ , and the total weight is  $(x + 2/3) \cdot 15/11$ . Since we require the average weight to equal 1, we get  $x = (11n - 2)/3$ , which yields a total final length of  $5n$ , as expected. A similar computation can be conducted for the rightmost branch.

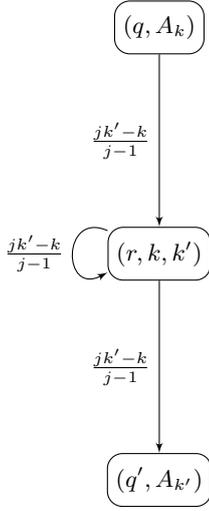


Fig. 8. Updating counters

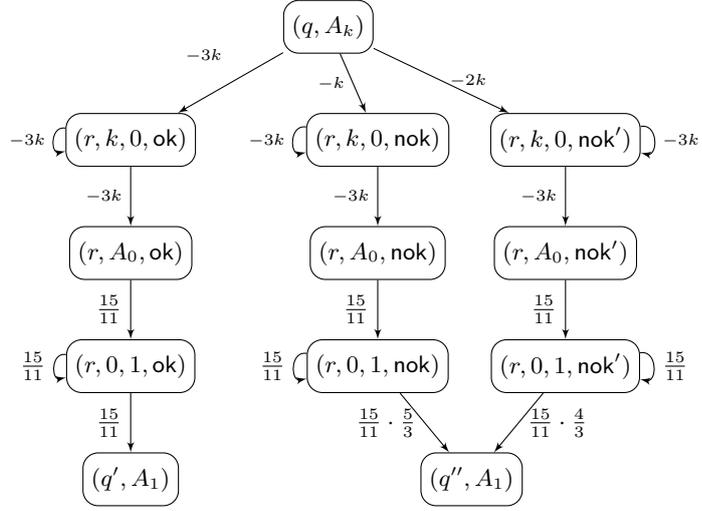


Fig. 9. Testing counters (here  $c_2$ )

We now formalize the proof. We consider a two-counter machine  $\mathcal{M} = \langle Q, R \rangle$  where  $Q$  is the set of states containing two special states *init* and *halt*, and  $R \subseteq Q \times (\{c_1 ++, c_1 --, c_2 ++, c_2 --\} \times Q \cup \{c_1 > 0, c_2 > 0\} \times Q \times Q)$  is the set of transitions. From  $\mathcal{M}$ , we define a weighted Kripke structure  $\mathcal{K} = \langle S, T, \ell \rangle$  as follows:

- the set  $S$  of states is the disjoint union of three sets:
  - the set of *main states*, which is the product of  $Q$  with the set  $\{A_1, A_2\}$ ;
  - the set of *auxiliary states*, used to encode transitions of  $\mathcal{M}$ :
    - \* for each instruction  $r = (q, \text{op}, q')$  of  $R$  incrementing or decrementing one of the counters, and for each  $k \in \{1, 2\}$  and  $k' = 3 - k$ , there is a state  $(r, k, k')$ ;
    - \* for each instruction  $r = (q, \text{test}_1, q', q'')$  of  $R$  testing whether  $c_1$  equals 0, there are six auxiliary states, which we name  $(r, k, 0, \text{ok})$ ,  $(r, A_0, \text{ok})$ ,  $(r, 0, 1, \text{ok})$  and  $(r, k, 0, \text{nok})$ ,  $(r, A_0, \text{nok})$ ,  $(r, 0, 1, \text{nok})$ .
    - \* finally, for each instruction  $r = (q, \text{test}_2, q', q'')$  of  $R$  testing whether  $c_1$  equals 0, there are nine auxiliary states, which we name  $(r, k, 0, \text{ok})$ ,  $(r, A_0, \text{ok})$ ,  $(r, 0, 1, \text{ok})$ , and  $(r, k, 0, \text{nok})$ ,  $(r, A_0, \text{nok})$ ,  $(r, 0, 1, \text{nok})$ , and  $(r, k, 0, \text{nok}')$ ,  $(r, A_0, \text{nok}')$ ,  $(r, 0, 1, \text{nok}')$ .
  - the set of *initializing states*, named  $(\text{init}, n)$  with  $n$  ranging from 1 to 5. Those states will be used to set the “initial” length of the path to 5, and the average to 1.

Before defining the set of transitions of  $\mathcal{K}$ , we define its labelling:

- states  $(\text{halt}, A_1)$  and  $(\text{halt}, A_2)$  are labelled with *halt*;
- each state whose name contains  $A_k$  is labelled with  $a_k$ ;

It remains to define the set of transitions and their weights.

- for all  $r = (q, \text{op}, q')$  in  $R$  (this does not include zero-tests), for each  $k \in \{1, 2\}$  and for  $k' = 3 - k$ , there are transitions from  $(q, A_k)$  to  $(r, k, k')$ , from  $(r, k, k')$  to itself, and from  $(r, k, k')$  to  $(q', A_{k'})$ . They all have weight  $\frac{jk' - k}{j-1}$  where  $j = 2$  (resp  $3, 5/2, 5/3$ ) if  $\text{op} = c_1 ++$  (resp.  $c_2 ++, c_1 --, c_2 --$ ).
- for all  $r = (q, \text{test}_1, q', q'')$  in  $R$ , for each  $k \in \{1, 2\}$ , we have two branches out of  $(q, A_k)$ : first, we have transitions from  $(q, A_k)$  to  $(r, k, 0, \text{ok})$ , from  $(r, k, 0, \text{ok})$  to itself, and from  $(r, k, 0, \text{ok})$  to  $(r, A_0, \text{ok})$ , all having weight  $-2k$ ; transitions from  $(r, A_0, \text{ok})$  to  $(r, 0, 1, \text{ok})$ , from  $(r, 0, 1, \text{ok})$  to itself, and from  $(r, 0, 1, \text{ok})$  to  $(q'', A_1)$ , all with weight  $10/7$ .  
Similarly, we have a transition from  $(q, A_k)$  to  $(r, k, 0, \text{nok})$  with weight  $-k$ , transitions from  $(r, k, 0, \text{nok})$  to itself and from  $(r, k, 0, \text{ok})$  to  $(r, A_0, \text{ok})$ , both having weight  $-2k$ ; transitions from  $(r, A_0, \text{ok})$  to  $(r, 0, 1, \text{ok})$  and from  $(r, 0, 1, \text{ok})$  to itself, both with weight  $10/7$ , and a transition from  $(r, 0, 1, \text{ok})$  to  $(q'', A_1)$  with weight  $10/7 \cdot 3/2$ .
- for all  $r = (q, \text{test}_2, q', q'')$  in  $R$ , for each  $k \in \{1, 2\}$ , we have three branches out of  $(q, A_k)$ : first, we have transitions from  $(q, A_k)$  to  $(r, k, 0, \text{ok})$ , from  $(r, k, 0, \text{ok})$  to itself, and from  $(r, k, 0, \text{ok})$  to  $(r, A_0, \text{ok})$ , all having weight  $-3k$ ; transitions from  $(r, A_0, \text{ok})$  to  $(r, 0, 1, \text{ok})$ , from  $(r, 0, 1, \text{ok})$  to itself, and from  $(r, 0, 1, \text{ok})$  to  $(q'', A_1)$ , all with weight  $15/11$ .  
Similarly, we have a transition from  $(q, A_k)$  to  $(r, k, 0, \text{nok})$  with weight  $-k$ , transitions from  $(r, k, 0, \text{nok})$  to itself and from  $(r, k, 0, \text{nok})$  to  $(r, A_0, \text{nok})$ , both having weight  $-3k$ ; transitions from  $(r, A_0, \text{nok})$  to  $(r, 0, 1, \text{nok})$  and from  $(r, 0, 1, \text{nok})$  to itself, both with weight  $15/11$ , and a transition from  $(r, 0, 1, \text{nok})$  to  $(q'', A_1)$ , with weight  $15/11 \cdot 5/3$ .  
Finally, we have a transition from  $(q, A_k)$  to  $(r, k, 0, \text{nok}')$  with weight  $-2k$ , transitions from  $(r, k, 0, \text{nok}')$  to itself and from  $(r, k, 0, \text{nok}')$  to  $(r, A_0, \text{nok}')$ , both having weight  $-3k$ ; transitions from  $(r, A_0, \text{nok}')$  to  $(r, 0, 1, \text{nok}')$  and from  $(r, 0, 1, \text{nok}')$  to itself, both with weight  $15/11$ , and a transition from  $(r, 0, 1, \text{nok}')$  to  $(q'', A_1)$ , all with weight  $15/11 \cdot 4/3$ .
- finally, there are transitions from  $(\text{init}, n)$  to  $(\text{init}, n + 1)$ , for  $1 \leq n \leq 4$ , and a transition from  $(\text{init}, 5)$  to  $(\text{init}, A_1)$ . They all have weight 1.

The  $\text{WCTL}_{\text{Avg}}$  has to enforce that the average weight is  $k$  when visiting states labelled with  $a_k$ , and that the halting state is reached. This is enforced by

$$\phi = \mathbf{E} \left( \bigwedge_{k \in \{0,1,2\}} (a_k \Rightarrow (\text{Avg} = k)) \right) \mathbf{U} \text{halt}.$$

We now claim that the two-counter machine halts if, and only if,  $\phi$  holds true in state  $(\text{init}, 1)$  of  $\mathcal{K}$ . This is easily proved, following the arguments above.

The undecidability proof for the unknown initial credit can now easily be obtained using formula

$$(\text{Avg} = 1) \wedge \phi$$

which enforces the initial weight be 1. □

## C Proofs of Section 5

### C.1 Decidability of $\text{WTL}_{\text{Sum}}^{\text{rf}}$ model checking

Consider a formula  $\phi \in \text{WTL}_{\text{Sum}}^{\text{rf}}$ , and a weighted Kripke structure  $\mathcal{K}$ . As for the branching-time case, we begin with moving numerical assertions from the formula into the model. We plug in each state of  $\mathcal{K}$  a module for checking the value of the accumulated weight: the module decrements (or increments) the accumulated weight until reaching value zero, and then increments (resp. decrements) it back to its original value; see Fig. 10. For this to work, we have to forbid decrementing the counter below zero, as otherwise there is always a valid path going through state  $r_M$ . We thus first move to the setting of one-counter automata, following the construction of Section 3.2: we first turn our weighted Kripke structure into a unitary one, and define a one-counter automaton made of two copies of this structure, as depicted on Fig. 2. The correspondence we defined in the proof of Prop. 8 shows that both models essentially exhibit the same behaviours.

Now, let  $M$  be the maximal (absolute value of the) constants involved in  $\phi$ . We replace the numerical assertions in  $\phi$  with LTL formulas that will be evaluated in an auxiliary module (depicted on Fig. 10). This module has  $2(M + 2)$  states  $\{r_k, r'_k \mid 0 \leq k \leq M + 1\}$ , and transitions

$$\{(r_k, -1, r_{k+1}), (r'_{k+1}, 1, r'_k), (r_k, = 0, r'_k) \mid 0 \leq k \leq M\} \cup \{(r_{M+1}, 0, r'_{M+1})\}.$$

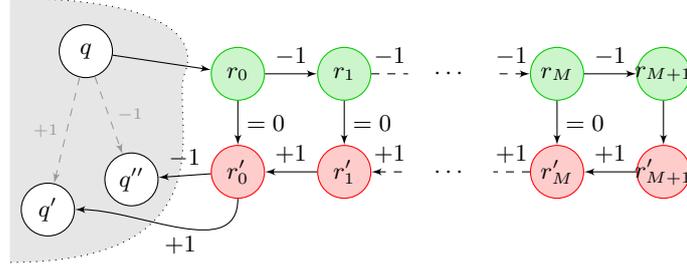
When entering this module in state  $r_0$  with counter value  $c \geq 0$ , then

- if  $c \leq M$ , there is exactly one possible run to  $r'_0$ , which takes the transition from  $r_c$  to  $r'_c$ ;
- if  $c > M$ , there is exactly one possible run to  $r'_0$ , which visits  $r_{M+1}$  and  $r'_{M+1}$ .

Each state  $q$  of the one-counter automaton above is equipped with one copy of this module: from  $q$ , there is only one possible transition, going to state  $r_0$  in the module associated with  $q$ . Then, from state  $r'_0$  of this module, we have all the transitions that were possible from  $q$  in the one-counter automaton. The states of the modules are labelled with  $n'$  (the same atomic propositions that we used in Section 3.2). We also require that the states in the “nonpositive” copy of the original Kripke structures, as well as all the states of the associated test modules, are labelled with  $\text{neg}$ . We write  $\mathcal{C}$  for the one-counter machine that we have built.

We now have to modify the  $\text{WTL}_{\text{Sum}}^{\text{rf}}$  formula into an “equivalent” LTL formula, encoding the numerical assertions as subformulas querying the run in the test modules. The inductive transformation is as follows:

$$\begin{aligned} \widehat{p} &= p \\ \widehat{\neg \phi} &= \neg \widehat{\phi} \\ \widehat{\phi \vee \psi} &= \widehat{\phi} \vee \widehat{\psi} \\ \widehat{\mathbf{X} \phi} &= \mathbf{X} (n' \mathbf{U} (\neg n' \wedge \widehat{\phi})) \\ \widehat{\phi \mathbf{U} \psi} &= (\neg n' \Rightarrow \widehat{\phi}) \mathbf{U} (\neg n' \wedge \widehat{\psi}) \end{aligned}$$



**Fig. 10.** From  $\text{WLTLSum}^{\text{rf}}$  to LTL (notice that  $q'$  and  $q''$  also have their corresponding test modules, which we omitted to draw for the sake of readability).

$$\widehat{\text{Sum}} = k = \begin{cases} \neg \text{neg} \wedge \mathbf{X}(n' \mathbf{U}(r_k \wedge \mathbf{X} r'_k)) & \text{if } k > 0 \\ \mathbf{X}(r_0 \wedge \mathbf{X} r'_0) & \text{if } k = 0 \\ \text{neg} \wedge \mathbf{X}(n' \mathbf{U}(r_{-k} \wedge \mathbf{X} r'_{-k})) & \text{if } k < 0 \end{cases}$$

$$\widehat{\text{Sum}} \geq k = \begin{cases} \neg \text{neg} \wedge \mathbf{X}(n' \mathbf{U} r'_k) & \text{if } k > 0 \\ \neg \text{neg} \vee \mathbf{X}((n' \wedge \neg r'_{-k+1}) \mathbf{U} r'_0) & \text{if } k \leq 0 \end{cases}$$

$$\widehat{\text{Sum}} \leq k = \begin{cases} \text{neg} \vee \mathbf{X}((n' \wedge \neg r'_{k+1}) \mathbf{U} r'_0) & \text{if } k \geq 0 \\ \text{neg} \wedge \mathbf{X}(n' \mathbf{U} r'_{-k}) & \text{if } k < 0 \end{cases}$$

The correspondence we defined in the proof of Prop. 8 can easily be adapted to a correspondence between the paths in the original structure and those in the one-counter automata with test modules. Writing  $\zeta$  for the correspondence mapping the paths of the original structure to the paths of the one-counter automaton with test modules, we have:

**Proposition 17.** *Let  $\phi \in \text{WLTLSum}^{\text{rf}}$ , and  $\pi$  be a path in  $\mathcal{K}$ . It holds that  $\pi \models \phi$  if, and only if,  $\zeta(\pi) \models \widehat{\phi}$ .*

We are now left with a model-checking problem for LTL over (unitary) one-counter automata. This can be done in PSPACE [DG09]. Applying this result naively would give an EXPSpace algorithm, since our transformation of the model is exponential. However, a more careful look at the size of the transformed model shows that it has size at most  $m \cdot |K| \cdot 2(M+2)$ , where  $m$  is the maximal constant (absolute value) appearing in  $\mathcal{K}$ . We build a Büchi automaton for  $\widehat{\phi}$ , whose size will be at most  $2^{O(|\phi|)}$ . The product of this automaton with the transformed Kripke structure is then in  $m \cdot |K| \cdot 2(M+2) \cdot 2^{O(|\phi|)}$ . Deciding whether a unitary Büchi one-counter automaton has an accepting run being NLOGSPACE-complete; this can be applied on-the-fly, without explicitly building the whole of our Büchi one-counter automaton, so that we get a PSPACE algorithm.

Thus:

**Theorem 14.** *Model checking  $\text{WLTLSum}^{\text{rf}}$  over weighted Kripke structures both with fixed and unknown initial credit is PSPACE-complete.*