PhD defense — October 30th, 2013

Algorithms and arithmetic for the implementation of cryptographic pairings

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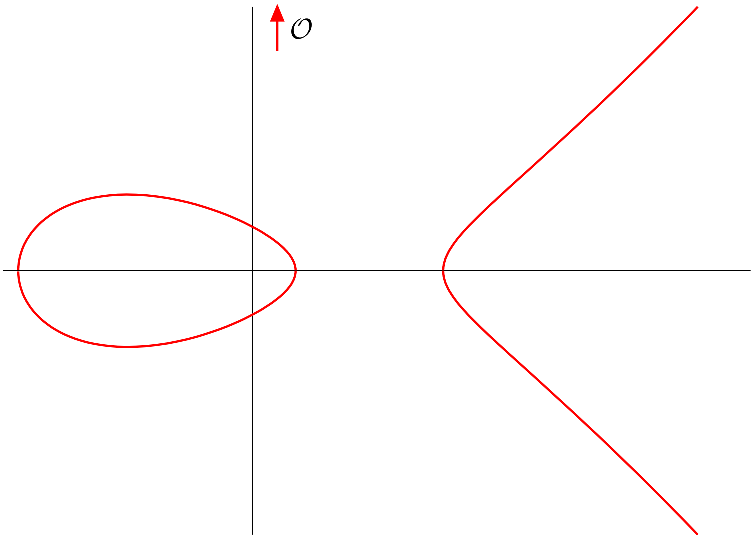
What is an elliptic curve?

$$E/K : y^2 + h(x)y = f(x)$$
with $\deg h \leq 1$ and $\deg f = 3$



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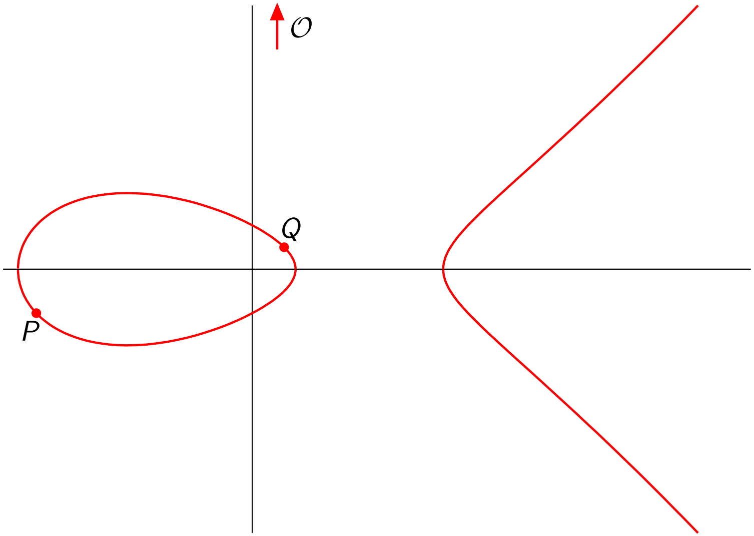

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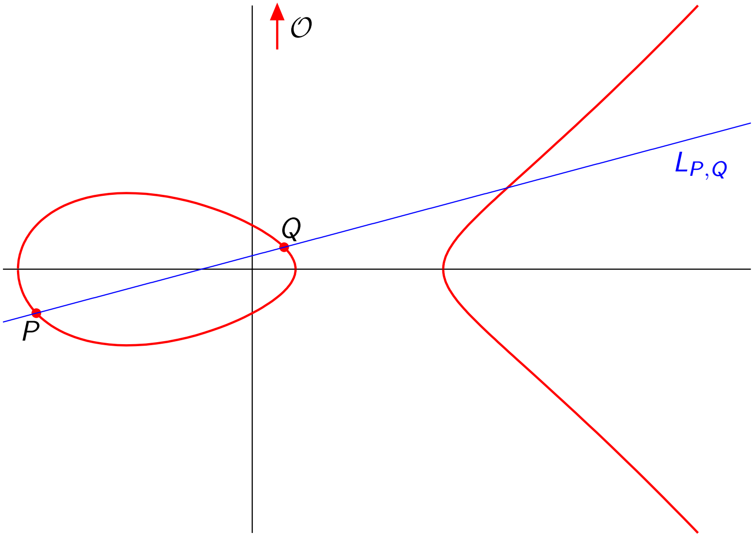

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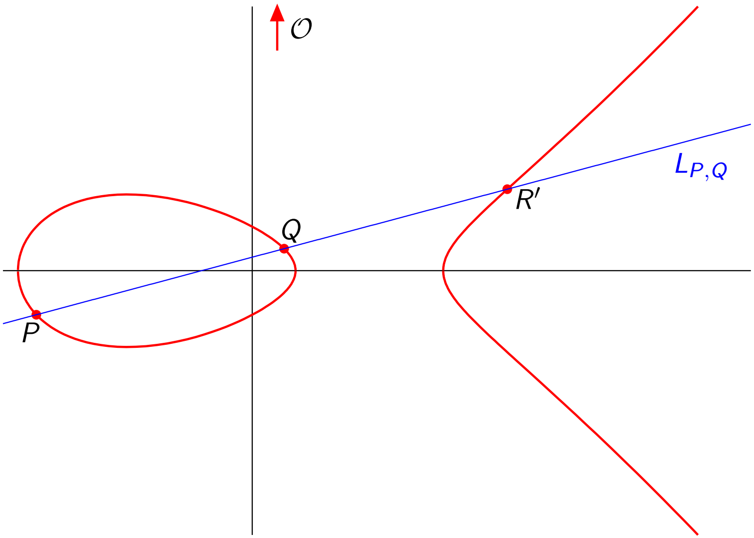

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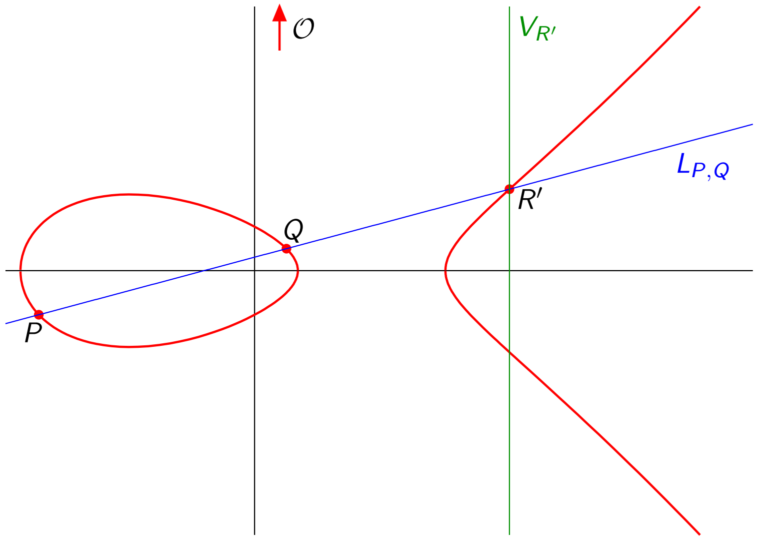

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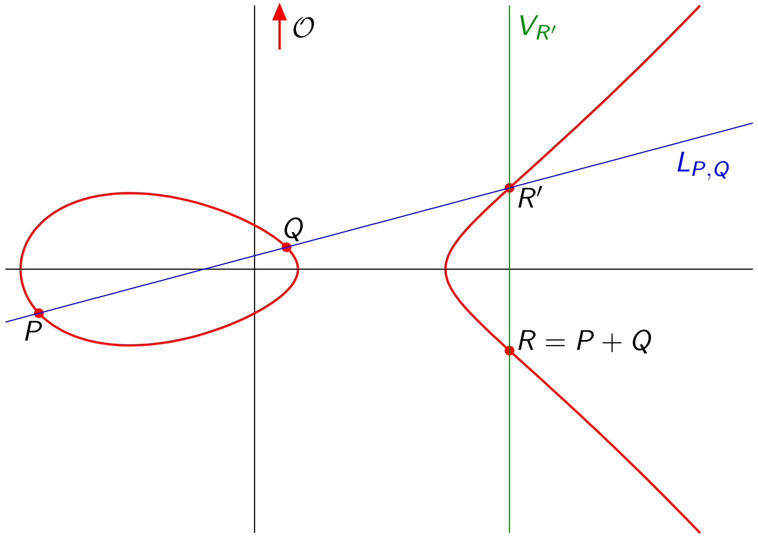

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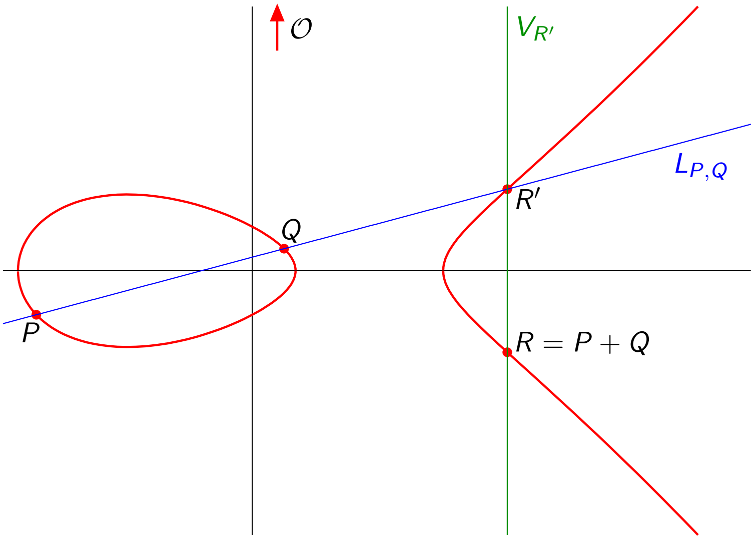

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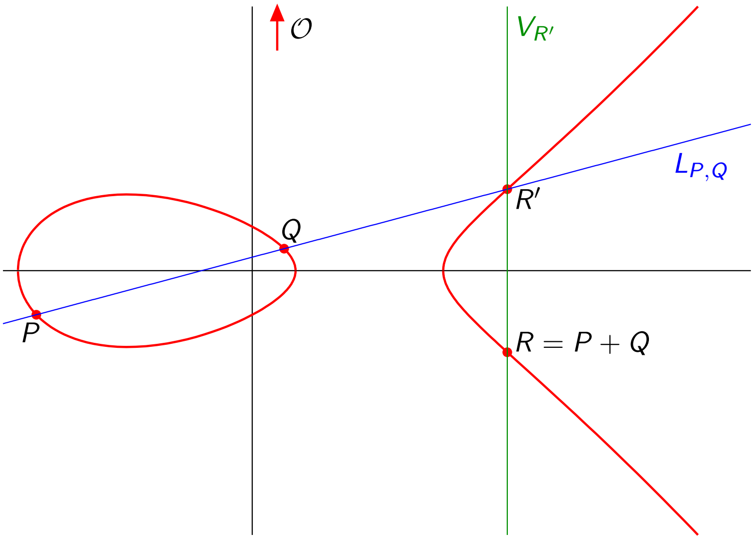

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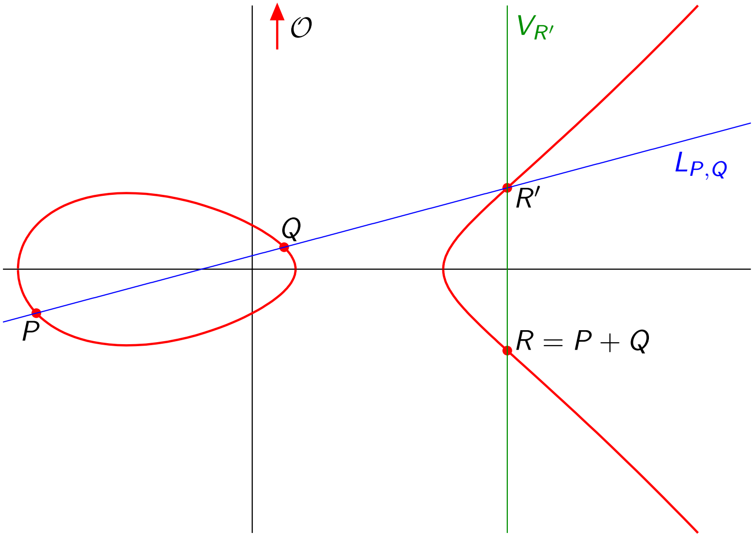

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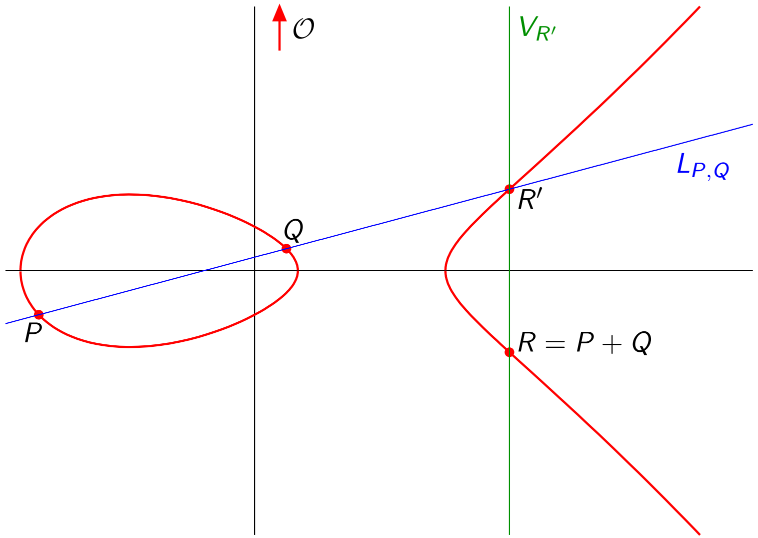

- Our favorite curves: $E_3 : y^2 = x^3 - x \pm 1$
  - characteristic 3
  - supersingular

Elliptic Curve Cryptography

Discrete Logarithm Problem (DLP)

Let $\mathbb{G}$ be a cyclic group, P a generator, given $Q \in \mathbb{G}$, it is supposed to be hard to compute a such that

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- Use this hard problem to design cryptographic protocols
- Diffie–Hellman key exchange:

  - Alice generates a secret integer a
  - Alice sends $[a]P$ to Bob
  - Alice computes $[a][b]P$

  - Bob generates a secret integer b
  - Bob sends $[b]P$ to Alice
  - Bob computes $[b][a]P$

They both share the same secret: $[ab]P$

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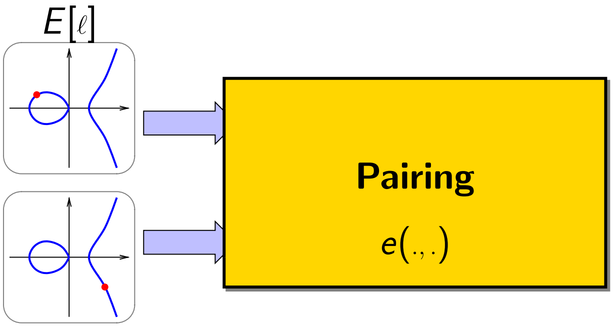

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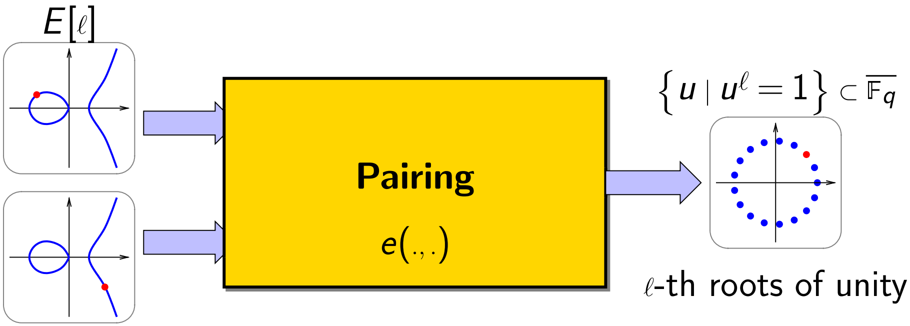

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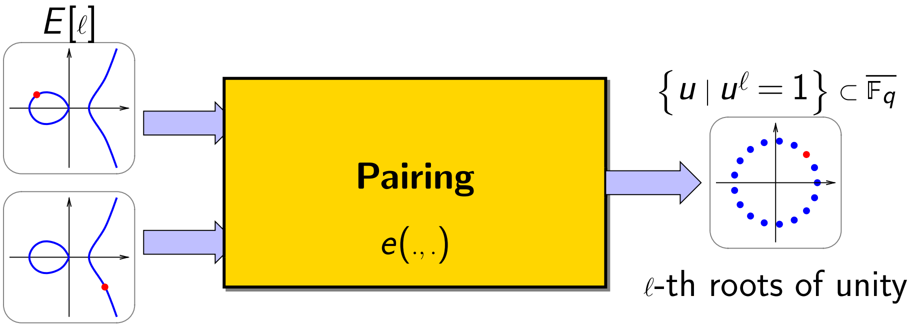

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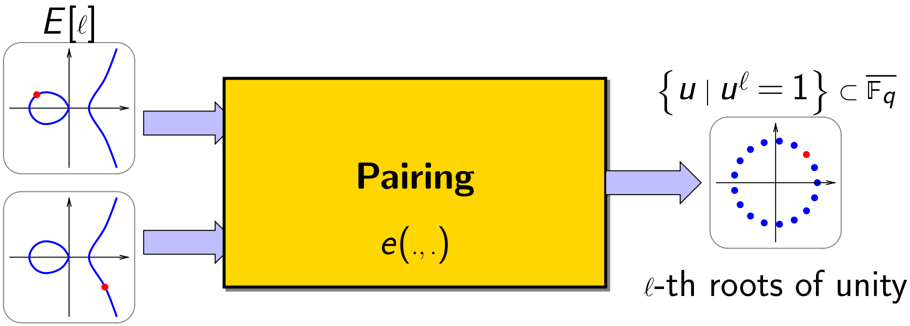

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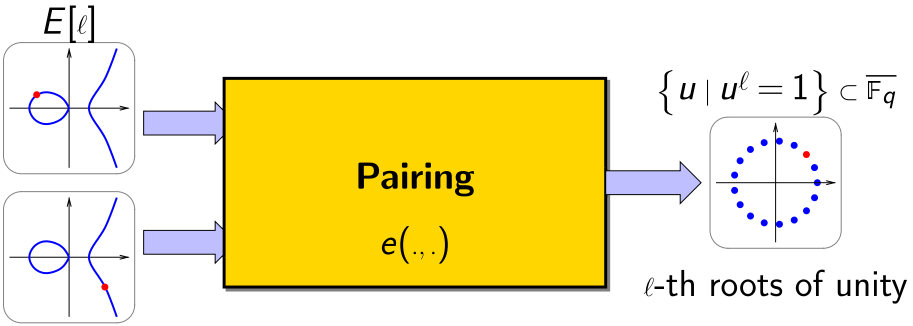

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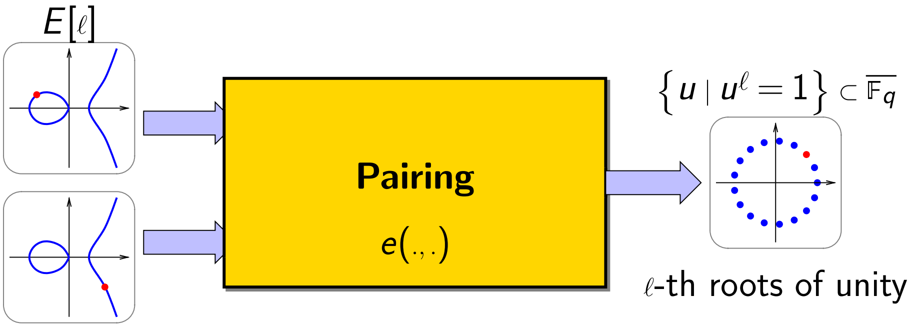

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- DLP should be hard on all the groups involved

Security considerations

- Security measurement
  - number of operations to break a cryptosystem
  - today's recommendation: 128-bit security

$$2^{128} \text{ operations}$$

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E_3 over $\mathbb{F}_{3^{509}}$

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$k = 6$, so DLP in $(\mathbb{F}_{3^{6\cdot 509}})^*$

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    - ⋆ very recent results (2013)

      Records by Joux and Göloğlu et al. records

      Joux

      Barbulescu, Gaudry, Joux, Thomé

      Adj, Menezes, Oliveira, Rodríguez-Henríquez — $\lessapprox 2^{75}$ operations

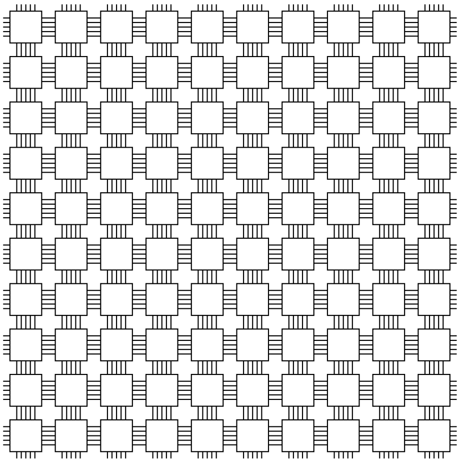
Why cryptography and hardware implementations?

- Growth of numeric exchanges
  - many applications
    - bank services
    - secure firmware updates
    - personal communications
    - ...
  - many targets
    - embedded electronics
    - smart cards
    - smartphones
    - computers, servers
- Security implies non-trivial computations
- Need for hardware implementations
  - CPUs may be inadequate
  - limited resources





Hardware implementation



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  - integrated circuit
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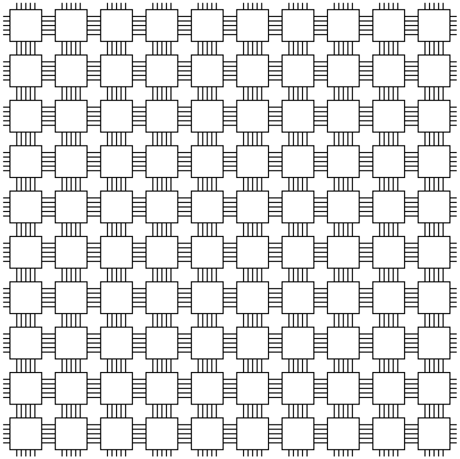

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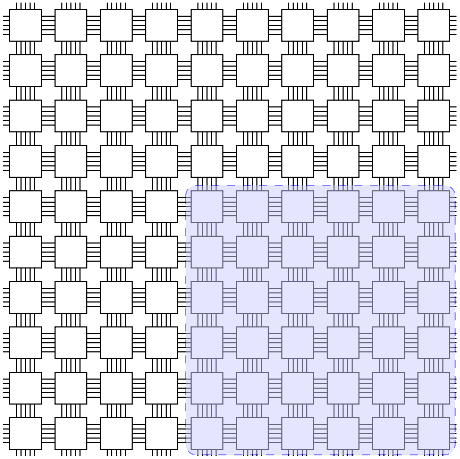

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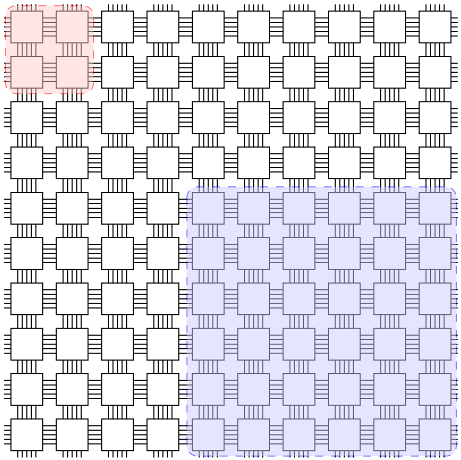

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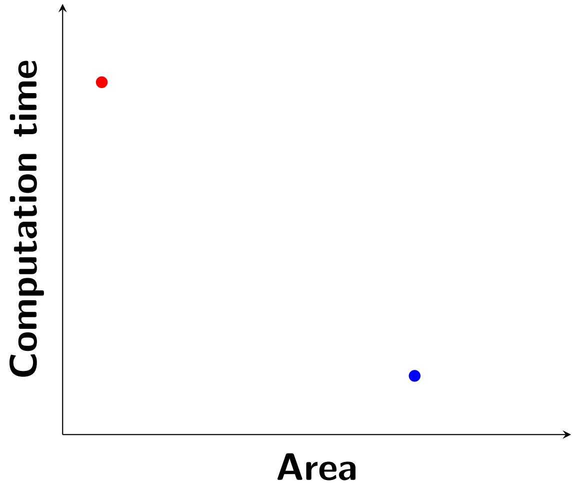

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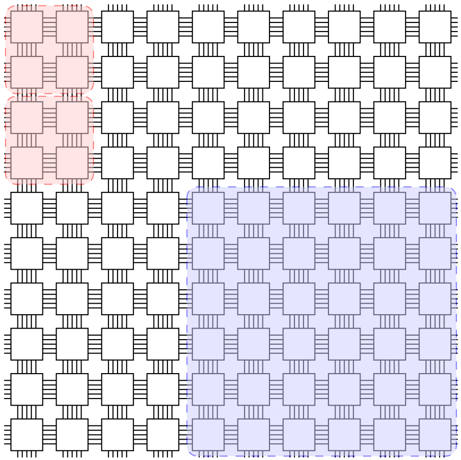

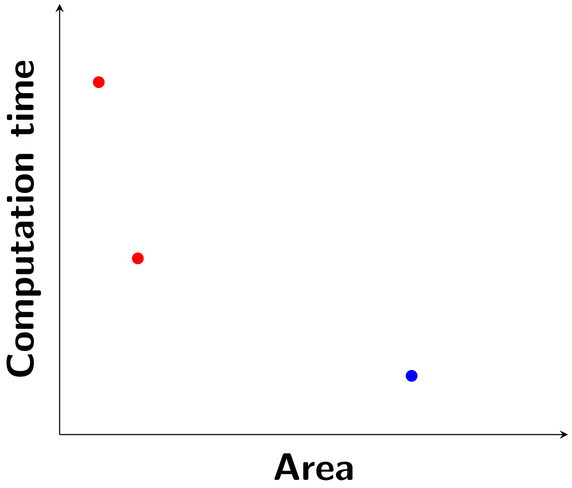

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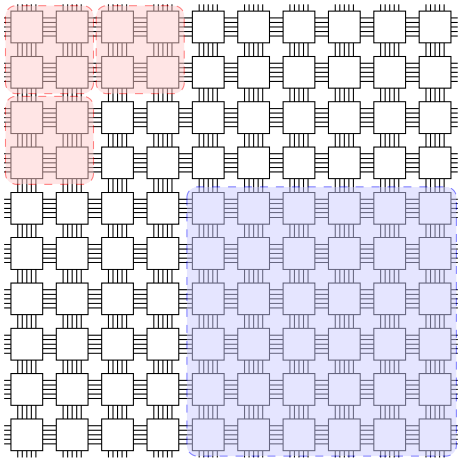

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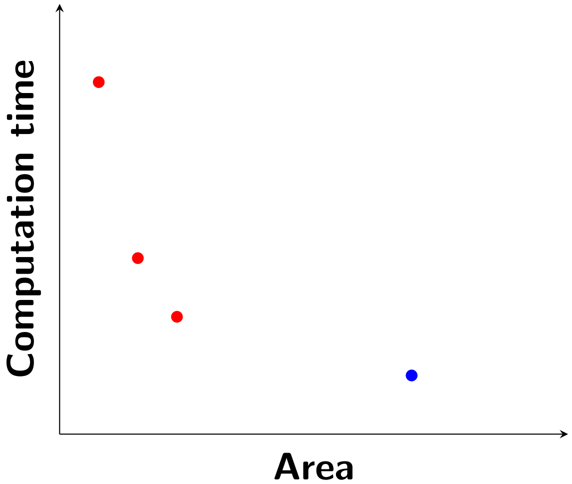

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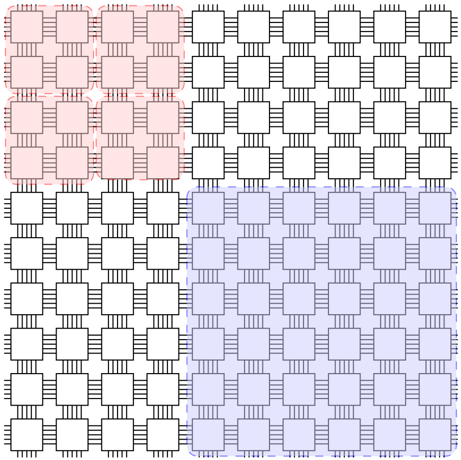

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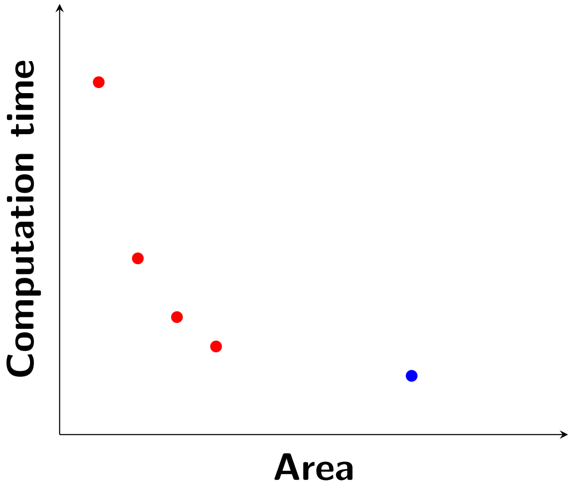

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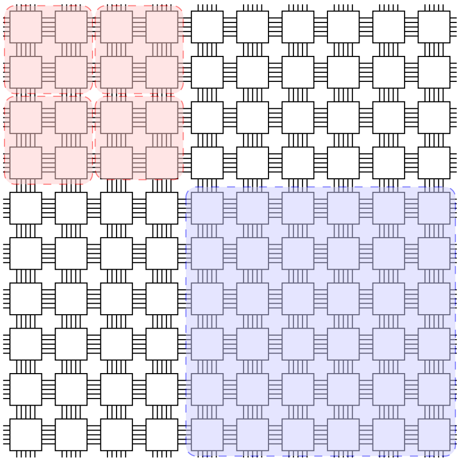

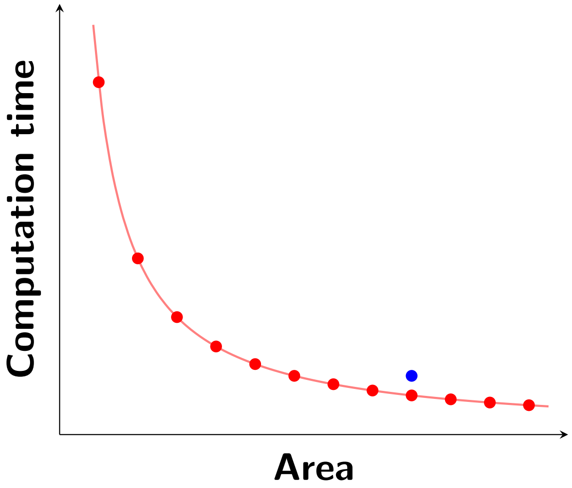

Contributions

- Fast accelerator for pairings [CHES 2009, IEEE TC 2011]
  Joint work with Beuchat, Detrey, Okamoto and Rodríguez-Henríquez
  - parallel architecture
  - pipelined subquadratic multiplier
- Compact design for pairings reaching 128-bit security
  - composite extension fields [Paring 2010]
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    Joint work with Aranha, Beuchat and Detrey
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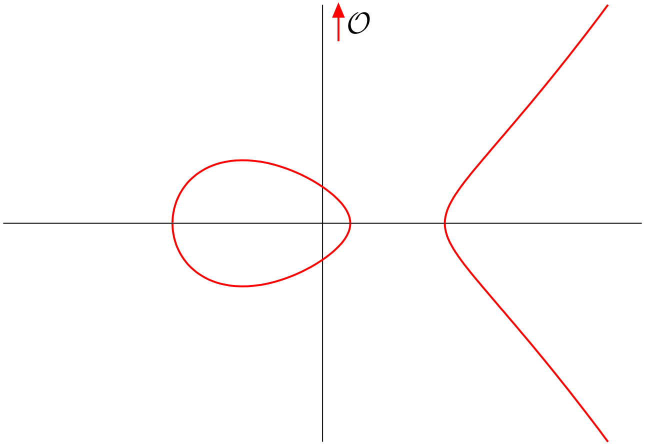
Outline of the talk

- Compact design through composite extension fields
- Pairing on genus-2 hyperelliptic curves
- Searching for efficient multiplication algorithms
- Conclusion and Perspectives

Computing the pairing: Miller's algorithm

- Computation of the pairing relies on Miller functions: $f_{n,P}$



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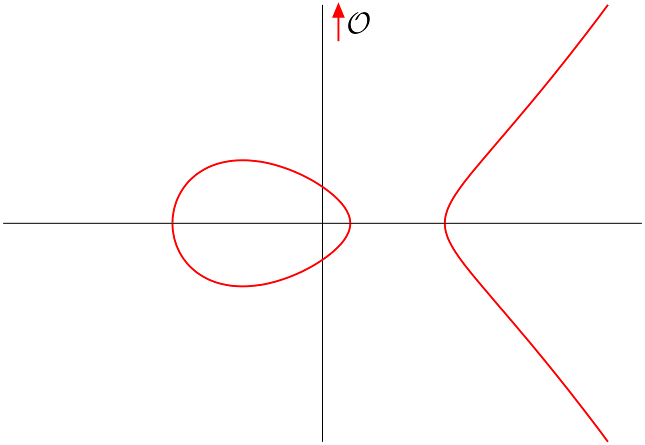

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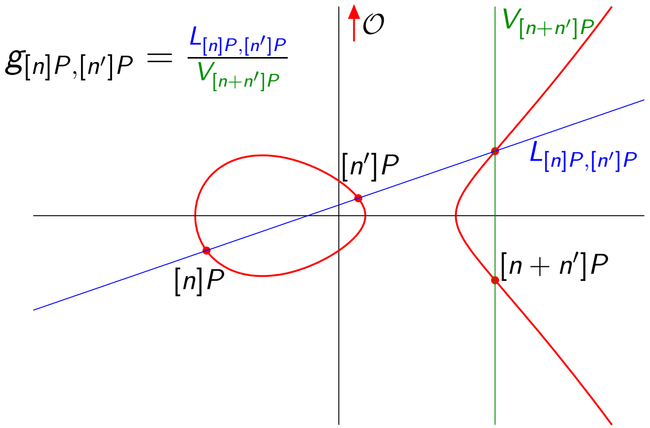

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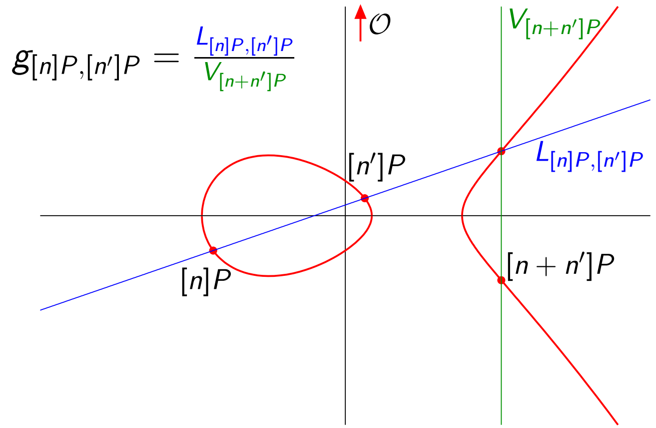

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  - in practice: double-and-add

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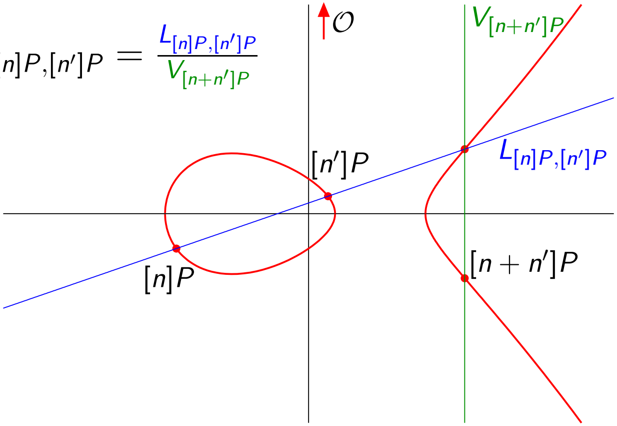
$$g_{[n]P,[n']P} = \frac{L_{[n]P,[n']P}}{V_{[n+n']P}}$$



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For $E_3(\mathbb{F}_{3^{509}})$	Tate pairing
# iterations	509
$\times$	10330
$+$	45170
$(.)^3$	8136
$(.)^{-1}$	2

- $\#E_3(\mathbb{F}_{3^{509}}) = 3^{509} + 3^{255} + 1$
  - triple-and-add algorithm

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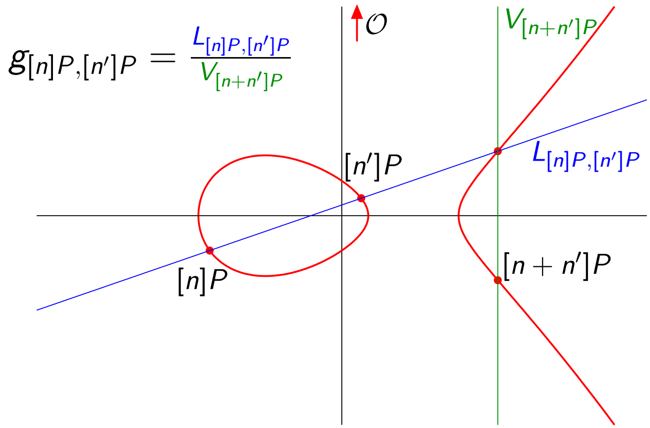
    $$\begin{aligned} f_{1,P} &= 1 \\ f_{n+n',P} &= f_{n,P} \cdot f_{n',P} \cdot g_{[n]P,[n']P} \end{aligned}$$
  - $g_{[n]P,[n']P}$ derived from the addition of $[n]P$ and $[n']P$

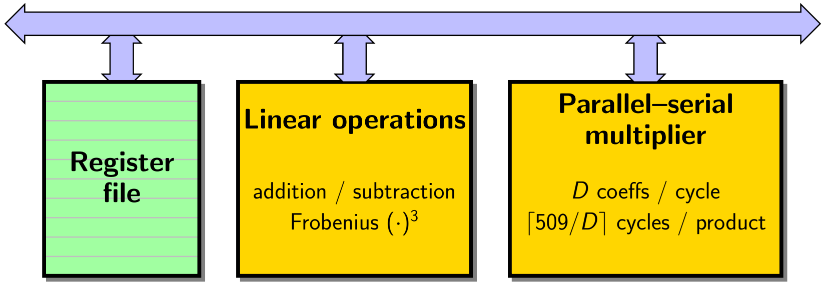


- Tate pairing: $f_{\#E(\mathbb{F}_q),P}$
  - use an addition chain
  - in practice: double-and-add
    $\log_2 \#E(\mathbb{F}_q)$ iterations

For $E_3(\mathbb{F}_{3^{509}})$	Tate pairing	Eta T
# iterations	509	254
$\times$	10330	3638
$+$	45170	17240
$(.)^3$	8136	4068
$(.)^{-1}$	2	1

- $\#E_3(\mathbb{F}_{3^{509}}) = 3^{509} + 3^{255} + 1$
  - triple-and-add algorithm
- Many improvements
  - vertical elimination
  - use of some curve endomorphisms
    - ⋆ Frobenius: Ate
    - ⋆ Verschiebung: Eta, Eta T

An arithmetic coprocessor



- Only need arithmetic operations in $\mathbb{F}_{3^{509}}$
  - implement a specialized processor
- Multiplication is critical
  - separate linear operations and multiplications
  - careful scheduling to keep multiplier busy

Operation count

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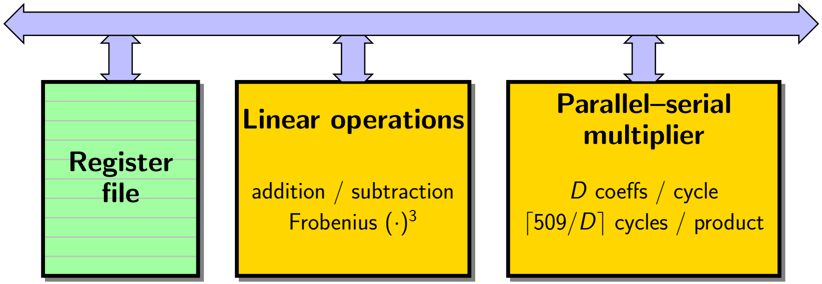

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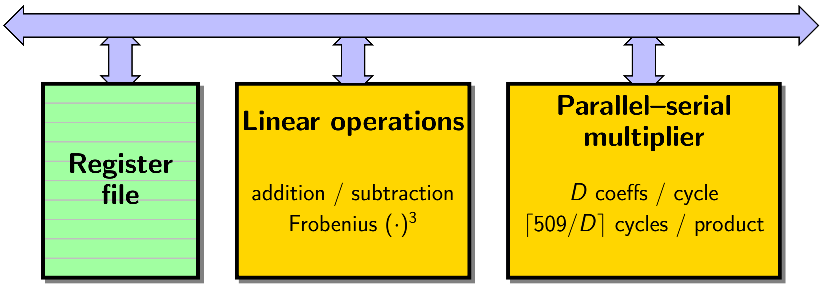
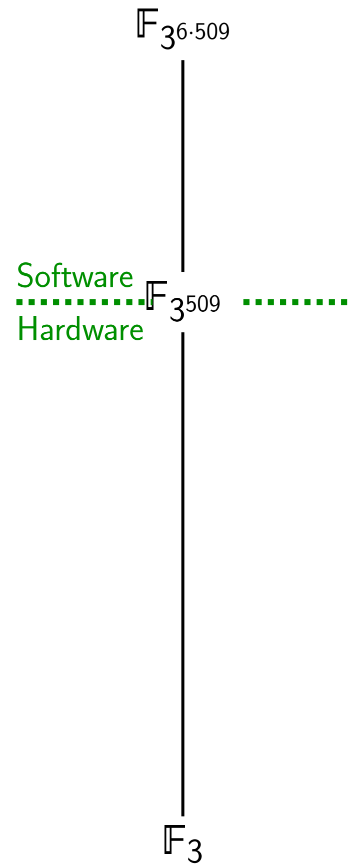


- Only need arithmetic operations in $\mathbb{F}_{3^{509}}$
  - implement a specialized processor
- Multiplication is critical
  - separate linear operations and multiplications
  - careful scheduling to keep multiplier busy
- Inverse is only needed once: Itoh–Tsujii algorithm
  - no need for hardware support
- Synthesis results for $\mathbb{F}_{3^{509}}$: 9625 slices
  - almost fully occupy a Virtex 6 LX 75 T (82%)
  - computation time: ≈ 4 ms

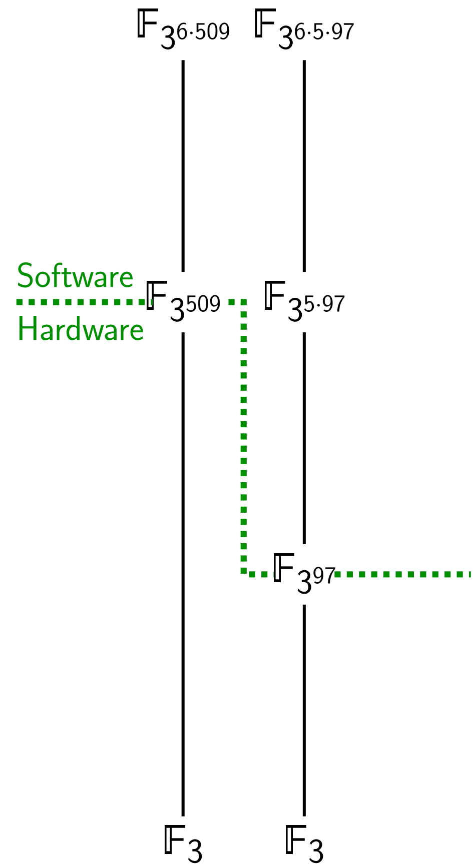
Operation count	
$\times$	3638
$+$	17240
$(.)^3$	4068
$(.)^{-1}$	1

Field of composite extension degree



Field of composite extension degree

- Provides some arithmetic advantages
  - smaller datapath



Field of composite extension degree

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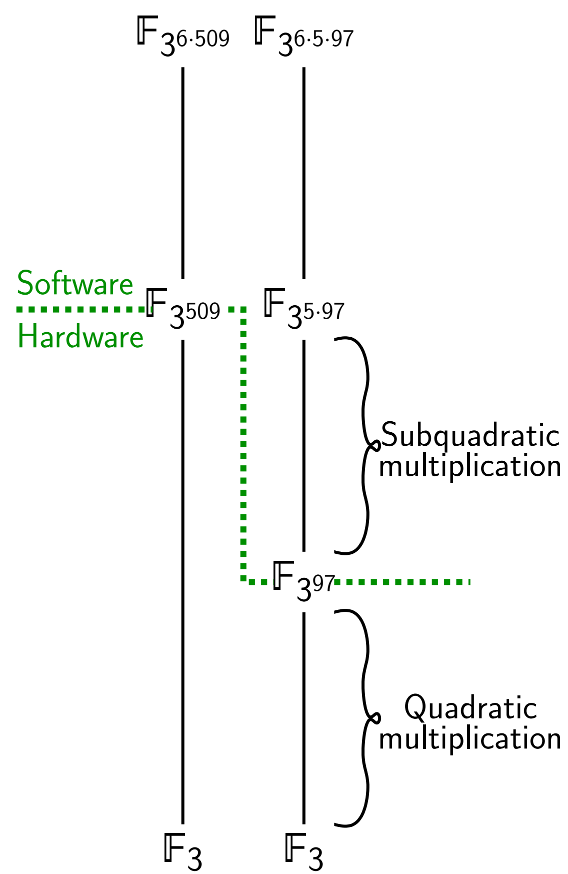

Field of composite extension degree

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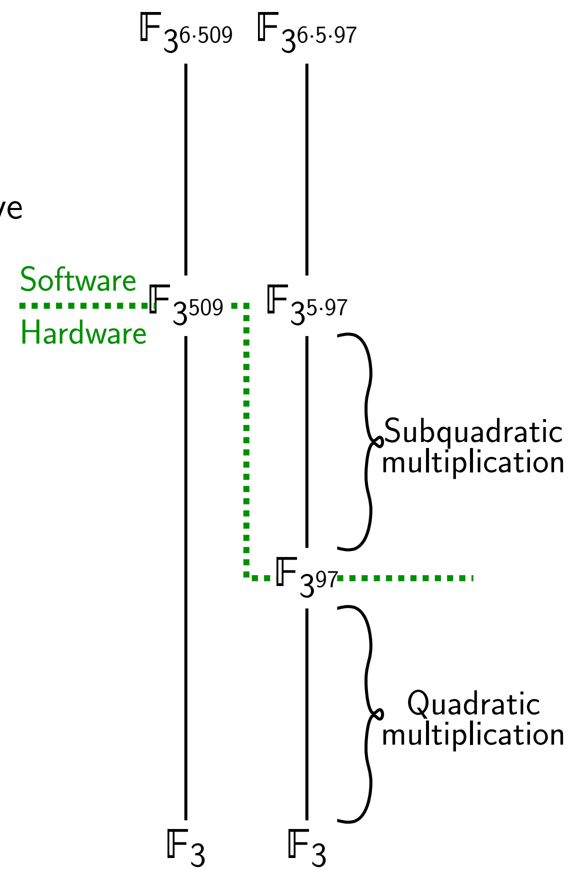
- Allows Weil Descent based attacks on the curve
  - GHS: using the composite extension degree

    $$\approx 2^{279} \text{ operations}$$

  - SDHP: Granger's algorithm

    $$\approx 2^{142} \text{ operations}$$

  - limited effect on security



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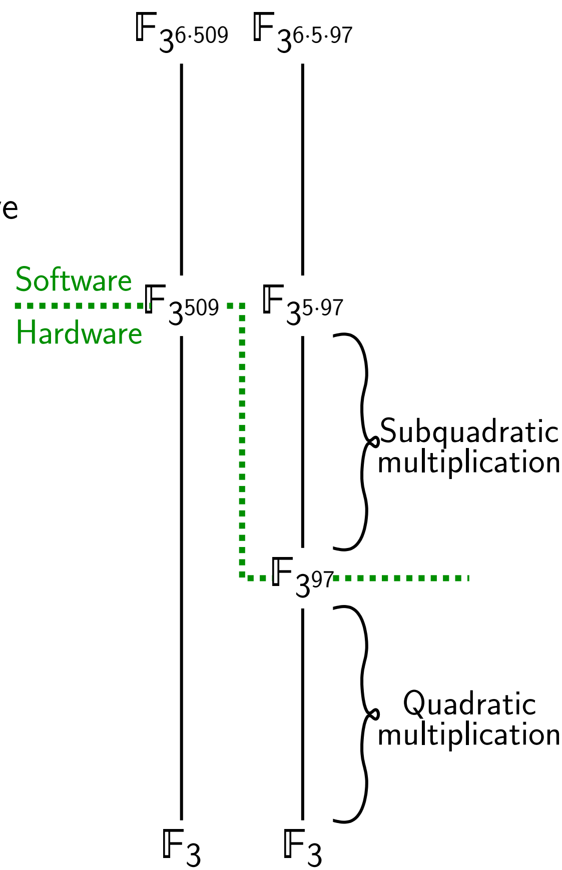
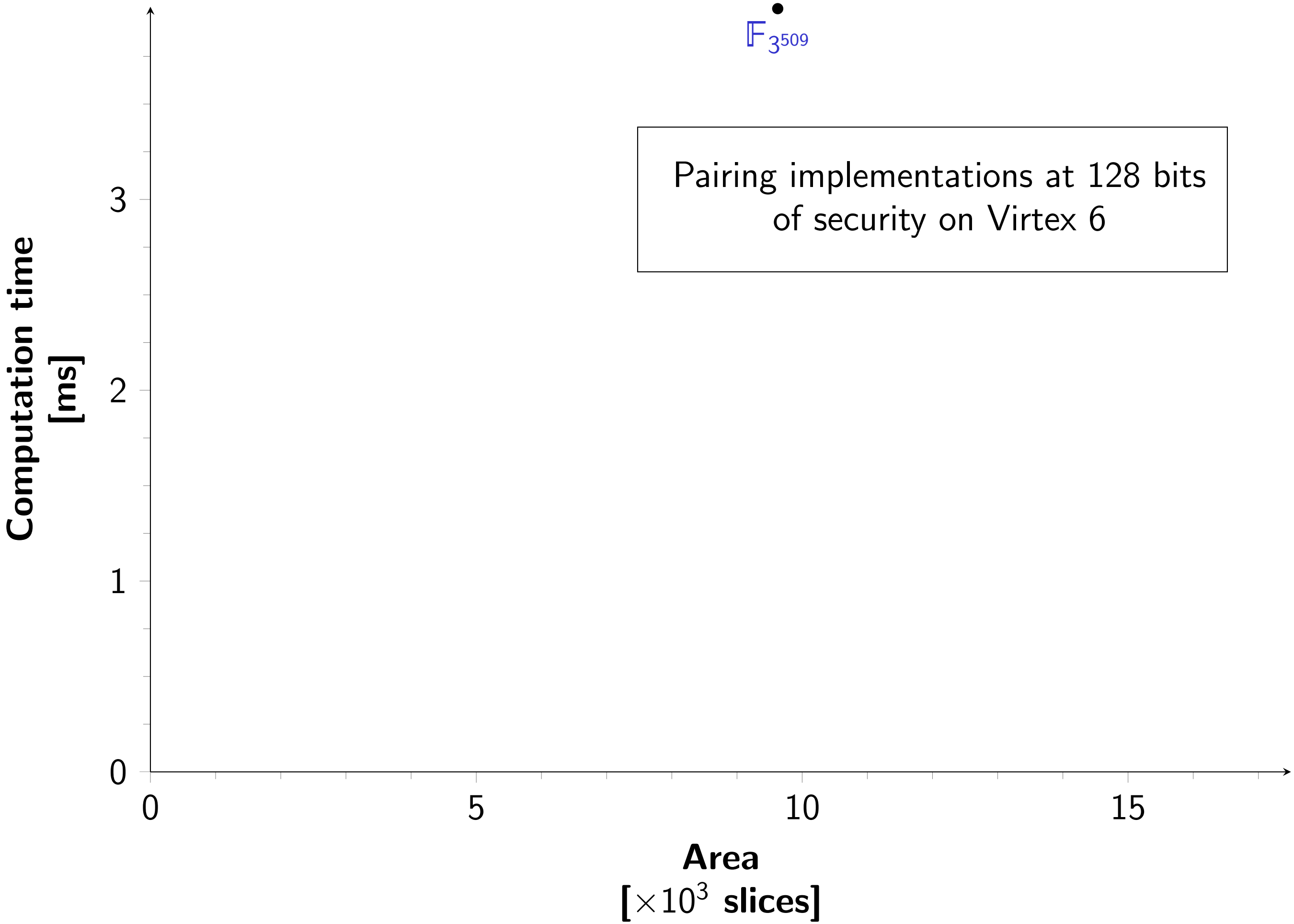
- Results
  - 1848 slices of the same Virtex 6 LX (15%)
    5.2 times smaller
  - compute a pairing in 1.6 ms
    2.5 times faster

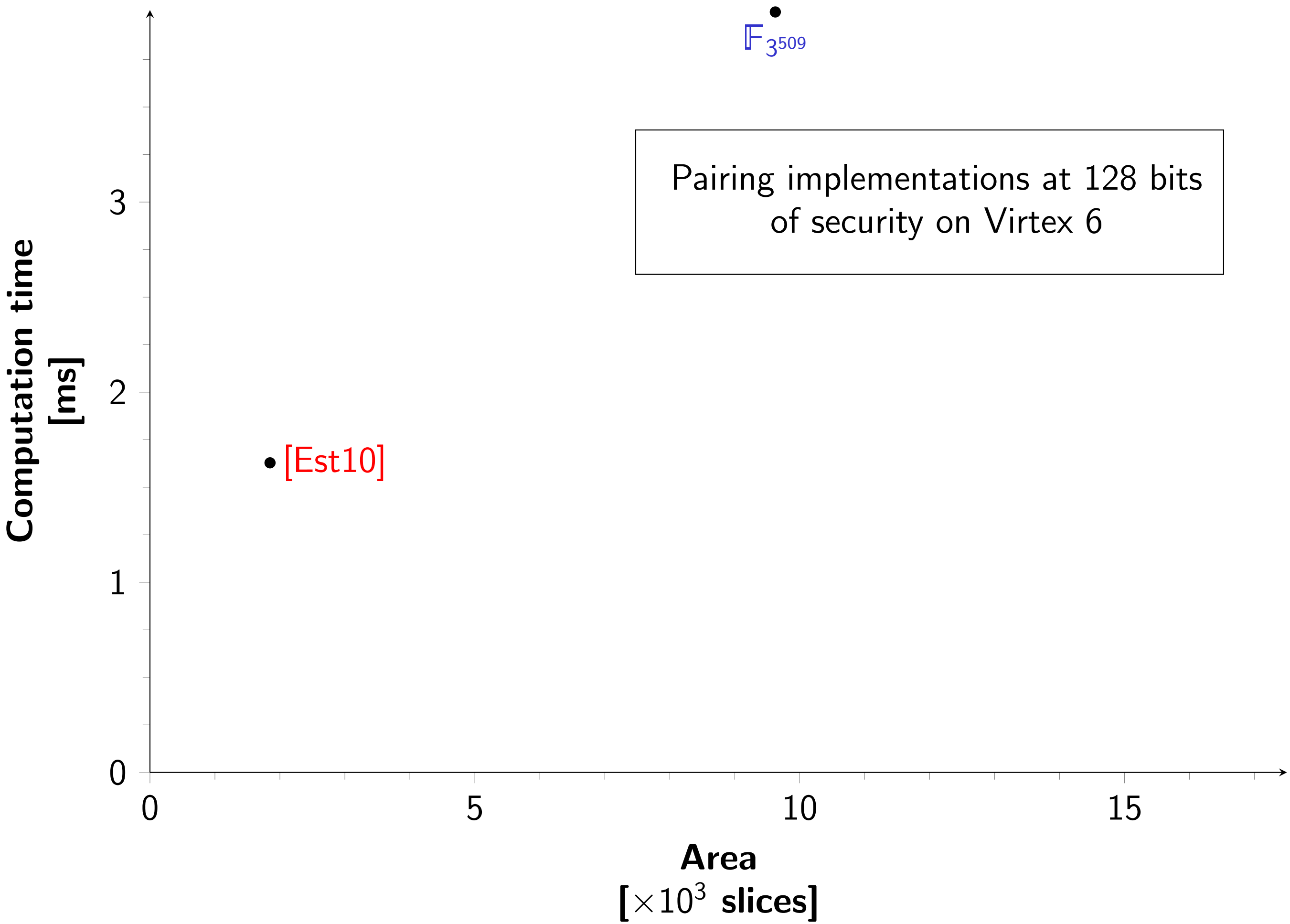


Benchmarks



Computation time [ms]
0
1
2
3
$\mathbb{F}_{3^{509}}$
Pairing implementations at 128 bits of security on Virtex 6
0
5
10
15
Area [$\times 10^3$ slices]

Benchmarks



Pairing implementations at 128 bits of security on Virtex 6
$\mathbb{F}_{3^{509}}$
[Est10]
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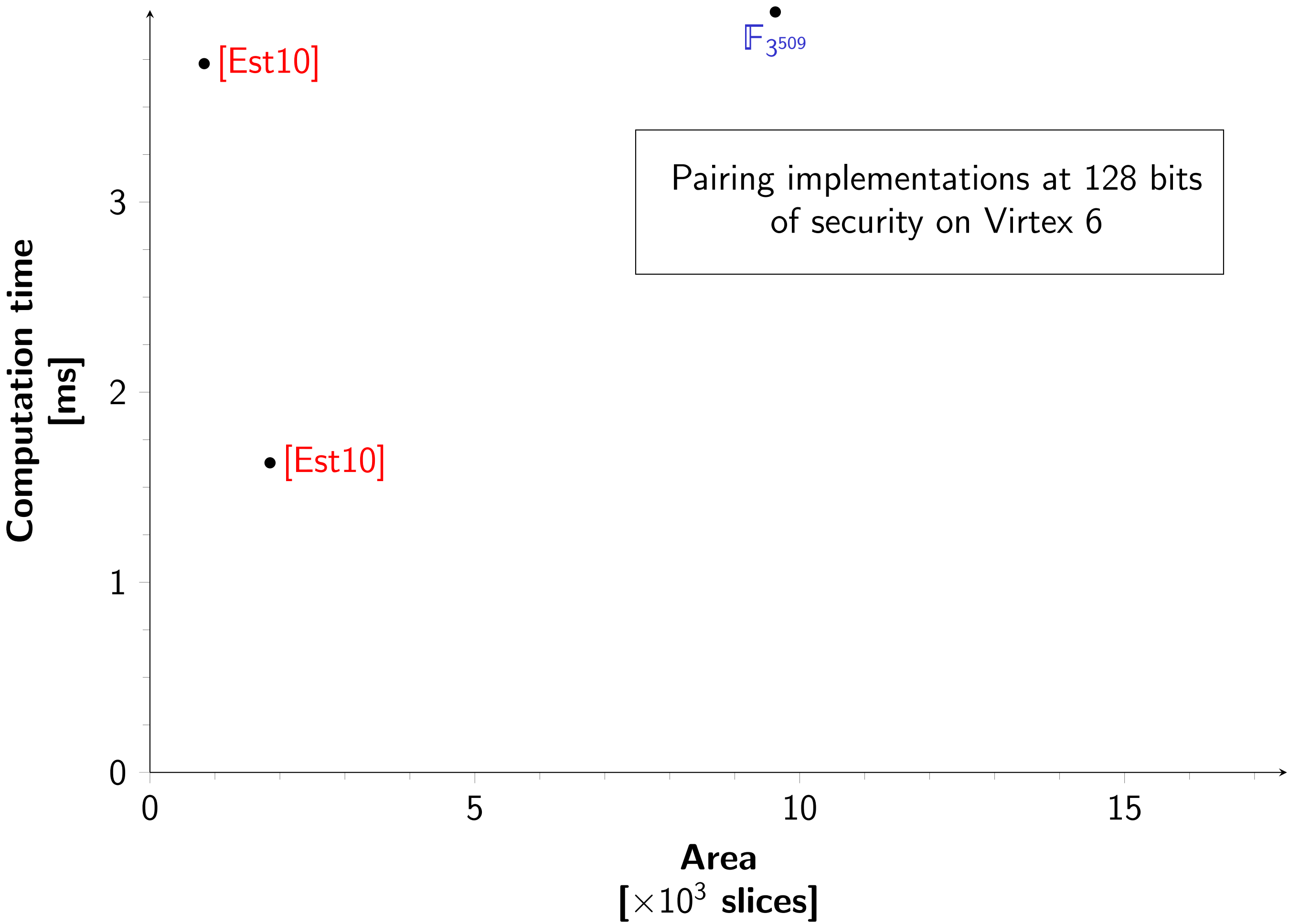
Benchmarks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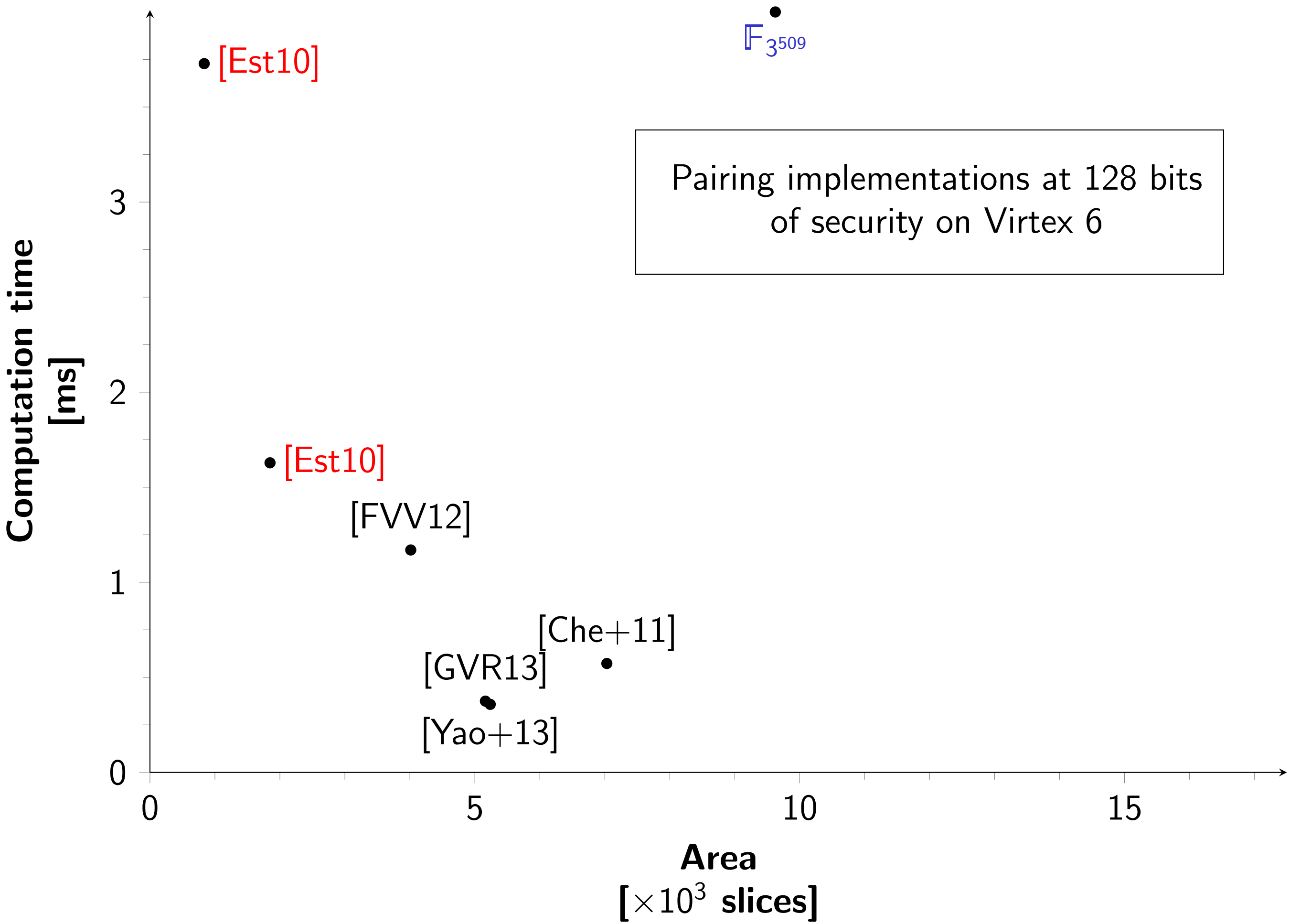

Computation time
[ms]
[Est10]
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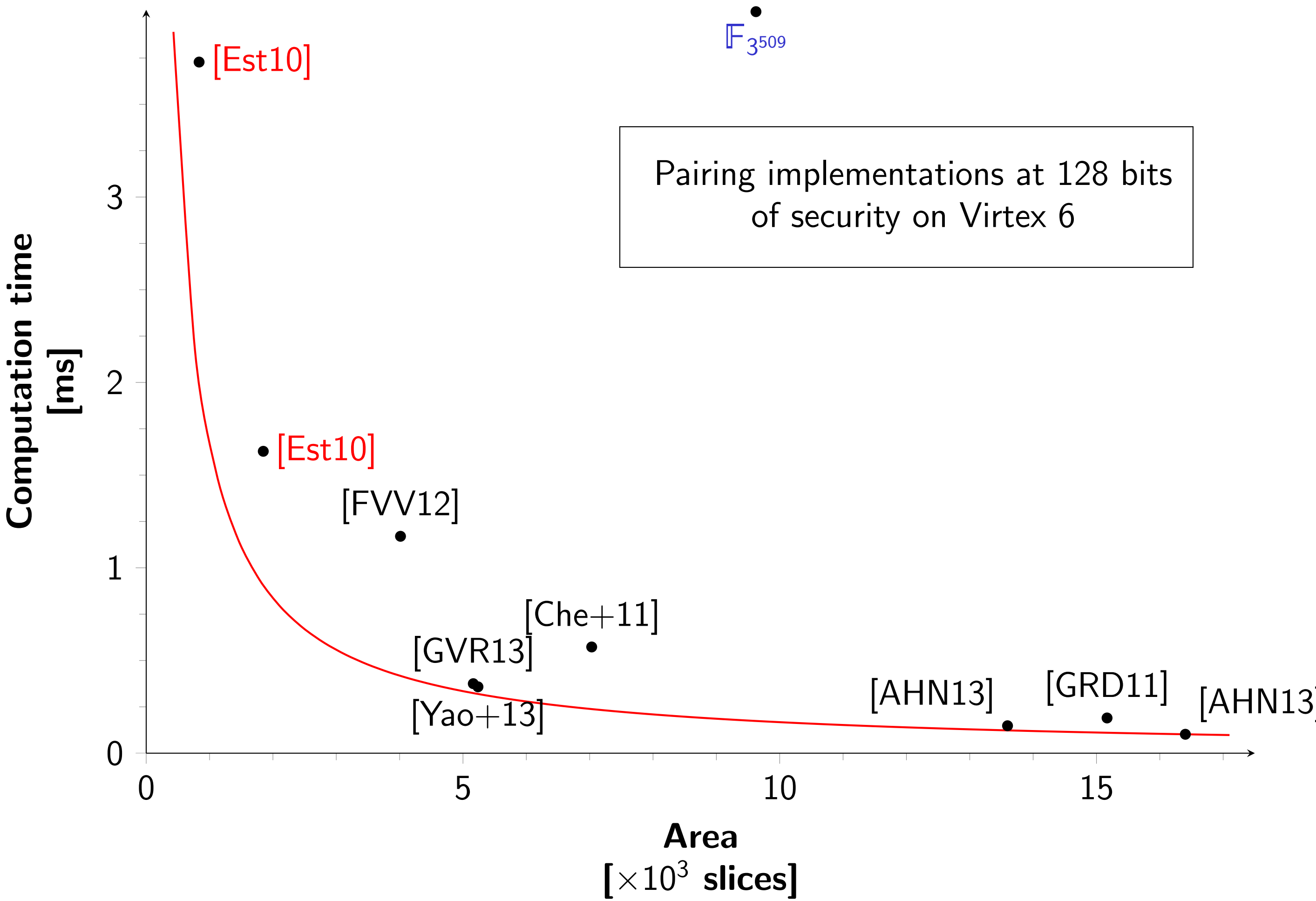
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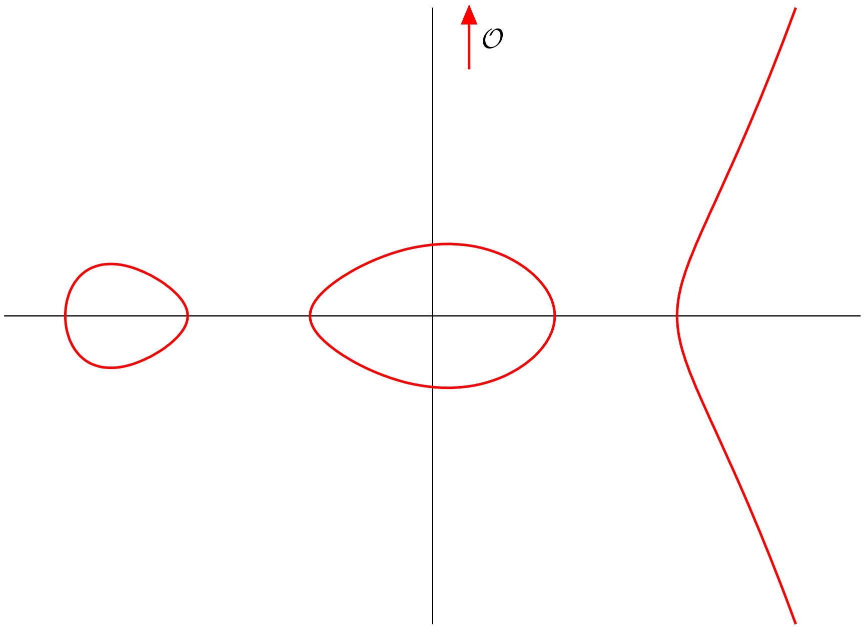
Outline of the talk

- Compact design through composite extension fields
- Pairing on genus-2 hyperelliptic curves
- Searching for efficient multiplication algorithms
- Conclusion and Perspectives

Genus-2 hyperelliptic curves

$$C/K : y^2 + h(x)y = f(x)$$
with $\deg h \leq 2$ and $\deg f = 5$



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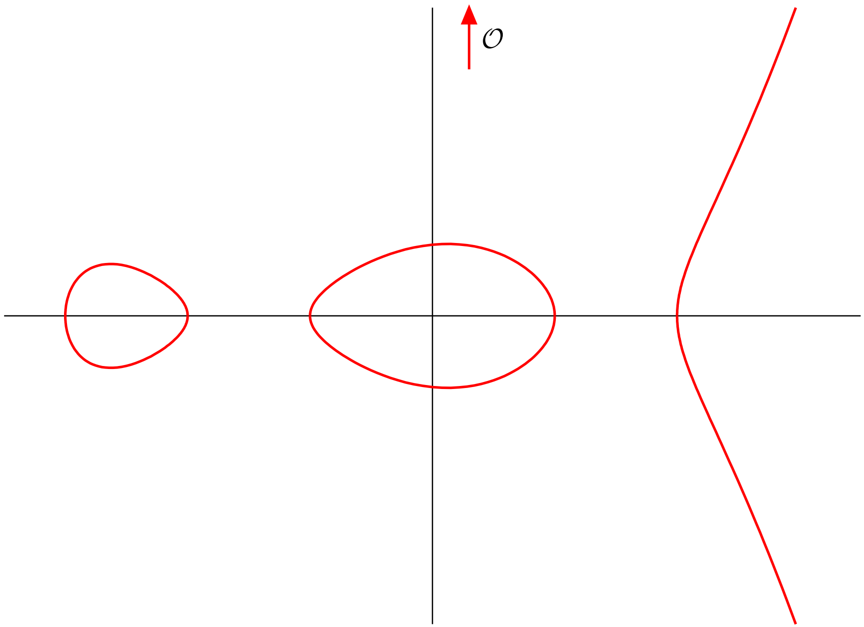

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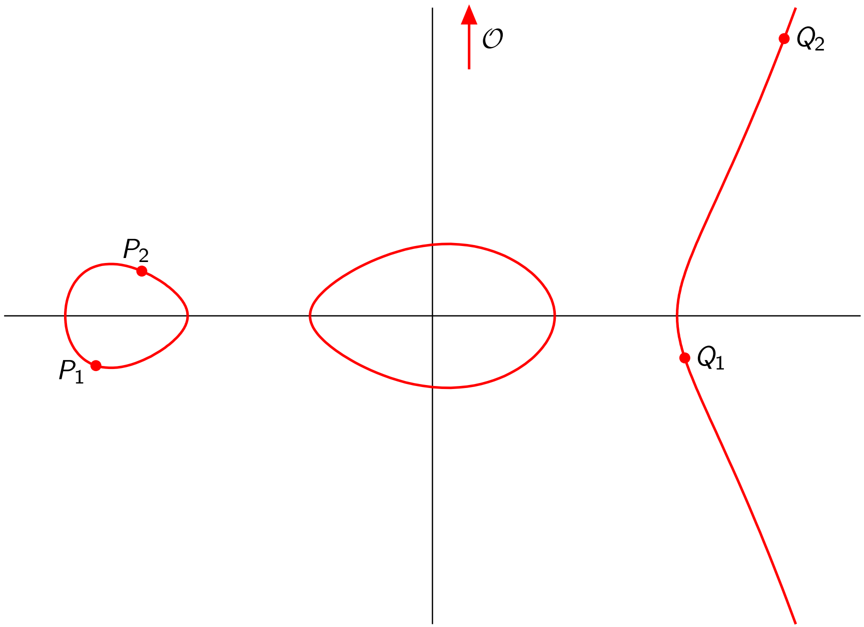

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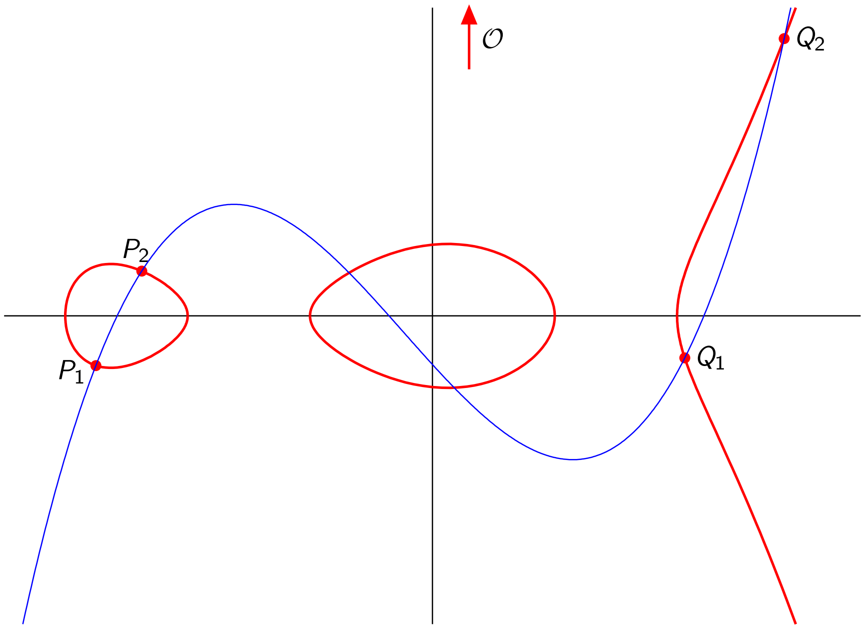

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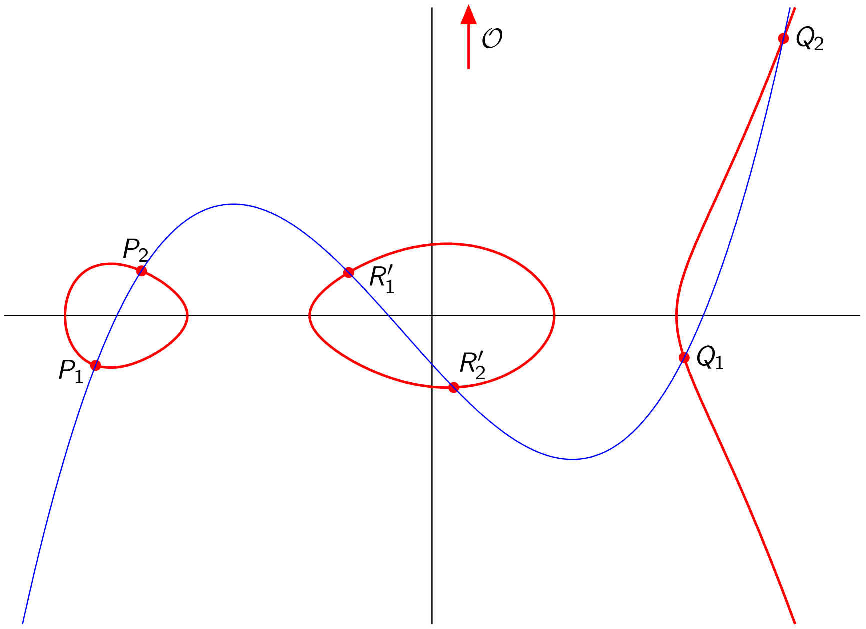

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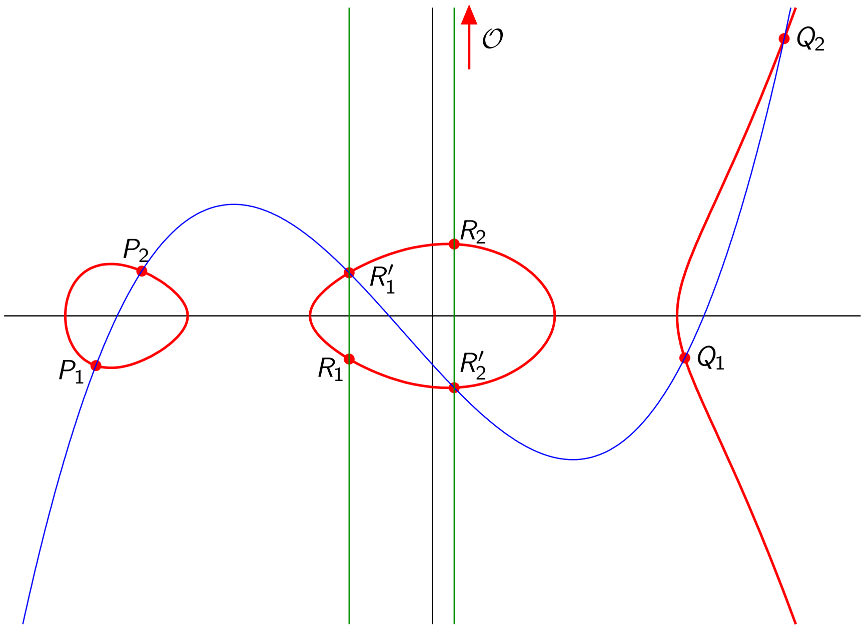

$$\{P_1, P_2\} + \{Q_1, Q_2\} = \{R_1, R_2\}$$

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- More formally
  - Jacobian of the curve
    $$\mathrm{Jac}_C$$
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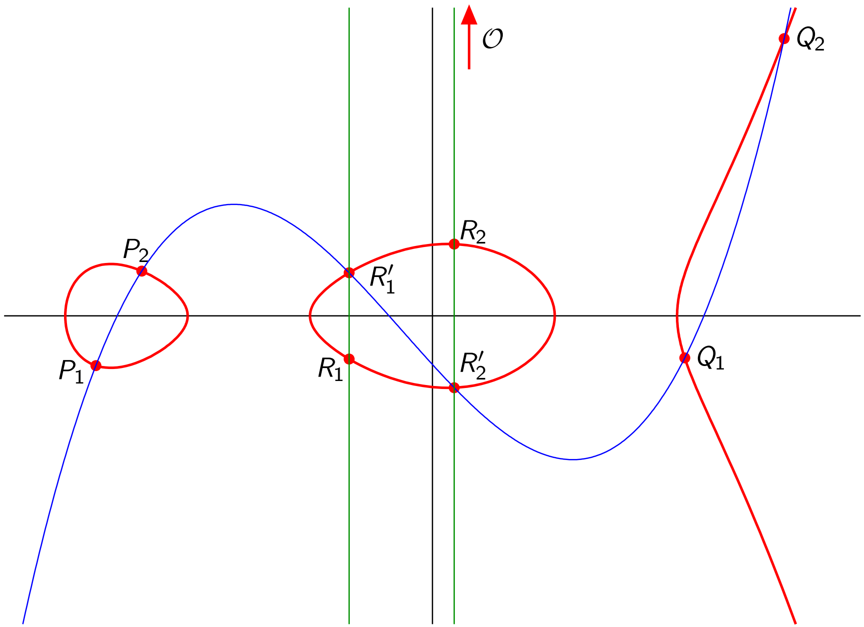

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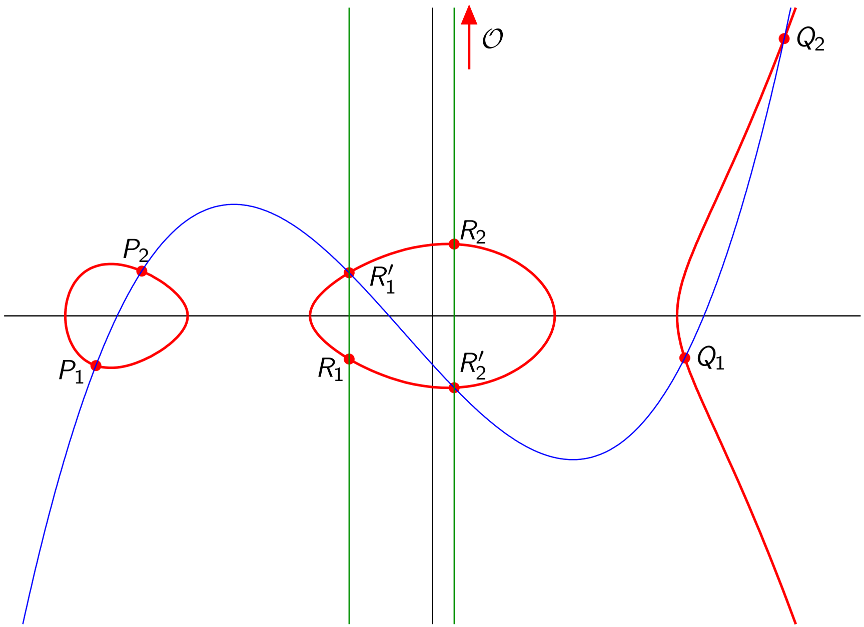
- More formally
  - Jacobian of the curve
    $$\mathrm{Jac}_C$$
  - is a group
- Chosen curves
  $$H_2 : y^2 + y = x^5 + x^3 + d, \text{ with } d \in \{0, 1\}$$
  - characteristic 2
  - supersingular

$$C/K : y^2 + h(x)y = f(x)$$
with $\deg h \leq 2$ and $\deg f = 5$



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Optimal Eta

- Parameters for 128-bit security
  - Embedding degree $k = 12$
  - Field: $\mathbb{F}_{2^{367}}$

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  - $\#\,\mathrm{Jac}_C(\mathbb{F}_{2^{367}}) = 2^{734} - 2^{551} - 2^{367} + 2^{184} + 1$
- Our pairing algorithm

Algorithm	Tate (double-and-add)
# iterations	734

Optimal Eta

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  - Efficient octupling formula: octuple-and-add

Algorithm	Tate (double-and-add)	Tate (octuple-and-add)
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Algorithm	Tate (double-and-add)	Tate (octuple-and-add)	Eta T	Optimal Eta
# iterations	734	245	184	123

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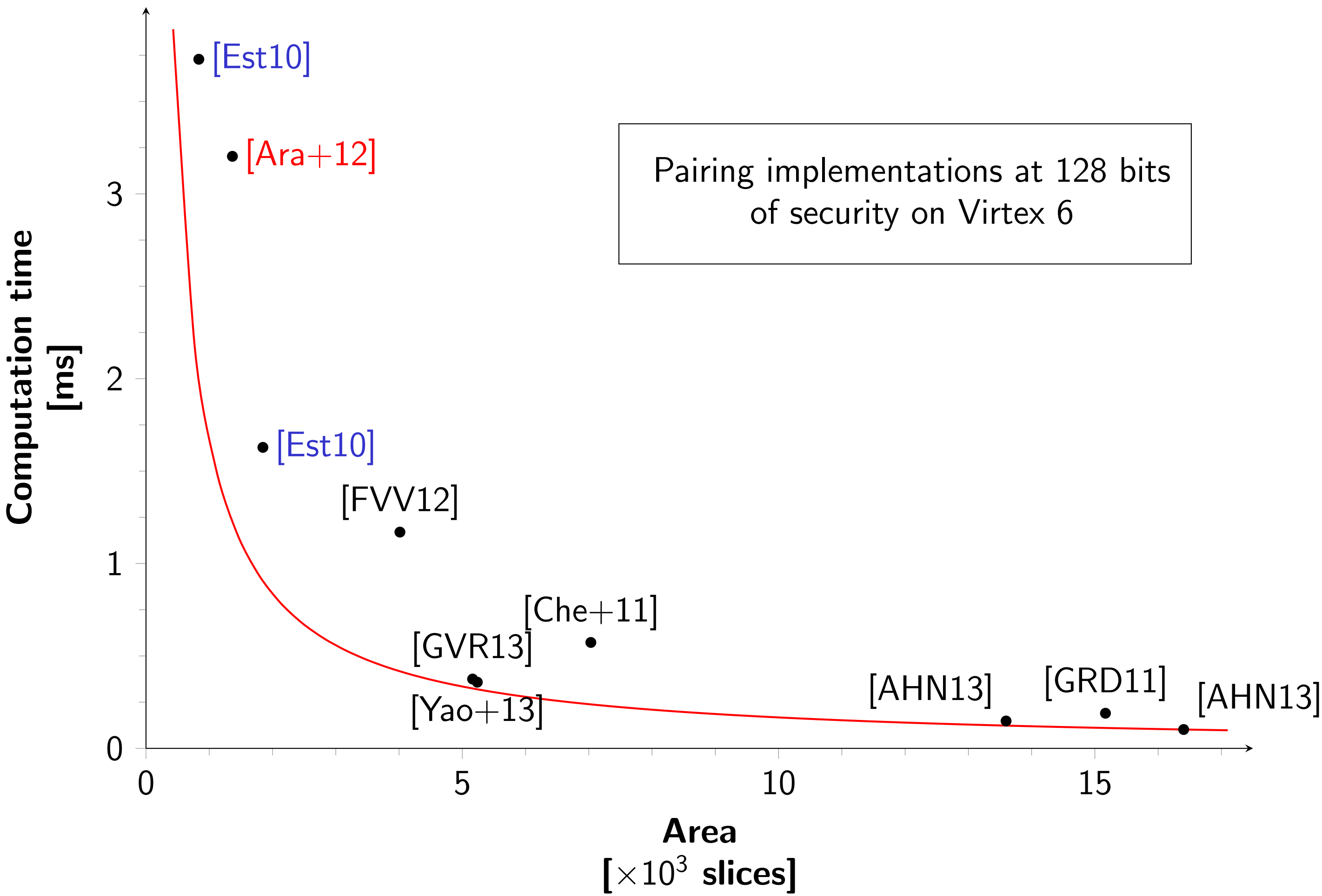
Algorithm	Tate (double-and-add)	Tate (octuple-and-add)	Eta T	Optimal Eta
# iterations	734	245	184	123

- Implementation on the previous coprocessor adapted for $\mathbb{F}_{2^{367}}$
  - 1366 slices on the same Virtex 6 LX (12%)
  - 3.2 ms
  - comparable performances with the elliptic case

Benchmarks



Pairing implementations at 128 bits of security on Virtex 6
Computation time [ms]
Area [$\times 10^3$ slices]
0
1
2
3
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5
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Outline of the talk

- Compact design through composite extension fields
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- Searching for efficient multiplication algorithms
- Conclusion and Perspectives

Origin of the problem

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$$
\begin{aligned}
(a_0 + a_1 X)(b_0 + b_1 X) &= a_0 b_0 + (a_0 b_1 + a_1 b_0) X + a_1 b_1 X^2 \\
&= a_0 b_0 + ((a_0 + a_1)(b_0 + b_1) - a_0 b_0 - a_1 b_1) X + a_1 b_1 X^2
\end{aligned}
$$

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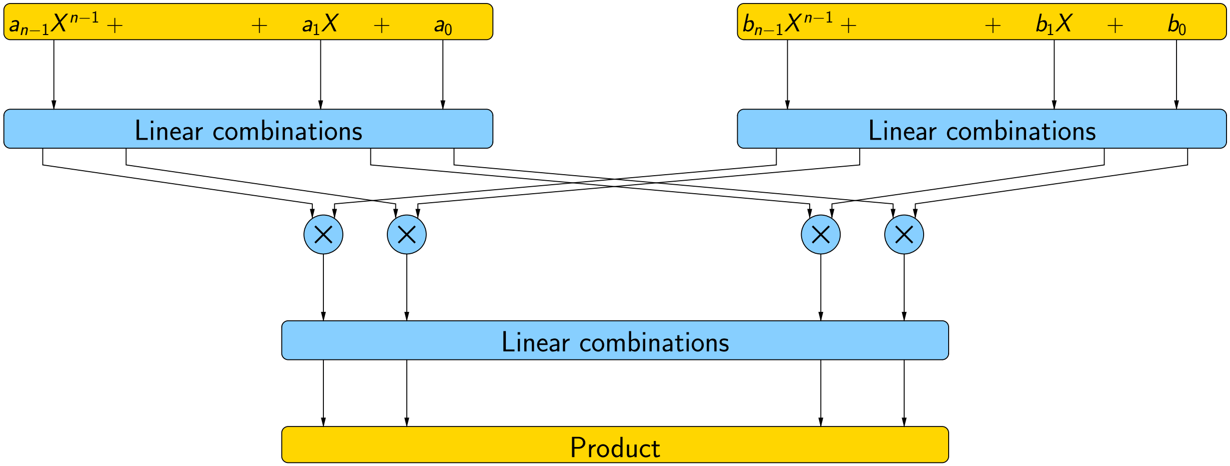
$$\begin{aligned}(a_0 + a_1 X)(b_0 + b_1 X) &= a_0 b_0 + (a_0 b_1 + a_1 b_0)X + a_1 b_1 X^2 \\ &= a_0 b_0 + ((a_0 + a_1)(b_0 + b_1) - a_0 b_0 - a_1 b_1)X + a_1 b_1 X^2\end{aligned}$$

- Well-studied problem
  - asymptotic complexity
  - theoretical bilinear complexity
  - small and "cryptographic" size
- *Five, six, and seven-term Karatsuba-like formulae*, P. Montgomery (2005)
  - ad-hoc formulae
  - exhaustive search for five-term multiplication
  - non-exhaustive search for six and seven-term multiplications
- Our approach: improve the search algorithm

Generalization of the problem

- Model of a multiplication algorithm



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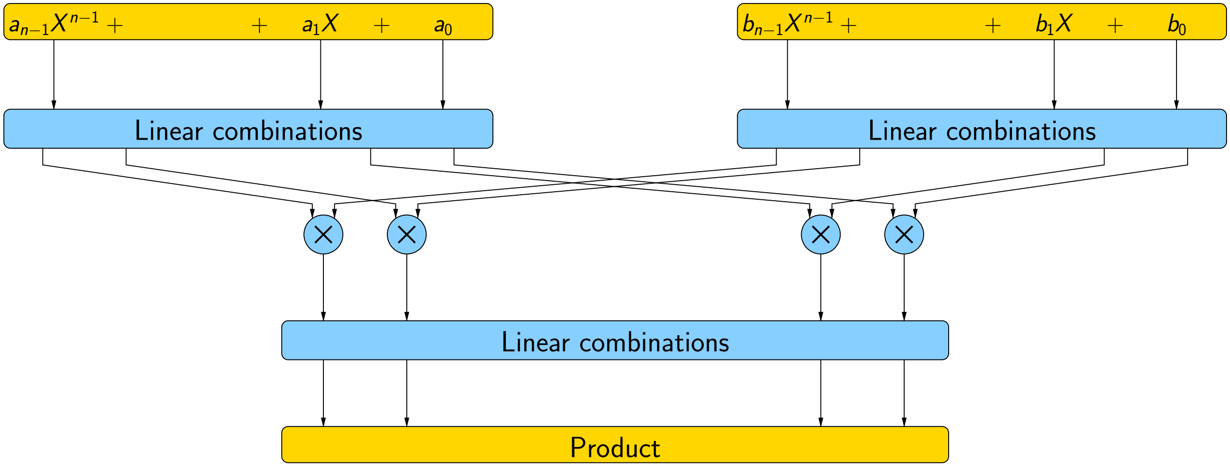

- Also true for any bilinear application
  - multiplication in extension fields
  - sparse products
  - matrix multiplications
  - ...

Formal framework

Formulation in terms of vector space for an $n \times m$ multiplication over a given field K

- Represent the coefficients of the result and the products as elements of

  $$V \text{ the } nm\text{-dimensional } K\text{-vector space generated by } \{a_i b_j\}_{0 \leq i < n, 0 \leq j < m}$$

  where the $a_i b_j$'s are seen as formal elements

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- Our target: the coefficients of the result is a family $\mathcal{T} \subset V$ that spans the target subspace $T = \operatorname{Span} \mathcal{T}$ of V
- The set $\mathcal{G}$ of the potential products to use in a formula: the generators
- Goal: find the optimal formulae (i.e. with a minimum number of products)
  - for increasing k until a solution is found
  - find each subset $\mathcal{W} \subset \mathcal{G}$ of exactly k products
  - which gives a valid formula (i.e. that lineary generates the coefficients of the result)

    $$\mathcal{T} \subset \operatorname{Span} \mathcal{W}$$

Resolution

- Naive approach: test each subset of k potential products

```
expand_family(∅, 𝒢)

procedure expand_family(𝒲, ℋ)
   if #𝒲 = k then
      if 𝒯 ⊂ Span 𝒲 then
         𝒲 is a solution
   else

      while ℋ ≠ ∅ do
         Pick a h in ℋ
         ℋ ← ℋ \ {h}
         expand_family(𝒲 ∪ {h}, ℋ)
end procedure
```

- Complexity depends on

$$\binom{\#\mathcal{G}}{k}$$

Resolution

- Naive approach: test each subset of k potential products
- Better approach: test each vector space of dimension k generated by potential products

```
expand_subspace({0}, 𝒢)

procedure expand_subspace(W, ℋ)
    if dim W = k then
        if 𝒯 ⊂ W then
            W is a solution
    else
        ℋ ← ℋ \ W
        while ℋ ≠ ∅ do
            Pick a h in ℋ
            ℋ ← ℋ \ {h}
            expand_subspace(W ⊕ Span(h), ℋ)
end procedure
```

- Complexity still depends on

$$\binom{\#\mathcal{G}}{k}$$

Resolution

- Naive approach: test each subset of k potential products
- Better approach: test each vector space of dimension k generated by potential products
- Even better approach: part of the solution is already known, use incomplete basis theorem

$\texttt{expand\_subspace}(\mathcal{T}, \mathcal{G})$

procedure $\texttt{expand\_subspace}(W, \mathcal{H})$
  if $\dim W = k$ **then**
    if $\operatorname{rank}(W \cap \mathcal{G}) = k$ **then**
      W is a solution
  else
    $\mathcal{H} \leftarrow \mathcal{H} \setminus W$
    while $\mathcal{H} \neq \emptyset$ **do**
      Pick a h in $\mathcal{H}$
      $\mathcal{H} \leftarrow \mathcal{H} \setminus \{h\}$
      $\texttt{expand\_subspace}(W \oplus \operatorname{Span}(h), \mathcal{H})$
end procedure

- Complexity now depends on

$$\binom{\#\mathcal{G}}{k - \operatorname{rank}\mathcal{T}}$$

Some results

- Multiplication of $n \times m$ term binary polynomials

Ring	$n \times m$	$\#\mathcal{G}$	k	**# of tests**	**# of solutions**	**# of formulae**	**Computation time (1 core)**
$\mathbb{F}_2[X]$	2×2	9	3	1	1	1	0
	3×3	49	6	9	3	9	0
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Some results

- Multiplication of $n \times m$ term binary polynomials

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- Optimal multiplication for the extensions $\mathbb{F}_{3^{5m}}$
  - 11 subproducts instead of 12 previously
  - yields a 5% improvement for the pairing on E_3

Outline of the talk

- Compact design through composite extension fields
- Pairing on genus-2 hyperelliptic curves
- Searching for efficient multiplication algorithms
- Conclusion and Perspectives

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- General method for cryptographic implementations
  - study mathematical structures
  - fix parameters thanks to cryptanalysis
  - algorithmic optimizations
  - choose the right arithmetic representation
  - implement different hardware accelerators

Perspectives

- Lower-level architecture
  - FPGA is a good prototyping platform
  - but with limited uses in real-life devices
  - develop skills in ASIC designs
  - power consumption awareness
- Integrate side-channel counter-measures
  - side-channel attacks are very effective threats
  - embedded systems need to be protected
- Use this method on different cryptographic primitives
  - scalar multiplication on hyperelliptic curves
  - lattice-based cryptography