

**Frequency domain local tests
for change detection**

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Context :

Quality control in mechanical engineering

- Pairs of scalar input and output **frequency domain data**,
- Nominal **input/output transfer functions** available.

Wanted:

- **Detect small deviations** in those transfer functions, using **nonparametric** techniques.
- Needed: **limit theorems** (LLN, CLT) for Fourier analysis.

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1. Some useful formulas

$$N = 2^q ; \quad \omega = e^{2i\pi n/N}, n = 0, \dots, N - 1$$

$$Y^N(\omega) \triangleq \frac{1}{\sqrt{N}} \sum_{k=1}^N y_k e^{-i\omega k}$$

DFT (de)correlations y_n, z_n stationary

$$\begin{aligned} E(Y^N(-\omega) Z^N(\omega)) &= \frac{1}{N} \sum_{k,l=1}^N E(y_k z_l) e^{-i\omega(l-k)} \\ &= \frac{1}{N} \sum_{r=1-N}^N e^{-i\omega r} (N - r) R_r^{yz} \\ &= S^{yz}(e^{i\omega}) + O\left(\frac{1}{N}\right) \quad (a) \end{aligned}$$

$$E(Y^N(-\xi) Z^N(\omega)) = O\left(\frac{1}{N}\right) \quad \text{for } \xi \neq \omega \quad (b)$$

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2. Limit theorems

For $x_n = G(z) u_n$

$$|X^N(\omega) - G(e^{i\omega}) U^N(\omega)| \leq O\left(\frac{\text{Cst}_G \text{Cst}_u}{\sqrt{N}}\right) \quad (c)$$

where $\text{Cst}_G \triangleq 2 \sum_{k=1}^{\infty} k |g(k)| < \infty$

and $|u_n| < \text{Cst}_u$

- No useful limit theorem when $N \rightarrow \infty$

(N estimated frequencies)

- Idea: **average over K successive blocks** of size N

(instead of windowing)

$$\left(Y_k^N(\omega), U_k^N(\omega) \right), \quad k = 1, \dots, K$$

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$$1. \text{ For } K \text{ large: } \frac{1}{K} \sum_{k=1}^K Y_k^N(-\omega) Z_k^N(\omega) = S^{yz}(e^{i\omega}) + O\left(\frac{1}{N}\right) \quad (1)$$

2. If y_n and z_n independent, and $S^{yy}(e^{i\omega})S^{zz}(e^{i\omega}) \neq 0$:
using CLT for triangular arrays of random variables:

$$\frac{\sum_{k=1}^K Y_k^N(-\omega) Z_k^N(\omega)}{\sqrt{K} \sqrt{S^{yy}(e^{i\omega})S^{zz}(e^{i\omega})}} = O\left(\frac{\sqrt{K}}{N}\right) + \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \mathcal{N}(0, 1)$$

$$(\sqrt{K}/N \rightarrow 0) \quad \frac{1}{\sqrt{K}} \sum_{k=1}^K Y_k^N(-\omega) Z_k^N(\omega) \sim \mathcal{N}\left(0, S^{yy}(e^{i\omega}) S^{zz}(e^{i\omega})\right) \quad (2)$$

3. For $\xi \neq \omega$:

$$\mathbb{E}\left(\frac{1}{\sqrt{K}} \sum_{k=1}^K Y_k^N(-\xi) Z_k^N(\xi)\right) \left(\frac{1}{\sqrt{K}} \sum_{k=1}^K Y_k^N(-\omega) Z_k^N(\omega)\right) = O\left(\frac{1}{N}\right) \quad (3)$$

$$4. (K, N \rightarrow \infty) \quad \frac{1}{\sqrt{K}} \sum_{k=1}^K Y_k^N(-\omega) \sim \mathcal{N}\left(0, S^{yy}(e^{i\omega})\right) \quad (4)$$

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3. Local approach

$$y_n = G(z) u_n + v_n, \quad S^{uu}(e^{i\omega}) S^{vv}(e^{i\omega}) \neq 0$$

$$G = G_0 + \frac{1}{\sqrt{K}} \tilde{G}$$

Frequency-wise complex **residual**:

$$\zeta_K^N(G_0, \omega) \triangleq \frac{1}{\sqrt{K}} \sum_{k=1}^K U_k^N(-\omega) \left(Y_k^N(\omega) - G_0(e^{i\omega}) U_k^N(\omega) \right)$$

Theorem: ($\sqrt{K}/N \rightarrow 0$)

$$\zeta_K^N(G_0, \omega) \sim \mathcal{N}\left(S^{uu}(e^{i\omega}) \tilde{G}(e^{i\omega}), S^{uu}(e^{i\omega}) S^{vv}(e^{i\omega})\right)$$

$$\text{For } \xi \neq \omega: \quad \mathbb{E}\left(\zeta_K^N(G_0, \xi) \zeta_K^N(G_0, \omega)\right) = 0 \quad (\text{use (3)})$$

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4. Testing

Hints for the CLT:

$$Y_k^N(\omega) = G(e^{i\omega}) U_k^N(\omega) + V_k^N(\omega) + O\left(\frac{1}{\sqrt{N}}\right)$$

$$\begin{aligned} \zeta_K^N(G_0, \omega) &= \tilde{G}(e^{i\omega}) \frac{1}{K} \sum_{k=1}^K |U_k^N(\omega)|^2 \quad (\text{use (1)}) \\ &+ \frac{1}{\sqrt{K}} \sum_{k=1}^K U_k^N(-\omega) V_k^N(\omega) \quad (\text{use (2)}) \quad (\dagger) \\ &+ O\left(\frac{1}{\sqrt{N}}\right) \frac{1}{\sqrt{K}} \sum_{k=1}^K U_k^N(-\omega) \quad (\text{use (4)}) \end{aligned}$$

$$H_0 : G = G_0 \text{ against } H_1 : G = G_0 + \frac{1}{\sqrt{K}} \tilde{G}$$

$$\chi_K^N(G_0, \omega) = \frac{|\zeta_K^N(G_0, \omega)|^2}{\hat{S}^{uu}(e^{i\omega}) \hat{S}^{vv}(e^{i\omega})}, \quad \chi^2(2) \text{ variable}$$

Noncentrality parameter:

$$\frac{S^{uu}(e^{i\omega})}{S^{vv}(e^{i\omega})} |\tilde{G}(e^{i\omega})|^2$$

Estimates of the spectra:

$$\hat{S}^{uu}(e^{i\omega}) = \frac{1}{K} \sum_{k=1}^K |U_{0,k}^N(\omega)|^2$$

$$\hat{S}^{vv}(e^{i\omega}) = \frac{1}{K} \sum_{k=1}^K |Y_{0,k}^N(\omega) - G_0(e^{i\omega}) U_{0,k}^N(\omega)|^2$$

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5. Implementations issues

How many blocks?

$$K, N \rightarrow \infty \text{ such that } \sqrt{K}/N \rightarrow 0$$

Closer look at (\dagger)

Distance between $\zeta_K^N(G_0, \omega)$ and its asymptotic behavior:

$$O\left(\frac{1}{\sqrt{K}} + \frac{1}{N} + \sqrt{\frac{K}{N}}\right)$$

$$\min_K \implies K \sim \sqrt{N}$$

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