

Contents

Model-based statistical signal processing

and **decision theoretic** approaches to **monitoring**

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Monitoring and **model-based statistical** processing

Stochastic models (static, dynamic) \longleftrightarrow **uncertainties**

Parameterized models (physical interpretation, diagnostics)

Fault in the system \longleftrightarrow **deviation** in the parameter vector

(Fault, damage, deviation, change)

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Monitoring and model-based **statistical** processing

Key concepts - **Detection** - Independent case

Key concepts - **Detection** - Dependent case

Key concepts - **Isolation** and diagnostics

Example: Structural Health Monitoring

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Problem

Parameterized nonlinear state-space models

$$\begin{cases} x_{k+1} = f(\theta, x_k, u_k, v_k) & x_k, v_k : \text{unknown states and inputs} \\ y_k = h(\theta, x_k, u_k, v_k) + \epsilon_k & u_k, y_k : \text{measured inputs and outputs} \end{cases}$$

θ : unknown parameters

Wanted:

- **Detect** and **diagnose** (small) **deviations** in θ .
- **Not to detect** events/features of no interest !

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Three detection problems

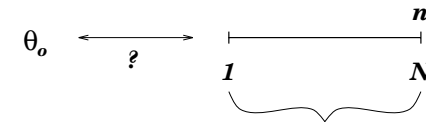
Which model(s) ?

Physical models, black-box models

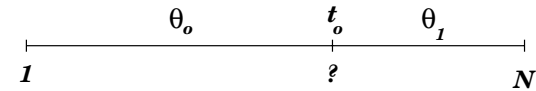
Neural networks, wavelet networks

Approximate models

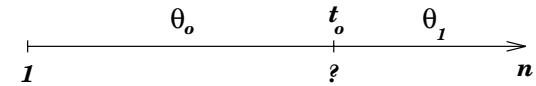
– Model **validation**



– **Off-line** change **detection**



– **On-line** change **detection**



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Key concepts - Independent data

Simple hypotheses, composite hypotheses

Hypotheses H_0 H_1

Simple θ_0 θ_1 **Known** parameter values

Composite Θ_0 Θ_1 **Unknown** parameter values

Likelihood	$p_{\theta}(y_i)$	Likelihood ratio	$\frac{p_{\theta_1}(y_i)}{p_{\theta_0}(y_i)}$
Log-likelihood	$l_{\theta}(y_i) \triangleq \ln p_{\theta}(y_i)$		
Log-likelihood ratio	$s_i \triangleq \ln \frac{p_{\theta_1}(y_i)}{p_{\theta_0}(y_i)} = l_{\theta_1}(y_i) - l_{\theta_0}(y_i)$		
		$E_{\theta_0}(s_i) < 0$	
		$E_{\theta_1}(s_i) > 0$	
Likelihood ratio	$\Lambda_N \triangleq \frac{p_{\theta_1}(\mathcal{Y}_1^N)}{p_{\theta_0}(\mathcal{Y}_1^N)} = \prod_i p_{\theta_1}(y_i) / \prod_i p_{\theta_0}(y_i)$		
Log-likelihood ratio	$S_N \triangleq \ln \Lambda_N = \sum_{i=1}^N s_i$		

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Hypothesis testing: likelihood ratio, GLR

Simple hypotheses : likelihood ratio test

$$\begin{cases} \text{If } \Lambda_N < \lambda / S_N < h : H_0 \text{ is chosen} \\ \text{If } \Lambda_N \geq \lambda / S_N \geq h : H_1 \text{ is chosen} \end{cases}$$

Composite hypotheses : Generalized likelihood ratio (GLR)

Maximize the likelihoods / **unknown** values of θ_0 and θ_1 :

$$\widehat{\Lambda}_N = \frac{\sup_{\theta_1 \in \Theta_1} p_{\theta_1}(\mathcal{Y}_1^N)}{\sup_{\theta_0 \in \Theta_0} p_{\theta_0}(\mathcal{Y}_1^N)} = \frac{p_{\widehat{\theta}_1}(\mathcal{Y}_1^N)}{p_{\widehat{\theta}_0}(\mathcal{Y}_1^N)}$$

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Scalar parameter - Simple case: known θ_1



Hypothesis H_0 $\theta = \theta_0 \quad (1 \leq i \leq k)$

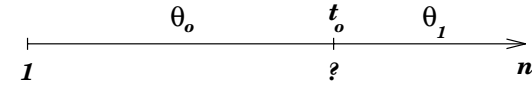
Hypothesis H_1 $\exists t_0 \text{ s.t. } \theta = \begin{cases} \theta_0 & (1 \leq i < t_0) \\ \theta_1 & (t_0 \leq i \leq k) \end{cases}$

Alarm time t_a : $t_a = \min \{k \geq 1 : g_k \geq h\}$

Estimated onset time: \widehat{t}_0

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On-line detection: CUSUM, GLR



Unknown onset time t_0 ; θ_0 assumed known

Alarm time t_a : $t_a = \min \{k \geq 1 : g_k \geq h\}$

Problem: design of the decision function g_k

Simple case (known θ_1): CUSUM

Composite case (unknown θ_1): modified CUSUM, GLR

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CUSUM algorithm

Ratio of likelihoods under H_0 and H_1 :

$$\frac{\prod_{i=1}^{t_0-1} p_{\theta_0}(y_i) \cdot \prod_{i=t_0}^k p_{\theta_1}(y_i)}{\prod_{i=1}^k p_{\theta_0}(y_i)} = \frac{\prod_{i=t_0}^k p_{\theta_1}(y_i)}{\prod_{i=t_0}^k p_{\theta_0}(y_i)} = \Lambda_{t_0}^k$$

Maximize over the unknown onset time t_0 :

$$\begin{aligned} (\widehat{t_0})_k &\triangleq \arg \max_{1 \leq j \leq k} \prod_{i=1}^{j-1} p_{\theta_0}(y_i) \cdot \prod_{i=j}^k p_{\theta_1}(y_i) \\ &= \arg \max_{1 \leq j \leq k} \Lambda_j^k \\ &= \arg \max_{1 \leq j \leq k} S_j^k, \quad S_j^k = \ln \Lambda_j^k \end{aligned}$$

$$g_k \triangleq \max_{1 \leq j \leq k} S_j^k = \ln \Lambda_{\widehat{t_0}}^k$$

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CUSUM algorithm (Contd.)

$$g_k \triangleq \max_{1 \leq j \leq k} S_j^k$$

$$= S_1^k - \min_{1 \leq j \leq k} S_j^j = S_1^k - m_k, \quad m_k \triangleq \min_{1 \leq j \leq k} S_1^j$$

$$t_a = \min \{k \geq 1 : g_k \geq h\}$$

$$t_a = \min \{k \geq 1 : S_1^k \geq m_k + h\} \quad \text{Adaptative threshold}$$

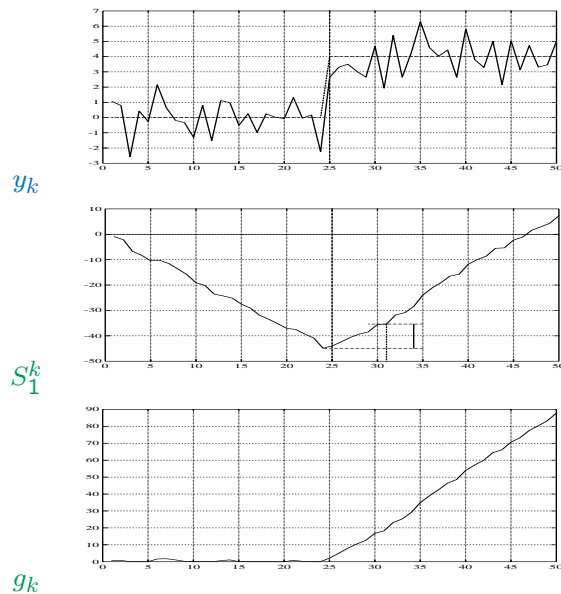
$$g_k = (g_{k-1} + s_k)^+$$

$$g_k = (S_{k-N_k+1}^k)^+, \quad N_k \triangleq N_{k-1} \cdot I(g_{k-1}) + 1$$

$$(\hat{t}_0)_k = t_a - N_{t_a} + 1 \quad \text{Sliding window with adaptive size}$$

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Log. likelihood ratio - CUSUM Algorithm



g_k

CUSUM algorithm (Contd.) - Gaussian example

$$\mathcal{N}(\mu, \sigma^2), \quad \theta \triangleq \mu, \quad p_\theta(y) \triangleq \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$s_i = \ln \frac{p_{\mu_1}(y_i)}{p_{\mu_0}(y_i)}$$

$$= \frac{1}{2\sigma^2} ((y_i - \mu_0)^2 - (y_i - \mu_1)^2)$$

$$= \frac{\nu}{\sigma^2} \left(y_i - \mu_0 - \frac{\nu}{2} \right), \quad \nu = \mu_1 - \mu_0$$

$$S_1^k \text{ involves } \sum_{i=1}^k y_i : \text{Integrator (with adaptive threshold)}$$

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Composite case: unknown θ_1

Modified CUSUM algorithms

Minimum magnitude of change

Weighted CUSUM

GLR algorithm

Double maximization

$$g_k = \max_{1 \leq j \leq k} \sup_{\theta_1} S_j^k(\theta_1)$$

Gaussian case, additive faults: second maximization explicit.

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Key concepts - Dependent data

Conditional likelihood $p_{\theta}(y_i | \mathcal{Y}_1^{i-1})$

Log-likelihood $l_{\theta}(y_i | \mathcal{Y}_1^{i-1}) \triangleq \ln p_{\theta}(y_i | \mathcal{Y}_1^{i-1})$

Log-likelihood ratio $s_i \triangleq \ln \frac{p_{\theta_1}(y_i | \mathcal{Y}_1^{i-1})}{p_{\theta_0}(y_i | \mathcal{Y}_1^{i-1})}$

$$E_{\theta_0}(s_i) < 0$$

$$E_{\theta_1}(s_i) > 0$$

Likelihood ratio $\Lambda_N \triangleq \frac{p_{\theta_1}(\mathcal{Y}_1^N)}{p_{\theta_0}(\mathcal{Y}_1^N)} = \prod_i \frac{p_{\theta_1}(y_i | \mathcal{Y}_1^{i-1})}{p_{\theta_0}(y_i | \mathcal{Y}_1^{i-1})}$

Log-likelihood ratio $S_N \triangleq \ln \Lambda_N = \sum_{i=1}^N s_i$

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Fisher information

$$\begin{aligned} \text{(Independent)} \quad I(\theta) &= E_{\theta} \left(\frac{\partial l_{\theta}(y_i)}{\partial \theta} \right)^2 = \text{cov}_{\theta} \left(\frac{\partial l_{\theta}(y_i)}{\partial \theta} \right) = \text{cov}_{\theta}(z_i) \\ &= - E_{\theta} \left(\frac{\partial^2 l_{\theta}(y_i)}{\partial \theta^2} \right) \end{aligned}$$

$$\begin{aligned} \text{(Dependent)} \quad I_N(\theta) &= \frac{1}{N} \text{cov}_{\theta} \left(\frac{\partial l_{\theta}(\mathcal{Y}_1^N)}{\partial \theta} \right) \\ &= \text{cov}_{\theta}(\mathcal{Z}_N) \end{aligned}$$

$$I(\theta) = \lim_{N \rightarrow \infty} I_N(\theta)$$

Fisher information = inverse of the **curvature** of the likelihood

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Key concepts - Dependent data (Contd.)

Efficient score $z_i^* \triangleq \left. \frac{\partial l_{\theta}(y_i | \mathcal{Y}_1^{i-1})}{\partial \theta} \right|_{\theta=\theta^*}$

$$\zeta_N^* \triangleq \frac{1}{\sqrt{N}} \left. \frac{\partial l_{\theta}(\mathcal{Y}_1^N)}{\partial \theta} \right|_{\theta=\theta^*} = \frac{1}{\sqrt{N}} \sum_{i=1}^N z_i^*$$

$$E_{\theta^*}(z_i^*) = 0$$

$$E_{\theta_0}(z_i^*) < 0 \quad (\theta_0 = \theta^* - \Upsilon)$$

$$E_{\theta_1}(z_i^*) > 0 \quad (\theta_1 = \theta^* + \Upsilon)$$

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Residuals and sufficient statistics

$S(\theta, \mathcal{Y}_1^N)$ s.t. $P_{\theta}(\mathcal{Y}_1^N | S)$ independent of θ .

Fisher information maintained.

- **Likelihood ratio** for both additive and non-additive faults,
- **Innovation** for additive faults,
- **Efficient score** for non-additive faults,
- Other **estimating functions** for non-additive faults.

Innovation not sufficient for monitoring the **dynamics**.

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Fault detection

Local approach (**small** deviations)

θ_0 : reference parameter, known (or identified)

Y_k : N -size sample of new measurements

Build a **residual** ζ **significantly non zero** when fault

Test $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_0 + \frac{\delta\theta}{\sqrt{N}}$

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First order **Taylor expansion** of the **efficient score**

$$\zeta_N(\theta) \approx \zeta_N(\theta_0) + \frac{1}{N} \left. \frac{\partial^2 \ln p_\theta(\mathcal{Y}_1^N)}{\partial \theta^2} \right|_{\theta=\theta_0} \delta\theta$$

$$E_{\theta_0} \zeta_N(\theta) \approx -I(\theta_0) \delta\theta$$

Efficient score = **ML estimating function**

$$\text{Characterized by: } E_{\theta_0} \zeta_N(\theta) = 0 \iff \theta = \theta_0$$

Caution: Efficient score \neq innovation!

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Dependent data - Small deviations - Likelihood

Second order **Taylor expansion** of the **log-likelihood ratio**

$$\theta = \theta_0 + \frac{\delta\theta}{\sqrt{N}}$$

$$S_N(\theta_0, \theta) \triangleq \ln \frac{p_\theta(\mathcal{Y}_1^N)}{p_{\theta_0}(\mathcal{Y}_1^N)} \approx \delta\theta^T \zeta_N(\theta_0) - \frac{1}{2} \delta\theta^T I(\theta_0) \delta\theta$$

$$E_{\theta_0} S_N \approx -\frac{1}{2} \delta\theta^T I(\theta_0) \delta\theta$$

$$E_\theta S_N \approx +\frac{1}{2} \delta\theta^T I(\theta_0) \delta\theta \approx -E_{\theta_0} S_N$$

$$\text{cov}_{\theta_0} S_N \approx \delta\theta^T I(\theta_0) \delta\theta \approx \text{cov}_\theta S_N$$

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Example: Gaussian scalar **AR process**

$$Y_k = \sum_{i=1}^p a_i Y_{k-i} + E_k, \quad \theta^T = (a_1 \dots a_p)$$

$$\zeta_N(\theta) = \frac{1}{\sqrt{N}} \frac{1}{\sigma^2} \sum_{k=1}^N \mathcal{Y}_{k-1,p}^- \varepsilon_k(\theta)$$

$$I(\theta) = \frac{1}{\sigma^2} T_p$$

Efficient score ζ : **vector-valued** function

Innovation ε : **scalar** function

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Hypotheses testing - Asymptotic Gaussianity

For i.i.d. variables, stationary Gaussian processes, stationary Markov processes, ...:

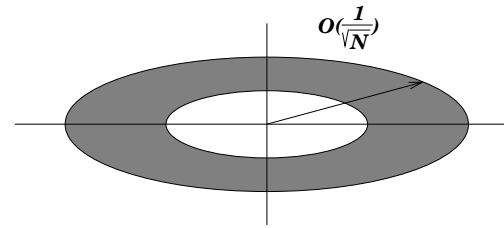
$$S_N(\theta_0, \theta) \rightarrow \begin{cases} \mathcal{N}(-\frac{1}{2} \delta\theta^T I(\theta_0) \delta\theta, \delta\theta^T I(\theta_0) \delta\theta) & \text{under } P_{\theta_0} \\ \mathcal{N}(+\frac{1}{2} \delta\theta^T I(\theta_0) \delta\theta, \delta\theta^T I(\theta_0) \delta\theta) & \text{under } P_{\theta_0 + \frac{\delta\theta}{\sqrt{N}}} \end{cases}$$

$$\zeta_N(\theta_0) \rightarrow \begin{cases} \mathcal{N}(0, I(\theta_0)) & \text{under } P_{\theta_0} \\ \mathcal{N}(I(\theta_0) \delta\theta, I(\theta_0)) & \text{under } P_{\theta_0 + \frac{\delta\theta}{\sqrt{N}}} \end{cases}$$

The **efficient score** ζ_N is asymptotically a **sufficient** statistics.

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Asymptotically optimum and equivalent tests for composite hypotheses



$$\text{GLR } \frac{\sup_{\theta \in \Theta_1} p_{\theta}(\mathcal{Y}_1^N)}{\sup_{\theta \in \Theta_0} p_{\theta}(\mathcal{Y}_1^N)} \geq \lambda$$

$$\frac{p_{\hat{\theta}}(\mathcal{Y}_1^N)}{p_{\theta_0}(\mathcal{Y}_1^N)} \geq \lambda$$

$$N (\hat{\theta} - \theta_0)^T I(\theta_0) (\hat{\theta} - \theta_0) \geq \lambda$$

$$\zeta_N^T(\theta_0) I^{-1}(\theta_0) \zeta_N(\theta_0) \geq \lambda$$

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Small deviations - Other estimating functions

Residual \leftrightarrow **Estimating function**

$$\zeta_N(\theta_0) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \mathcal{K}(\theta_0, Y_k)$$

Characterized by: $E_{\theta_0} \mathcal{K}(\theta, Y_k) = 0 \iff \theta = \theta_0$

Mean **sensitivity** (Jacobian) and covariance

$$\mathcal{J}(\theta_0) \triangleq -E_{\theta_0} \frac{\partial \mathcal{K}(\theta_0, Y_k)}{\partial \theta}, \quad \Sigma(\theta_0) \triangleq \lim_{N \rightarrow \infty} E_{\theta_0} \zeta_N(\theta_0) \zeta_N^T(\theta_0)$$

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First order Taylor expansion of this **residual**

$$\theta = \theta_0 + \frac{\delta\theta}{\sqrt{N}}$$

$$\zeta_N(\theta) \approx \underbrace{\zeta_N(\theta_0)}_{\text{CLT}} + \sqrt{N} \underbrace{\frac{1}{N} \left(\sum_{k=1}^N \frac{\partial}{\partial \theta} \mathcal{K}(\theta, Y_k) \Big|_{\theta=\theta_0} \right)}_{\text{LLN}} \frac{\delta\theta}{\sqrt{N}}$$

CLT \downarrow under θ_0

LLN \downarrow under θ_0

$$\mathcal{N}(0, \Sigma(\theta_0))$$

$$E_{\theta_0} \frac{\partial}{\partial \theta} \mathcal{K}(\theta, Y_k) \Big|_{\theta=\theta_0}$$

$$= -\mathcal{J}(\theta_0)$$

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The **residual** is asymptotically **Gaussian**

$$\zeta_N(\theta_0) \rightarrow \begin{cases} \mathcal{N}(0, \Sigma(\theta_0)) & \text{under } P_{\theta_0} \\ \mathcal{N}(\mathcal{J}(\theta_0) \delta\theta, \Sigma(\theta_0)) & \text{under } P_{\theta_0 + \frac{\delta\theta}{\sqrt{N}}} \end{cases}$$

(On-board) χ^2 -test for composite hypotheses

$$\zeta_N^T \underbrace{\Sigma^{-1} \mathcal{J} (\mathcal{J}^T \Sigma^{-1} \mathcal{J})^{-1} \mathcal{J}^T \Sigma^{-1}}_{\mathbf{I}^{-1}} \zeta_N \geq h$$

Invariant / pre-multiplication of ζ with invertible gain.

Noises and **uncertainty** on θ_0 taken into account.

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Nonlinear dynamic systems - LS residuals

State estimation: (full order) observers

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\theta, \hat{x}, u, y) \\ \hat{y}_k(\theta) = h(\theta, \hat{x}(k\tau), u(k\tau)) \end{cases}$$

$$\hat{f}(\theta, \hat{x}, u, y) \triangleq f(\theta, \hat{x}, u) + K_o (y - h(\theta, \hat{x}, u))$$

$$\zeta_N : \mathcal{K}(\theta_0, Y_k) = \left(\frac{\partial \hat{y}_k(\theta)}{\partial \theta} \Big|_{\theta=\theta_0} \right)^T (y_k - \hat{y}_k(\theta_0))$$

$$\begin{cases} \frac{\partial \hat{x}}{\partial \theta} = \frac{\partial}{\partial \hat{x}} \hat{f}(\theta_0, \hat{x}, u, y) \frac{\partial \hat{x}}{\partial \theta} + \frac{\partial}{\partial \theta} \hat{f}(\theta_0, \hat{x}, u, y) \\ \frac{\partial \hat{y}_k(\theta)}{\partial \theta} = \frac{\partial}{\partial \hat{x}} h(\theta_0, \hat{x}(k\tau), u(k\tau)) \frac{\partial \hat{x}_k}{\partial \theta} + \frac{\partial}{\partial \theta} h(\theta_0, \hat{x}(k\tau), u(k\tau)) \end{cases}$$

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Isolation (1) - Nuisance approach

Which components of θ ?

$$\zeta \sim \mathcal{N}(\mathcal{J} \theta, \Sigma), \quad \theta = \begin{pmatrix} \theta_a \\ \theta_b \end{pmatrix}, \quad \mathcal{J} = (\mathcal{J}_a \quad \mathcal{J}_b), \quad p_{\theta_a, \theta_b}(\zeta)$$

Decide between $\theta_a = 0$ and $\theta_a \neq 0$; θ_b unknown

$$\mathbf{I} \triangleq \mathcal{J}^T \Sigma^{-1} \mathcal{J} \triangleq \begin{pmatrix} \mathbf{I}_{aa} & \mathbf{I}_{ab} \\ \mathbf{I}_{ba} & \mathbf{I}_{bb} \end{pmatrix}$$

$$\mathbf{I}_a^{*-1} : \text{upper-left block of } \mathbf{I}^{-1}; \quad \mathbf{I}_a^* = \mathbf{I}_{aa} - \mathbf{I}_{ab} \mathbf{I}_{bb}^{-1} \mathbf{I}_{ba}$$

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Nonlinear dynamic systems - Simulation methods

Monte Carlo sequential simulation

Particle filters

Numerical approximation of the **efficient score**

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Statistical **projection** (sensitivity)

$$2 \ln \frac{\max_{\theta_a} p_{\theta_a,0}(\zeta)}{p_{0,0}(\zeta)} = \zeta_a^T I_{aa}^{-1} \zeta_a, \quad \zeta_a : \text{partial score}$$

Statistical **rejection** (minmax)

$$2 \ln \frac{\max_{\theta_a, \theta_b} p_{\theta_a, \theta_b}(\zeta)}{\max_{\theta_b} p_{0, \theta_b}(\zeta)} = \zeta_a^{*T} I_a^{*-1} \zeta_a^*, \quad \zeta_a^* : \text{effective score}$$

$$\zeta_a^* = \zeta_a - I_{ab} I_{bb}^{-1} \zeta_b$$

Regression of **informative** score over **nuisance** score (Neyman, 1954).

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Dealing with **nuisance** parameters - Contd.

Other approaches :

- Reparameterization : generalization of sensitivity approach
- Invariant tests
- Minimax tests

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Isolation (2) - Multiple hypotheses approach

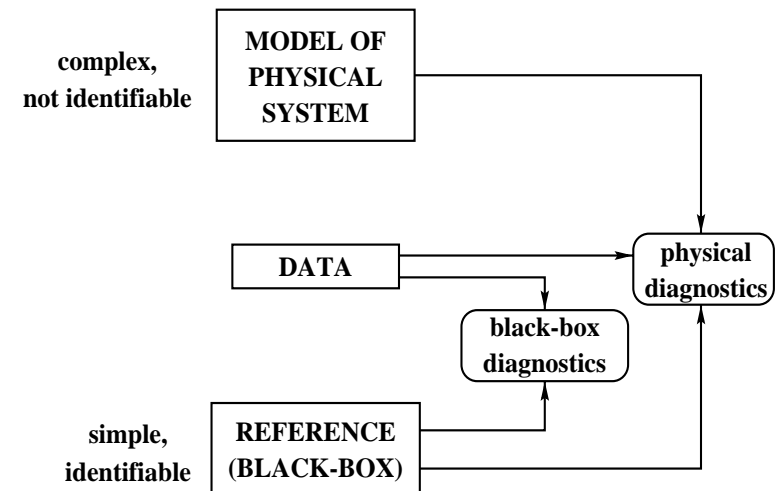
$$H_0 : \theta \in \Theta_0$$

$$H_i : \theta \in \Theta_i, (i = 1 : m)$$

- Bayesian approach
- Invariant tests

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(On-Board) **Diagnostics**/localization : don't solve the inverse problem!



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Optimal **sensor location** for monitoring

Maximize the **power** of the detection algorithms :

$$\text{Trace} (\mathcal{J}^T \Sigma^{-1} \mathcal{J})$$

Physical model necessary,
'compensate' for the number of d.o.f. of χ^2 tests.

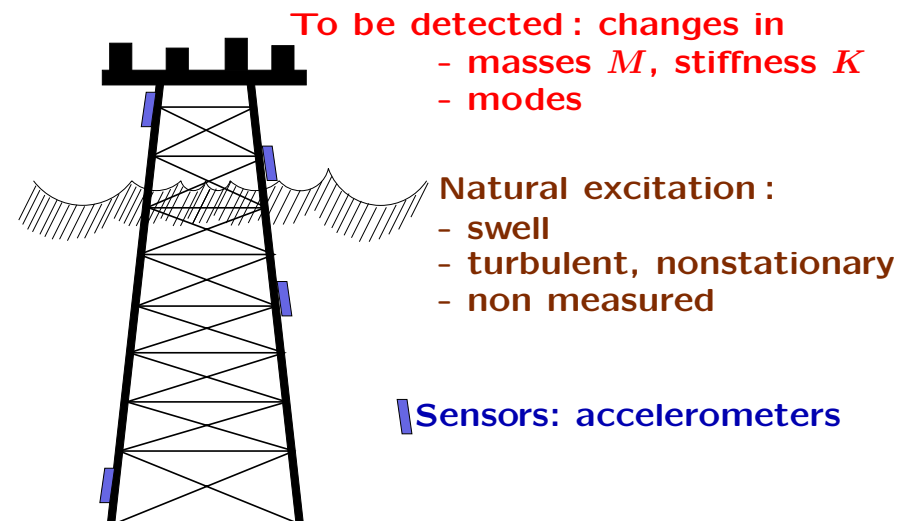
Two possible uses :

- For a **given set of faults** : how many sensors, and where?
- For a **given sensor pool** : which faults are detectable?

Fault **detectability**

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VIBRATIONS : OFFSHORE STRUCTURES



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EXAMPLE

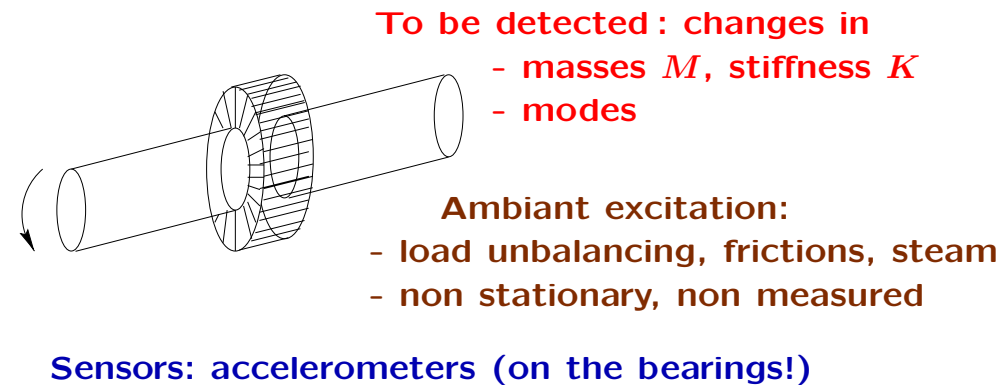
Structural health monitoring

With *Laurent Mével, Maurice Goursat, Albert Benveniste*

Toolboxes: (free) Scilab; LMS CADA-X

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VIBRATIONS : ROTATING MACHINES



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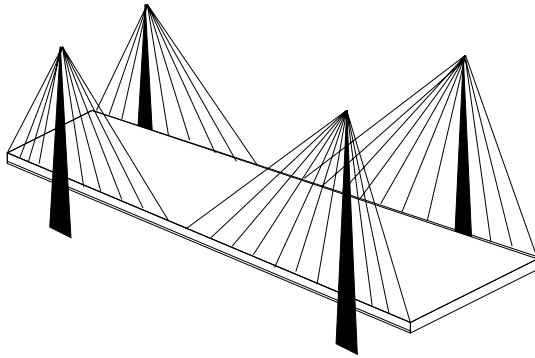
VIBRATIONS : BRIDGES

Ambiant excitation:

- wind
- traffic UNDER
- traffic ON

To be detected: changes in

- masses M , stiffness K
- modes



Nuisance:

- traffic ON
- temperature

Sensors: accelerometers, laser

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Modelling

FE model:
$$\begin{cases} M\ddot{Z}(s) + C\dot{Z}(s) + KZ(s) = \nu(s) \\ Y(s) = LZ(s) \end{cases}$$

$(M\mu^2 + C\mu + K)\Psi_\mu = 0$, $\psi_\mu = L\Psi_\mu$

State space:
$$\begin{cases} X_{k+1} = FX_k + V_k \\ Y_k = HX_k \end{cases}$$

$F\Phi_\lambda = \lambda \Phi_\lambda$, $\varphi_\lambda \triangleq H\Phi_\lambda$

$$\underbrace{e^{\delta\mu}}_{\text{modes}} = \lambda, \quad \underbrace{\psi_\mu}_{\text{mode shapes}} = \varphi_\lambda$$

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Problems: In-operation **modal identification**
and **damage detection** and **localization**

- The **excitation** is typically:
 - natural, **not controlled**.
 - **not measured**:
 - * buildings, bridges, offshore structures,
 - * rotating machinery,
 - * cars, trains, aircrafts.
 - **nonstationary** (e.g., turbulent).
- How to **detect** and **localize** small damages? **Early?** **On-board?** (without re-identification)
- **Output-only** damage detection and localization

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Output-only **covariance-based** **subspace** identification

$$R_i \triangleq \mathbb{E}(Y_k Y_{k-i}^T), \quad \mathcal{H} = \begin{pmatrix} R_0 & R_1 & R_2 & \dots \\ R_1 & R_2 & R_3 & \dots \\ R_2 & R_3 & R_4 & \dots \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

ok if stationary!

$$R_i = H F^i G, \quad G \triangleq \mathbb{E}(X_k Y_k^T)$$

$$\mathcal{O} \triangleq \begin{pmatrix} H \\ HF \\ HF^2 \\ \vdots \end{pmatrix}, \quad \mathcal{C} \triangleq (G \quad FG \quad F^2G \quad \dots)$$

$$\mathcal{H} = \mathcal{O} \mathcal{C}, \quad \mathcal{H} \longrightarrow \mathcal{O} \longrightarrow (H, F) \longrightarrow (\lambda, \varphi_\lambda)$$

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Implementation

$$\hat{R}_i \triangleq \frac{1}{N} \sum_{k=1}^N Y_k Y_{k-i}^T, \quad \hat{\mathcal{H}} = \begin{pmatrix} \hat{R}_0 & \hat{R}_1 & \hat{R}_2 & \dots \\ \hat{R}_1 & \hat{R}_2 & \hat{R}_3 & \dots \\ \hat{R}_2 & \hat{R}_3 & \hat{R}_4 & \dots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

ok when nonstationary!

SVD($\hat{\mathcal{H}}$) + truncation $\longrightarrow \hat{\mathcal{O}} \longrightarrow (\hat{H}, \hat{F}) \longrightarrow (\hat{\lambda}, \hat{\varphi}_\lambda)$

$$\hat{\mathcal{H}} = U \Delta W^T = U \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_0 \end{pmatrix} W^T; \quad \hat{\mathcal{O}} = U \Delta_1^{1/2}$$

$$\mathcal{O}_p^\dagger(H, F) = \mathcal{O}_p(H, F) F$$

$$\det(F - \lambda I) = 0, \quad F \Phi_\lambda = \lambda \Phi_\lambda, \quad \varphi_\lambda = H \Phi_\lambda$$

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Introducing the parameter vector

FE model:
$$\begin{cases} M \ddot{Z}(s) + C \dot{Z}(s) + K Z(s) = \nu(s) \\ Y(s) = L Z(s) \end{cases}$$

$$(M \mu^2 + C \mu + K) \Psi_\mu = 0, \quad \psi_\mu = L \Psi_\mu$$

State space:
$$\begin{cases} X_{k+1} = F X_k + V_k \\ Y_k = H X_k \end{cases}$$

$$F \Phi_\lambda = \lambda \Phi_\lambda, \quad \varphi_\lambda \triangleq H \Phi_\lambda$$

Parameter:
$$\underbrace{e^{\delta \mu}}_{\text{modes}} = \lambda, \quad \underbrace{\psi_\mu}_{\text{mode shapes}} = \varphi_\lambda; \quad \theta \triangleq \begin{pmatrix} \Lambda \\ \text{vec } \Phi \end{pmatrix}$$

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Robustness to nonstationary excitation

The estimates are **consistent**.

Combination of:

- the key **factorization** property of the covariances,
- the **averaging** operation underlying covariance computation,

allows to cancel out nonstationarities in the excitation.

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Eigenstructure monitoring

$$\begin{cases} X_{k+1} = F X_k + V_k & F \varphi_\lambda = \lambda \varphi_\lambda \\ Y_k = H X_k & \Phi_\lambda \triangleq H \varphi_\lambda \end{cases}$$

Canonical parametrization: $\theta \triangleq \begin{pmatrix} \Lambda \\ \text{vec } \Phi \end{pmatrix}$

Observability in modal basis: $\mathcal{O}_{p+1}(\theta) = \begin{pmatrix} \Phi \\ \Phi \Delta \\ \vdots \\ \Phi \Delta^p \end{pmatrix}$

System parameter characterization:

$\mathcal{H}_{p+1,q}$ and $\mathcal{O}_{p+1}(\theta)$ have the **same left kernel**.

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Back to structural **subspace** identification

$$\exists S, \quad S^T S = I_s, \quad S^T \mathcal{O}_{p+1}(\theta_0) = 0; \quad \text{say } S(\theta_0)$$

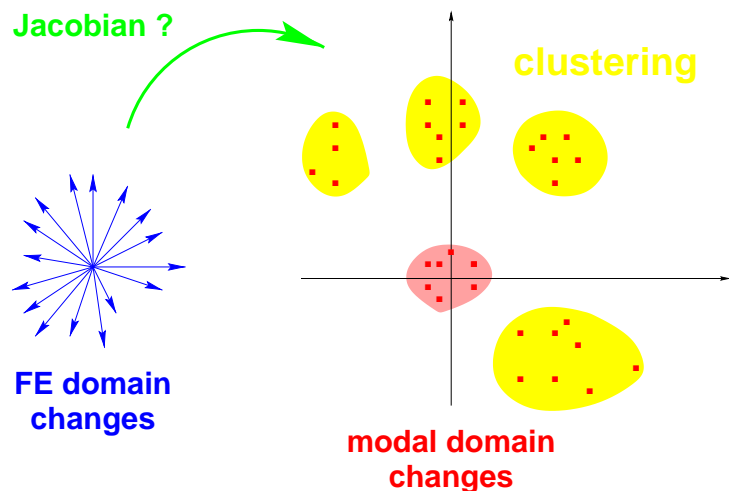
$$\theta_0 \leftrightarrow (R_i^0)_i \text{ characterized by: } S^T(\theta_0) \hat{\mathcal{H}}_{p+1,q}^0 = 0$$

Residual for structural damage monitoring

$$\zeta_N(\theta_0) \triangleq \text{vec}(S^T(\theta_0) \hat{\mathcal{H}}_{p+1,q})$$

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On-board damage **diagnostics**: **projecting** changes



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Relation to **parity space**

$$\zeta_{\text{parity}} = \mathcal{G} \mathcal{Y}_{k,p+1}^+, \quad \mathcal{G} \mathcal{O}_{p+1} = 0$$

$$\zeta_{\text{subspace}} = S^T \hat{\mathcal{H}}_{p+1,q}, \quad S^T \mathcal{O}_{p+1} = 0$$

First order **statistics** \longleftrightarrow **Second** order **statistics**

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Damage **diagnostics**: (local) **sensitivity** approach

$$\zeta \sim \mathcal{N}(\mathcal{J} \delta\theta, \Sigma), \quad \delta\theta = \mathcal{I} \mathcal{J}_{(M_0^*, K_0^*)} \begin{pmatrix} \delta M \\ \delta K \end{pmatrix}$$

(M_0^*, K_0^*) : **design** model

$$\text{Jacobian} : (\delta M, \delta K) \xrightarrow{\mathcal{J}_{(M_0^*, K_0^*)}} (\delta\mu, \delta\psi_\mu)$$

Reduction: \mathcal{I} matching computed/identified modes

$$\text{Problem} : \dim \begin{pmatrix} M \\ K \end{pmatrix} \gg \dim \theta$$

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Computing Jacobian

1. $(\delta M, \delta K) \xrightarrow[\text{mode selection}]{\mathcal{IJ}(M_0^*, K_0^*)} (\delta \mu, \delta \psi_\mu)$
2. Apply \mathcal{IJ} to **unit** vectors $(\delta M, \delta K)$
3. **Truncate small** vectors $(\delta \mu, \delta \psi_\mu)$
4. **Cluster** the remaining vectors $(\delta \mu, \delta \psi_\mu)$, using the χ^2 -metric.

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Results on real data

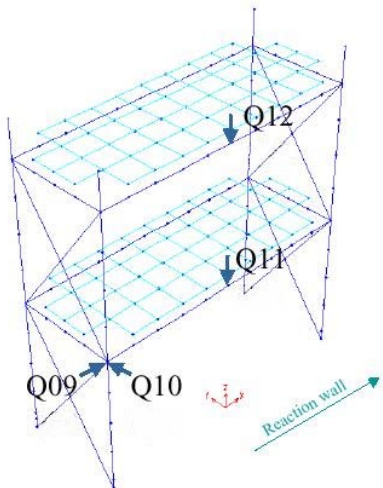
Steelquake

Z24 bridge

Flutter monitoring

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Steelquake



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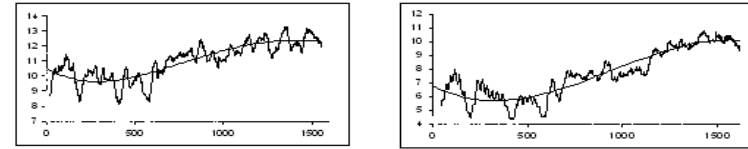
Scenario	Undamaged	Damaged
Q09 /39	$2.81 \cdot 10e2$	$3.78 \cdot 10e6$
Q10 /40	$1.53 \cdot 10e2$	$2.20 \cdot 10e7$
Q11 /41	$6.75 \cdot 10e2$	$2.18 \cdot 10e4$
Q12 /42	$2.88 \cdot 10e2$	$1.62 \cdot 10e4$

Z24 bridge

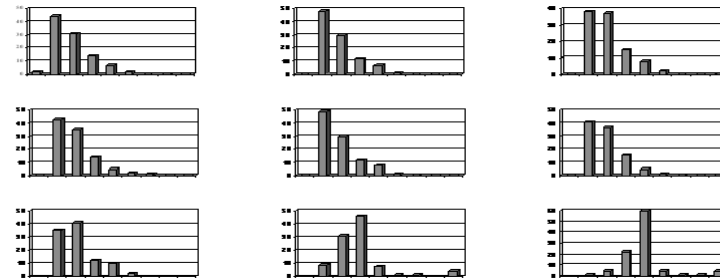
	Mode	1	2	3	4	χ^2
Undamaged	Freq.(Hz)	3.88	5.01	9.80	10.30	$8.80 \cdot 10e2$
Damaged (1)	Freq.(Hz)	3.87	5.06	9.79	10.32	$8.00 \cdot 10e5$
Damaged (2)	Freq.(Hz)	3.76	4.93	9.74	10.25	$3.96 \cdot 10e6$

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Z24 bridge (Contd.)



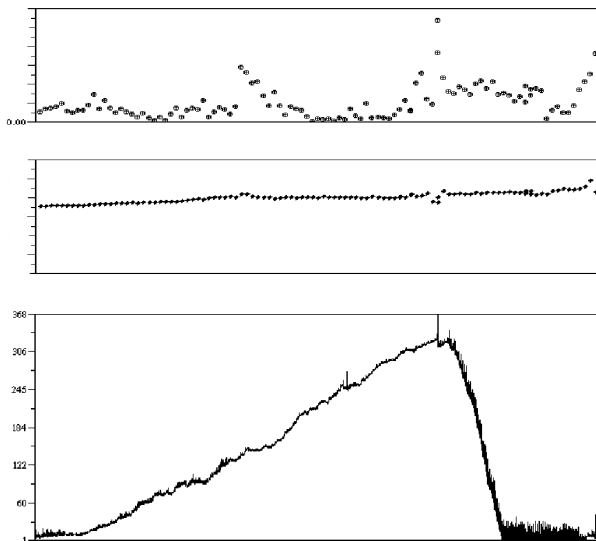
The test values over three months, log-scale. Two sensors sets.



Distribution of the test values for each of the nine months.

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Flutter monitoring



Damping coefficient (top), frequency (middle), CUSUM test (bottom).

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Conclusion

A **statistical** framework

enlightens the meaning and increases the power

of a number of **familiar operations**

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