Advanced numerical computation of $\chi^2$ tests for fault detection and isolation

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Introduction

- Typical fault detection and isolation (FDI) procedure:
  - residual generation
  - residual evaluation

- Evaluation of Gaussian residuals
  - for parametric change in linear systems
  - for small parametric change in nonlinear systems
    (local asymptotic approach to change detection)
- Changes in the mean of a Gaussian vector — $\chi^2$-tests

Gaussian residual evaluation — $\chi^2$-tests

Consider $m$-dimensional residual $z \sim \mathcal{N}(M \eta, \Sigma)$ with $M \in \mathbb{R}^{m \times n}$, $\eta \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{m \times m}$, $m \geq n$.

Fault detection
$H_0 : \eta = 0$ against $H_1 : \eta \neq 0$

Fault isolation for some partition $\eta = \begin{bmatrix} \eta_a \\ \eta_b \end{bmatrix}$
$H_0 : \eta_a = 0$ against $H_1 : \eta_a \neq 0$

These hypothesis testing problems lead to $\chi^2$-tests.

Basic formulas for the $\chi^2$-tests

$z \sim \mathcal{N}(M \eta, \Sigma)$ $I = M^T \Sigma^{-1} M$ $\eta = \begin{bmatrix} \eta_a \\ \eta_b \end{bmatrix}$ $M = [M_a \ M_b]$ $I = \begin{bmatrix} I_{aa} & I_{ab} \\ I_{ba} & I_{bb} \end{bmatrix}$

Fault detection (global test)
$t = z^T \Sigma^{-1} M (M^T \Sigma^{-1} M_a)^{-1} M_a^T \Sigma^{-1} z$

Fault isolation by sensitivity test
$\tilde{t}_a = z^T \Sigma^{-1} M_a (M_a^T \Sigma^{-1} M_a)^{-1} M_a^T \Sigma^{-1} z$

or equivalently
$\tilde{\xi}_a = M_a^T \Sigma^{-1} z$
$\tilde{t}_a = \tilde{\xi}_a I_{aa} \tilde{\xi}_a$
Basic formulas of the \( \chi^2 \)-tests (contd.)

Fault isolation by min-max test

\[
\begin{align*}
\tilde{\zeta}_a &= M_a^T \Sigma^{-1} z \\
\tilde{\zeta}_b &= M_b^T \Sigma^{-1} z \\
\zeta_a^* &= \zeta_a - I_{ab} \Gamma_{bb}^{-1} \zeta_b \\
\Gamma_a^* &= \Gamma_{aa} - I_{ab} \Gamma_{bb}^{-1} I_{ba} \\
t_a^* &= \zeta_a^T \Gamma_a^{-1} \zeta_a^*
\end{align*}
\]

Numerical difficulties: when the matrices to be inverted are badly conditioned, these basic formulas can lead to large numerical errors. It is thus important to develop advanced numerical algorithms.

Advanced numerical computation – Global test

\[
t = z^T \Sigma^{-1} M (M^T \Sigma^{-1} M)^{-1} M^T \Sigma^{-1} z
\]

- Use pseudo-inverse for \( \Sigma^{-1} \) if badly conditioned
- Compute \( t \) as a square: \( t = \|(M^T \Sigma^{-1} M)^{-\frac{1}{2}} M^T \Sigma^{-1} z\|^2 \)
- Avoid the inverse involving \( M \).

Proposed solution:
- Let \( \Gamma = \Sigma^{-\frac{1}{2}} \) (with pseudo-inverse if necessary),
- QR decomposition of \( \Gamma M: \Gamma M = QR \) with \( QTQ = I \),
- Then
  \[
  t = z^T \Gamma Q R (R^T Q R)^{-1} R^T Q^T \Gamma z
  = z^T \Gamma Q R (R^T R)^{-1} R^T Q^T \Gamma z
  = \|Q^T \Gamma z\|^2
  \]

Advanced numerical computation – Sensitivity test

\[
t_a = z^T \Sigma^{-1} M_a (M_a^T \Sigma^{-1} M_a)^{-1} M_a^T \Sigma^{-1} z
\]

Same as global test (\( M \) being replaced by \( M_a \)).

Advanced numerical computation:
- QR decomposition of \( \Gamma M_a: \Gamma M_a = Q_a R_a \) with \( Q_a^T Q_a = I \), then
  \[
  \tilde{t}_a = \|Q_a^T \Gamma z\|^2
  \]

Advanced numerical computation – Minmax test

More computations are involved.

QR decompositions:

\[
\begin{align*}
\Gamma M_a &= Q_a R_a \\
\Gamma M_b &= Q_b R_b \\
(I - Q_b Q_b^T) Q_a &= Q_c R_c
\end{align*}
\]

then

\[
t_a^* = \|Q_c^T \Gamma z\|^2
\]

A non-trivial step for deriving the algorithm:

\[
(I - Q_a^T Q_b Q_b^T Q_a) = Q_a^T (I - Q_b Q_b^T) (I - Q_b Q_b^T) Q_a
\]

Remark: SVD can be used instead of QR decomposition.
A numerical example
Gaussian vector: \( \dim(z) = 9, \dim(\eta) = 5, \dim(\eta_a) = 2 \).
Condition number of \( \Sigma \): \( 3.4 \times 10^{10} \).
Histograms are based on 10000 random realizations.

Basic minmax test (solid line) and \( \chi^2(2, 44.955) \) density function (dashed line).

Conclusion

- \( \chi^2 \)-tests are frequently used for residual evaluation.
- Advanced numerical algorithms can significantly improve the numerical accuracy of badly conditioned problems.