Information criteria

for fault detection and isolation

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Faults in linear dynamical systems

Basic statistical isolation tools

Linking and mixing deterministic and stochastic approaches

Information criteria for residual optimization

Faults in linear dynamical systems

$$\begin{cases} X_{k+1} &= FX_k &+ GU_k &+ \Gamma\,\Upsilon_x &+ W_k \\ Y_k &= HX_k &+ JU_k &+ \Xi\,\Upsilon_y &+ V_k \end{cases}$$

Reducing the problem to a static one

ℓ-size window, $\mathcal{Y}_k^{(\ell)}$ made of $Y_{k-\ell+1},\ldots,Y_k$.

$$\underline{\mathcal{Y}}_k^{(\ell)} = \mathcal{Y}_k^{(\ell)} - \mathcal{M}_\ell(G,J)\,\mathcal{U}_k^{(\ell)}$$
$$\underline{\mathcal{V}}_k^{(\ell)} = \mathcal{M}_\ell(I_n,0)\,\mathcal{W}_k^{(\ell)} + \mathcal{V}_k^{(\ell)}$$

$$\mathcal{Y} = \underline{\mathcal{Y}}_k^{(\ell)} = \mathcal{O}_\ell\; X_{k-\ell+1} + \underline{\mathcal{M}}_\ell(\Gamma)\;\Upsilon_x + (\mathbb{1}_\ell \otimes \Xi)\;\Upsilon_y + \underline{\mathcal{V}}_k^{(\ell)}$$

Problems

$$Z = H\;X + M\;\eta + \epsilon, \quad \mathrm{cov}(\epsilon) = \Sigma = \bar{\Sigma}\;\bar{\Sigma}^T, \bar{\Sigma} > 0$$

$$Z = H\;X + M_a\;\eta_a + M_b\;\eta_b + \epsilon$$

Detection

Decide between $\eta = 0$ and $\eta \neq 0$.

Diagnosis

Decide between $\eta_a = 0$ and $\eta_a \neq 0$.

Residual generation and evaluation

Find *transform* $\mathcal{G}$, *residual* $\zeta = \mathcal{G}\;Z$: $\|\zeta_0\|$ min., $\|\zeta_\eta\|$ max.

Deterministic approaches: 'parity checks'

Deterministic rejection

$$v^T\;\mathcal{O}_\ell = 0; \quad n_\ell = \ell r - \mathrm{rank}(\mathcal{O}_\ell)$$

$\mathcal{P}_{\ell,s}$: orthonormal (sub-)basis of parity space, $s \leq n_\ell$

$$\mathcal{G}_{\ell,s} \triangleq \mathcal{P}_{\ell,s}^T, \quad \zeta = \mathcal{G}_{\ell,s}\;\mathcal{Y}$$

Parity space through SVD - Corollary

$$\text{Minimize} \quad \|\mathcal{G}_{\ell,s}\;\mathcal{O}_\ell\|_F, \quad s.t.\;\; \mathcal{G}_{\ell,s}\;\mathcal{G}_{\ell,s}^T = I_s$$

Criteria for residual optimization

$$J_r = \frac{\|\mathcal{G}_{\ell,s}\;\mathcal{O}_\ell\|_F^2}{\|\mathcal{G}_{\ell,s}\;\mathcal{M}_\ell\|_F^2}, \quad J_\alpha = \alpha\;\|\mathcal{G}_{\ell,s}\;\mathcal{O}_\ell\|_F^2 + (1-\alpha)\;\|\mathcal{G}_{\ell,s}\;\mathcal{M}_\ell\|_F^2$$

Basic statistical tools: Gaussian vector $Z \sim \mathcal{N}(M\eta, \bar{\Sigma}\bar{\Sigma}^T)$

$$l_\eta(Z) \triangleq -2\ \ln p_\eta(z) = (Z - M\eta)^T\ \Sigma^{-1}\ (Z - M\eta)$$

Projection and rejection

$$P_M = I - \bar{\Sigma}^{-1}\ M(M^T\ \Sigma^{-1}\ M)^{-1} M^T\ \bar{\Sigma}^{-T}$$

$$Z - M\hat{\eta} = \tilde{P}_M\ Z, \quad \tilde{P}_M \triangleq \bar{\Sigma}\ P_M\ \bar{\Sigma}^{-1}, \quad \tilde{P}_M\ M = 0$$

Transformation and information

$$\mathbf{F}_Z = M^T\ \Sigma^{-1}\ M$$

$$\zeta = \mathcal{G}\ Z \sim \mathcal{N}(L\ \Upsilon, R),\ L = \mathcal{G}\ M, R = \mathcal{G}\ \Sigma\ \mathcal{G}^T$$

$$\mathbf{F}_\zeta = L^T R^{-1} L \neq \mathbf{F}_Z$$

$$\mathbf{F}_{B\zeta} = \mathbf{F}_\zeta, \quad B \text{ invertible}$$

Detection: global test

$$t_1 = 2\ \ln \frac{\max_\eta p_\eta(Z)}{p_0(Z)} = l_0(Z) - \min_\eta l_\eta(Z)$$

Projection

$$t_1 = Z^T\ \bar{\Sigma}^{-T}\ (I - P_M)\ \bar{\Sigma}^{-1}\ Z$$

Transformation

$$\zeta = \mathcal{G}\ Z \triangleq M^T\ \Sigma^{-1}\ Z; \quad \mathcal{G}\ M = \mathbf{F} = \mathcal{G}\ \Sigma\ \mathcal{G}^T; \quad \zeta \sim \mathcal{N}\ (\mathbf{F}\ \eta, \mathbf{F})$$

$$t_1 = \zeta^T\ \mathbf{F}^{-1}\ \zeta = Z^T\ \Sigma^{-1}\ M\ \mathbf{F}^{-1} M^T\ \Sigma^{-1}\ Z$$

Isolation: $Z \sim \mathcal{N}(M\ \eta, \Sigma), \quad M = \begin{pmatrix} M_a & M_b \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_a \\ \eta_b \end{pmatrix}$

$$\mathbf{F} = M^T\ \Sigma^{-1}\ M = \begin{pmatrix} M_a^T\ \Sigma^{-1}\ M_a & M_a^T\ \Sigma^{-1}\ M_b \\ M_b^T\ \Sigma^{-1}\ M_a & M_b^T\ \Sigma^{-1}\ M_b \end{pmatrix} \triangleq \begin{pmatrix} \mathbf{F}_{aa} & \mathbf{F}_{ab} \\ \mathbf{F}_{ba} & \mathbf{F}_{bb} \end{pmatrix}$$

$$\mathbf{F}^{-1} = \begin{pmatrix} \mathbf{F}_a^{*\,-1} & -\mathbf{F}_{aa}^{-1}\ \mathbf{F}_{ab}\ \mathbf{F}_b^{*\,-1} \\ -\mathbf{F}_{bb}^{-1}\ \mathbf{F}_{ba}\ \mathbf{F}_a^{*\,-1} & \mathbf{F}_b^{*\,-1} \end{pmatrix}, \quad \mathbf{F}_i^* = \mathbf{F}_{ii} - \mathbf{F}_{ij}\ \mathbf{F}_{jj}^{-1}\ \mathbf{F}_{ji}$$

1.? Whitening the data: $\Sigma = \breve{\mathcal{G}}^T\ \Delta\ \breve{\mathcal{G}}$

$$\breve{\zeta} = \breve{\mathcal{G}}\ Z \sim \mathcal{N}(\breve{\mathcal{G}} M\ \eta, \Delta), \quad \breve{\mathbf{F}} = M^T\ \breve{\mathcal{G}}^T\ \Delta^{-1}\ \breve{\mathcal{G}}\ M = \mathbf{F}$$

2. Sensitivity method

$$\tilde{t}_a = 2\ \ln \frac{\max_{\eta_a} p_{\eta_a,0}(Z)}{p_{0,0}(Z)} = l_{0,0}(Z) - \min_{\eta_a} l_{\eta_a,0}(Z)$$

Projection

$$\tilde{t}_a = Z^T\ \bar{\Sigma}^{-T}\ (I - P_{M_a})\ \bar{\Sigma}^{-1}\ Z$$

Transformation

$$\tilde{\zeta}_a = \tilde{\mathcal{G}}_a\ Z \triangleq M_a^T\ \Sigma^{-1}\ Z; \quad \tilde{\zeta}_a \sim \mathcal{N}\ (\mathbf{F}_{aa}\ \eta_a, \mathbf{F}_{aa})$$

Decoupling, decorrelation and information

$$\tilde{\zeta} \triangleq \begin{pmatrix} \tilde{\zeta}_a \\ \tilde{\zeta}_b \end{pmatrix} \sim \mathcal{N}(\mathbf{F}\ \eta, \mathbf{F}); \quad \tilde{\mathbf{F}} = \mathbf{F}$$

3. (Minmax) rejection method

$$t_a^* = 2\ \ln \frac{\max_{\eta_a,\eta_b} p_{\eta_a,\eta_b}(Z)}{\max_{\eta_b} p_{0,\eta_b}(Z)} = \min_{\eta_b} l_{0,\eta_b}(Z) - \min_{\eta_a,\eta_b} l_{\eta_a,\eta_b}(Z)$$

Projection

$$t_a^* = Z^T\ \bar{\Sigma}^{-T}\ (P_{M_b} - P_M)\ \bar{\Sigma}^{-1}\ Z$$

Transformation

$$\zeta_a^* = \mathcal{G}_a^*\ Z \triangleq M_a^T\ \Sigma^{-1}\ \tilde{P}_{M_b}\ Z; \quad \mathcal{G}_a^*\ M_b = 0; \quad \zeta_a^* \sim \mathcal{N}\left(\mathbf{F}_a^*\ \eta_a, \mathbf{F}_a^*\right)$$

Decoupling, decorrelation and information

$$\zeta^* \triangleq \begin{pmatrix} \zeta_a^* \\ \zeta_b^* \end{pmatrix} \sim \mathcal{N}(L^*\ \eta, \Sigma^*), \quad L^* = \begin{pmatrix} \mathbf{F}_a^* & 0 \\ 0 & \mathbf{F}_b^* \end{pmatrix}; \quad \mathbf{F}^* = \mathbf{F}$$

4. Mixed method: rejection and sensitivity

Decoupling, decorrelation and information

$$\bar{\zeta} \triangleq \begin{pmatrix} \zeta_a^* \\ \tilde{\zeta}_b \end{pmatrix} \sim \mathcal{N}(\bar{L}\ \eta, \bar{\Sigma}), \quad \bar{L} = \begin{pmatrix} \mathbf{F}_a^* & 0 \\ \mathbf{F}_{ba} & \mathbf{F}_{bb} \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \mathbf{F}_a^* & 0 \\ 0 & \mathbf{F}_{bb} \end{pmatrix}; \quad \bar{\mathbf{F}} = \mathbf{F}$$

Decomposition of global test

$$t_1 = t_a^* + \tilde{t}_b$$

Additive decomposition (general distribution, more than 2 faults),
Orthogonal decomposition (Gaussian case, only 2 faults).

Linking and mixing deterministic and stochastic approaches

Regression model $\quad Z = H\ X + M_a\ \eta_a + M_b\ \eta_b + \epsilon$

Deterministic/stochastic link: factorization

Global test based on $\quad \mathcal{G}_M^* = M^T\ \Sigma^{-1}\ \tilde{P}_H, \quad \mathcal{G}_M^*\ H = 0$

QR fact. of H: $\quad PH = 0,\ NH$ full rank, $N^T N + P^T P = I$

P factorizes $\tilde{P}_H$ and $\mathcal{G}_M^*$

Mixed algorithms

Estimate X,η_a,η_b: $\min_{X,\eta_b}\ l_{X,0,\eta_b}(Z) - \min_{X,\eta_a,\eta_b}\ l_{X,\eta_a,\eta_b}(Z)$;

Reject (determinist.) η_b, estimate X,η_a;

Reject (determinist.) X,η_b, estimate η_a.

Multiple faults - Optimality criteria

$$\mathcal{H}_j:\ \eta_j = 0 \cdot e_j, \quad \mathcal{K}_j:\ \eta_j = \alpha_j \cdot e_j; \quad \widetilde{\mathcal{H}}_j = \mathcal{K}_j \bigcap \left(\cap_{i \neq j}\, \mathcal{H}_i\right)$$

Test	Hypotheses	'Power'
global t_1	$\mathcal{H}_0 = \cap_j\, \mathcal{H}_j\ /\ \mathcal{H}_0^c = \cap_j\, \mathcal{K}_j$	β
minmax t_j^*	$\mathcal{H}_j\ /\ \mathcal{K}_j$	β_j^*
sensitivity $\tilde{t}_j$	$\mathcal{H}_0/\widetilde{\mathcal{H}}_j$	$\tilde{\beta}_j$
$(\tilde{t}_1, \ldots, \tilde{t}_q)$	$\bigcup_j \left(\mathcal{H}_0/\widetilde{\mathcal{H}}_j\right)$	$\min_j \tilde{\beta}_j$
$(t_1^*, \ldots, t_q^*)$	$\bigcup_j (\mathcal{H}_j/\mathcal{K}_j)$	$\min_j \beta_j^*, \quad \sum_{j=1}^q \beta_j^*$

For small faults: $\sum_{j=1}^{q} \beta_j^* \approx \text{cst} \cdot \sum_{j=1}^{q} {\rho_j^*}^2\ \mathbf{F}_{jj}$

Information criteria for residual optimization

$$\mathcal{Y}_\ell = \mathcal{O}_\ell\, X + \mathcal{M}_\ell\, \Upsilon + \mathcal{E}$$

$$\zeta = \mathcal{G}_{s,\ell}\, \mathcal{Y}_\ell$$

$$\mathbf{F}_\zeta = \mathcal{M}_\ell^T\, \mathcal{G}_{s,\ell}^T\, \left(\mathcal{G}_{s,\ell}\, \Sigma_\ell\, \mathcal{G}_{s,\ell}^T\right)^{-1}\, \mathcal{G}_{s,\ell}\, \mathcal{M}_\ell$$

Optimization

$$\begin{cases} \min_{\mathcal{G}_{s,\ell},\, s,\, \ell} \;\; C(\mathbf{F}), \\ \text{s.t. } \mathcal{G}_{s,\ell}\, \mathcal{O}_\ell = 0,\, \mathcal{G}_{s,\ell}\, \mathcal{G}_{s,\ell}^T = I_s \end{cases}$$

? (Scalar) criterion C based on (matrix) Fisher info.

Existing criteria

(alg. performances, design of experiments, optimal sensor location)

$$c_z(\mathbf{F}) = \left(\frac{\operatorname{tr} \mathbf{F}^z}{q}\right)^{1/z} \quad (z < 0)$$

$$\lim_{z\to 0} c_z(\mathbf{F}) = |\mathbf{F}|^{1/q}, \quad c_{-1}(\mathbf{F}) = \frac{q}{\operatorname{tr}\left(\mathbf{F}^{-1}\right)}, \quad \lim_{z\to-\infty} c_z(\mathbf{F}) = \lambda_{\min}(\mathbf{F})$$

Distance between matrices

(array calibration, source separation)

$$\mathbf{K}(\Sigma_1,\Sigma_2) = \frac{1}{2}\left(\operatorname{tr}\left(\Sigma_1\, \Sigma_2^{-1} - I_q\right) - \ln\left|\Sigma_1\, \Sigma_2^{-1}\right|\right)$$

$$\mathbf{K}(A\, \Sigma_1\, A^T, A\, \Sigma_2\, A^T) = \mathbf{K}(\Sigma_1,\Sigma_2), \;\; A \text{ invertible}$$

$$\mathbf{K}(\Sigma, I) \approx \|\Sigma - I\|_F^2 \approx \|\Sigma^{-1} - I\|_F^2$$

Criteria

$$C_1(\mathbf{F}) \;=\; \mathbf{K}(\mathbf{F}, I_q) \;=\; \tfrac{1}{2}\left(\operatorname{tr}\mathbf{F} - \ln|\mathbf{F}| - q\right)$$

$$C_2(\mathbf{F}) \;=\; \min_{\delta>0}\, \mathbf{K}(\mathbf{F}, \delta\, I_q) \;=\; \tfrac{1}{2} \ln \frac{\left(\frac{\operatorname{tr}\mathbf{F}}{q}\right)^q}{|\mathbf{F}|} = \max_{AA^T=I} C_0\left(A\, \mathbf{F}\, A^T\right)$$

$$C_3(\mathbf{F}) \;=\; \min_{\Delta_q>0}\, \mathbf{K}\left(\mathbf{F}, \Delta_q\right) \;=\; -\tfrac{1}{2}\, \ln \frac{|\mathbf{F}|}{\prod_{i=1}^q \mathbf{F}_{ii}} = C_0(\mathbf{F})$$

Solution (Lagrangian)

$$\left(\mathcal{G}_{s,\ell}^*\, \Sigma_\ell\, \mathcal{G}_{s,\ell}^{*T}\right)^{-1}\, \mathcal{G}_{s,\ell}^*\, \mathcal{M}_\ell\, \frac{\partial C_i}{\partial \mathbf{F}}\, \mathcal{M}_\ell^T\, \left(I_{\ell r} - \mathcal{G}_{s,\ell}^{*T}\, \left(\mathcal{G}_{s,\ell}^*\, \Sigma_\ell\, \mathcal{G}_{s,\ell}^{*T}\right)^{-1}\, \mathcal{G}_{s,\ell}^*\, \Sigma_\ell\right) - \Lambda\, \mathcal{O}_\ell^T = 0$$

$$\frac{\partial C_1}{\partial \mathbf{F}} = \frac{1}{2}\, \left(I_q - \mathbf{F}^{-1}\right); \quad \frac{\partial C_2}{\partial \mathbf{F}} = \frac{1}{2}\, \left(\frac{q}{\operatorname{tr}\mathbf{F}}\, I_q - \mathbf{F}^{-1}\right); \quad \frac{\partial C_3}{\partial \mathbf{F}} = \frac{1}{2}\, \left(\sum_i \frac{J_i}{\mathbf{F}_{ii}} - \mathbf{F}^{-1}\right)$$

Perspectives

- Effective residual generation!
- Optimal sensor location for monitoring (isolation)
- Fault detectability?