Topics On Discretizations of Stochastic Differential Equations
Notes
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1 Basics on Probability

Random variables always take value in Polish (metric complete separable) topological spaces. A random pair (say) is the class of measurable functions

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X,Y} E \times F$$

defined up to events of probability 0. The law of (X, Y) is a probability distribution on $E \times F$ (a Borel measure ≥ 0 of total integral 1); and we denote:

$$\mathrm{Law}(X, Y) \in \mathrm{Proba}(E \times F).$$

Using descrpitive set theory, it is possible to check that there always exists a (non-unique) mesurable function $F : E \times A \to F$ and a randome variable $U \in A$ independent of X such that:

$$Y = F(X, U).$$

It is then easy to condition Y with respect to X for instance one can define rigorously:

$$\mathrm{Law}(Y \mid X) \stackrel{\mathrm{def}}{=} p(X, \mathrm{d}y) \stackrel{\mathrm{def}}{=} \mathrm{Law}(F(x, U)) \,|_{x=X}$$

p is called a probability transition kernel and is a measurable map from E towards $\mathrm{Proba}(F)$:

$$p : E \to \mathrm{Proba}(F).$$

p exists but is *not unique*, however the random distribution $p(X, \mathrm{d}y)$ is uniquely defined.

Our main example is the following: consider a continuous stochastoc process, that is a random variable denoted

$$(X_t)_{t\in[0,T]} \in C([0, T], \mathbb{R}^d),$$

where $C([0, T], \mathbb{R}^d)$ is endowed with uniform convergence (making it a separable Banach space). Then one can easily consider:

$$\mathrm{Law}\left((X_s)_{s\in[t,T]} \mid (X_s)_{s\in[t,T]}\right)$$

Note that if $(X_t)_{t\in[0,T]}$ is *cadlag (left limit and right continuous)* there is a Polish topology (called Skorohkod topology) that extends the uniform convergence. There is thus *no measurability problem* in that case.

2 Processes and Stochastic calculus

See also blog Lowther

(Time) Filtration When dealing with processes one consider a so-called filtration defined by *all the events than can be generated by a* random seed *before time* t. Think about a random number generator in simulation in the case t is discrete.

One usually then only considers *adapted processes.* X is adapted if and only if it exists a measurable function F such that:

$$X_t = F((\mathrm{Seed}_s)_{s\in[0,t]}),$$

where Seed_t encodes all the random variables before time t. The events that can be generated by this mean are denoted $\mathcal{F}_t$. We will thus use the standard σ-algebra notation

$$\mathbb{E}\left[\quad \mid \mathcal{F}_t\right] \overset{\mathrm{def}}{=} \mathbb{E}\left[\quad \mid (\mathrm{Seed}_s)_{s\in[0,t]}\right].$$

For example 'strong' solutions of SDEs can be defined using a Brownian motion as a 'seed' in which case:

$$(\mathrm{Seed}_s)_{s\in[0,t]} = (W_s)_{s\in[0,t]}$$

but often one need to *add some indepedent random variables used to simulate/construct more complex models.*

L^p **Banach space of processes** If X is continuous it is useful to consider the separable Banach (hence complete) space $\mathbb{L}^p(C([0,T]))$ of random processes defined by the norm:

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right]^{1/p}$$

In what follows we will use *Cauchy sequence arguments* in this Banach space to define processes.

In particular, most stochastic processes can be:

- Defined in discrete time using a discrete (dyadic) time ladder.
- Extended in continuous time as a Cauchy sequence in the above Banach space.

Processes constructed as a Cauchy sequence are *naturally adapted.*

We will also use the fact that if X is in $\mathbb{L}^p(C([0,T]))$ then its random modulus of uniform continuity $\Gamma(\delta t)$ (Heine theorem) satisfies by dominated convergence:

$$\lim_{\delta t\to 0} \mathbb{E}[\Gamma(\delta t)^p] = 0.$$

Martingales The most important type of processes are *martingales.*

Definition 2.0.1. *A martingale M in $\mathbb{R}^d$ (with respect to some filtration) is an integrable process*

$$\mathbb{E}\,|M_t| < +\infty$$

such that its future average variation conditioned by the 'present' is 0

$$\mathbb{E}\left[M_{t+h} \mid \mathcal{F}_t\right] = M_t \qquad \forall t, h \geq 0.$$

Example 2.0.2. Random Walk in discrete time.

Brownian motion

Lemma 2.0.3. *W is said to be an adapted Brownian motion iff W is a continuous martingale with*

$$\mathrm{Law}(W_{t+h} - W_t \mid \mathcal{F}_t) = \mathcal{N}(0, h).$$

Note that this implies that increments are stationary and independent from the past $(W_{t+h} - W_t)_{h \geq 0}$ is independent from $\mathcal{F}_t$ for all t.

Remark 2.0.4. Using the central limit theorem, it is possible to show that Brownian motion is the unique distribution on $C(\mathbb{R}, \mathbb{R})$ which is, for its own filtration, i) a continuous martingale, ii) has independent increments and iii) variance 1 at time $t = 1$.

There are various ways to show existence of Brownian motion, but the most constructive, simulation-based one is due to Levy and work as follows:

- construct BM skeleton for time ladder $t_i^{(k)} = i\, 2^{-k}$, $i \in \mathbb{N}$.
- construct BM for time ladder $t_i^{(k+1)} = i 2^{-(k+1)}$, $i \in \mathbb{N}$ by using the *conditioning formula for Gaussian vectors.*
- Use a Cauchy sequence argument ot define the limit in $\mathbb{L}^2(C([0, T], \mathbb{R}))$ (as below). It requires an rough estimation of the supremum of a discrete Brownian bridge (a Brownian random walk conditioned by its initial and final values.

Doob's optional stopping theorem A fundamental property is that the martingale property still holds for so-called bounded 'stopping time'.

Lemma 2.0.5. *Let τ be a random time defined by (C is a closed set and X is a continuous adapted process) the first hitting time after some t and before some $T > t$:*

$$\tau \stackrel{\text{def}}{=} \inf(s \in [t, T] : \, X_s \in C).$$

Then

$$\mathbb{E}\left[M_\tau \mid \mathcal{F}_t\right] = M_t.$$

Maximal inequality The most important inequalities related to martingales is the so-called *maximal inequalities*, which state that maximum of martingales over time is controlled by final value. One version of which is:

Lemma 2.0.6.

$$\mathbb{E}\left[\sup_{s \in [0,t]} |M_s - M_0|^2\right] \leq 4\mathbb{E}\left[|M_t - M_0|^2\right]$$

Sketch of proof. Case $M_0 = 0$ and M continuous. Define the stopping time

$$\tau = \inf(t \in [0, T] : \, |M_t| \geq a).$$

Prove:

$$\mathbb{P}\left(\sup_{t \in [0,T]} |M_t| \geq a\right) \leq \frac{1}{a} \mathbb{E}[|M_T| \, \mathbb{1}_{\sup_{t \in [0,T]} |M_t| < a}].$$

USe that for any r.v. by IbP:

$$\mathbb{E}[|X|^2] = \int_0^\infty 2a\mathbb{P}(|X| \geq a)da$$

□

The above lemma has following nice consequence, which enables to construct easily Cauchy sequences of martingales as processes:

Corollary 2.0.7. *The L^2 norm of time supremum of martingales is equivalent to the L^2 norm of their final value.*

Stochastic integral If Z is an adapted process which is at least left continuous, and say bounded, and M an adapted martingale, it is possible to construct now the integral

$$\int_0^t Z_{s^-} dM_s = \mathbb{P} - \lim_{\delta t \to 0} \sum_{t_i \leq t} Z_{t_{i-1}} \left(M_{t_i} - M_{t_{i-1}} \right)$$

and show that the latter is a martingale.

Lemma 2.0.8. *Assume Z is piecewise continuous and the bound below is finite with $1/p + 1/q = 1$ (this condition is unecessary in the general theory), then $\int_0^t Z_{s^-} dM_s$ exists and:*

$$\mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t Z_{s^-} dM_s \right|^2\right] \leq 4\mathbb{E}\left[\sup_{t \leq T} |Z_t|^{2p}\right]^{1/p} \mathbb{E}\left[\sup_{t \leq T} |M_T - M_0|^{2q}\right]^{1/q}$$

Sketch of proof. If Z is piecewise constant the intergal $\int ZdM$ is again a martingale. One can then use the maximal martingale inequality and Hölder inequality to obtain the bound. When Z is note piecewise constant, one can construct a Cauchy sequence in the L^2 Banach space of continuous processes using piecewise constant approximations of Z; and the result follows. □

Quadratic variation

Lemma 2.0.9. *Let $\{t_i\}$ be a time discretization of $[0, t]$ with $t_{i+1} - t_i \leq \delta t$ and M a square integrable martingale. There exists an increasing (adapted) process $[M, M]_{0,t}$ called the quadratic variation defined by*

$$\mathbb{P} - \lim_{\delta t \to 0} \sum_i (M_{t_{i+1}} - M_{t_i})^2 = [M, M]_{0,t}.$$

Moreover:

- $\mathbb{E}([M, M]_{0,t}) = \mathbb{E}((M_t - M_0)^2) = \mathbb{V}\mathrm{ar}(M_t \mid \mathcal{F}_0)$
- *$[M, M]_{0,t}$ is continuous if M is continuous.*
- *$[W, W]_{0,t} = t$ if W is Brownian motion.*

Sketch of proof. Martingale property:

$$\mathbb{E}\left[(M_{t_{i+1}} - M_{t_i})^2\right] = \mathbb{E}(M_{t_{i+1}}^2) - \mathbb{E}(M_{t_i}^2) = \mathbb{V}\mathrm{ar}(M_{t_{i+1}} - M_{t_i}).$$

For continuity properties, remark that square integrable martingales are uniformly integrable by maximal martingale inequality and one can exchange $\mathbb{E}$ and time limits.

Now we assume for simplicity that M is continuous and $[M, M]_{0,t}$ is also square integrable (order 4 moment). Then the difference of two dyadic discrete approximations of $[M, M]_{0,t}$ is martingale ! Using maximal inequality this defines Cauchy sequences in the $\mathbb{L}^2(C([0,T), \mathbb{P})$ space of random processes.

□

Lemma 2.0.10. *Let Z be left continuous adapted and M be a continuous martingale.*

- $\int_0^t Z_{s^-} dM_s$ *is a continuous martingale.*
- *Moreover*

$$\left[\int_0 Z_{s^-} dM_s, \int_0 Z_{s^-} dM_s\right]_{0,t} = \int_0^t Z_{s^-}^2 d[M, M]_{0,t}$$

Sketch of proof. If Z is piecewise continuous on $[t_i, t_{i+1}]$, we trivially get that

$$\left(\int_{t_i}^{t_{i+1}} Z_s dM_s\right)^2 = Z_{t_i}^2 (M_{t_{i+1}} - M_{t_i})^2$$

which yields the result if $t_{i+1} - t_i \leq \delta t \to 0$.

When Z is note piecewise constant, and under some additional moment condition above, use a Cauchy sequence in the L^2 Banach space of continuous processes using piecewise constant approximations of Z. □

Semi-martingales and Itô formula

Definition 2.0.11 (Semi-martingale, informal)**.** *An adapted stochastic process that can be decomposed in the form:*

$$X_t = I_t + D_t + M_t$$

where I is increasing, D is decreasing, and M is a martingale is called a semi-martingale. Stochastic intrgation with respect to semi-martingales make sense. The quadratic variation of a semi-martingale satifies: $[X, X] = [M, M]$.

Example 2.0.12. if X is solution a SDE: $dX_t = b(X_t) + a(X_t)dW_t$ aénd φ is C^2, then $\varphi(X_t)$ is a semi-martingale. See below.

We can then state and formally prove Itô formula:

Lemma 2.0.13. *Let X be a* continuous *semi-martingale, and φ be C^2 then $\varphi(X_t)$ is a semi-martingale and in the sense of stochastic integration:*

$$d\varphi(X_t) = \partial_{x^i}\varphi(X_t)dX_t^i + \frac{1}{2}\partial_{x^i}\partial_{x^j}\varphi(X_t)d[X^i, X^j]_{0,t}$$

Sketch of proof. Dimension 1 and φ smooth with bounded derivatives, and $X, [X,X]$ have finite higher moments. On a time ladder with a Taylor expansion:

$$\varphi(X_{t_{i+1}}) = \varphi(X_{t_i}) + \varphi'(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \frac{1}{2}\varphi''(X_{t_i})(X_{t_{i+1}} - X_{t_i})^2 + O(\left|X_{t_{i+1}} - X_{t_i}\right|^3).$$

But by uniform continuity

$$\overset{\mathbb{L}^p}{\lim_{\delta t \to 0}} \sup_i \left|X_{t_{i+1}} - X_{t_i}\right| = 0.$$

Using e.g. Holder inequality and the definition of the quadratic variation $[X,X]$, we get the result. □

3 Strong solutions and approximations of SDEs

If F is measurbale function and U, a given random variable, a *strong* solution X of the equation

$$F(X, U) = 0$$

is a random variable that satisfies $X = f(U)$ (a.s.) for some measurable function f.

In the same way, one can consider the SDE

$$dX_t = b(X_t)dt + a(X_t)dW_t$$

and look for possibly unique strong solutions of the form

$$X_t = F((W_s)_{0 \le s \le t})$$

where F is Borel measurable as a function of continuous paths endowed with uniform convergence.

One may then look for discretizations *in the strong sense*, that is defined as functions of the underlying BM:

$$X_t^{\delta t} = f_{\delta t}((W_s)_{0 \le s \le t}).$$

Quite obviously, a straitforward solution is given by the so-called explicit Euler(-Maruyama) scheme:

$$dX_t^{\delta t} = b(X^{\delta t}_{\delta t \lfloor t/\delta t \rfloor})dt + a(X^{\delta t}_{\delta t \lfloor t/\delta t \rfloor})dW_t.$$

3.1 The Cauchy theory with the explicit Euler scheme

Using the *martingale maximal inequality* and a Gronwall inequality, one cane obtain a full Cauchy theory if a,b are globally Lipschitz.

Lemma 3.1.1. *Assume $\delta t/\delta t'$ is a positive integer. Then:*

$$\mathbb{E}\left[\sup_{s \le t} \left|X_s^{\delta t} - X_s^{\delta t'}\right|^2\right] \le (t\mathrm{Lip}(b)^2 + 4\mathrm{Lip}(a)^2)\mathbb{E}\int_0^t \left|X^{\delta t}_{\delta t \lfloor s/\delta t \rfloor} - X^{\delta t'}_{\delta t' \lfloor s/\delta t' \rfloor}\right|^2 ds$$

ONe can then control the difference between to schemes:

Lemma 3.1.2. $s \leq t$.

$$\mathbb{E}\left[\left|X^{\delta t}_{\delta t \lfloor s/\delta t \rfloor} - X^{\delta t'}_{\delta t' \lfloor s/\delta t' \rfloor}\right|^2\right] \leq 2\mathbb{E}\left[\sup_{s \leq t}\left|X^{\delta t}_s - X^{\delta t'}_s\right|^2\right] + 4\left\|b\right\|^2_\infty \delta t^2 + 4\left\|a\right\|^2_\infty \delta t$$

Using a Gronwall argument we thus get:

Proposition 3.1.3. *Assume $\delta t/\delta t'$ is a positive integer, and a, b are bounded and Lipschitz.*

$$\mathbb{E}\left[\sup_{s \leq t}\left|X^{\delta t}_s - X^{\delta t'}_s\right|^2\right] \leq C_T \delta t.$$

The SDE is well-posed in the strong sense, the explicit Euler scheme is strongly convergent, of strong order $1/2$.

Proof. Again use a Cauchy sequence argument. □

3.2 Stochastic Taylor expansions

One can obtain the usual Taylor expansion with integral rest term by iteratively using the substitution:

$$f(t) = f(t_0) + \int_{t_0}^t f'(t)dt.$$

If $F(t)$ is a continuous stochastic process adapted to a Brownian motion W, there is a chance that there exists adapted stochastic processes A_t, B_t such that

$$F(t) = F(t_0) + \int_{t_0}^t A^{(0)}_t dt + A^{(1)}_t dW_t.$$

If $A^{(0)}_t$ and $A^{(1)}_t$ are again semi-martingales, you can iterate the decomposition and obtain *stochastic Taylor decompositions* similar to usual Taylor expansions.

3.3 Stochastic Taylor schemes

Remember that if $dX_t = b(X_t)dt + a(X_t)dW_t$ is the solution of an SDE and $f \in C^2(\mathbb{R})$ a smooth funciont by Itô formula

$$df(X_t) = L[f](X_t)dt + a'(X_t)dW_t$$

where $L = b(x)\frac{d}{dx} + \frac{a^2(x)}{2}\frac{d^2}{dx^2}$ is the second order differential operator associated with X (the so-called 'Markov generator').

We can now consider the SDE as the first iteration of a Taylor expansion:

$$X_t = X_{t_0} + \underbrace{\int_{t_0}^t b(X_{t_1})dt_1}_{\stackrel{\text{def}}{=} I_{(0)}[b(X)]_{t_0,t}} + \underbrace{\int_{t_0}^t a(X_{t_1})dW_{t_1}}_{\stackrel{\text{def}}{=} I_{(1)}[a(X)]_{t_0,t}}$$

We iterate the stochastic Tayor expansion to obtain the explicit Euler-Maruyama schme with a rest term:

$$
\begin{aligned}
X_t = & X_{t_0} + b(X_{t_0}) \int_{t_0}^{t} dt_1 + a(X_{t_0}) \int_{t_0}^{t} dW_{t_1} \\
& + I_{(0,0)}[L(b)(X)]_{t_0,t} + I_{(1,0)}[b'\, a(X)]_{t_0,t} \\
& + I_{(0,1)}[L(a)(X)]_{t_0,t} + I_{(1,1)}[a'\, a(X)]_{t_0,t}
\end{aligned}
$$

where we have used the notation:

$$
\begin{aligned}
I_{(0,0)}[\varphi(X)]_{t_0,t} & \stackrel{\text{def}}{=} \int_{t_0}^{t} \int_{t_0}^{t_1} \varphi(X_{t_2}) dt_2 dt_1 \\
I_{(1,0)}[\varphi(X)]_{t_0,t} & \stackrel{\text{def}}{=} \int_{t_0}^{t} \int_{t_0}^{t_1} \varphi(X_{t_2}) dW_{t_2} dt_1 \\
I_{(0,1)}[\varphi(X)]_{t_0,t} & \stackrel{\text{def}}{=} \int_{t_0}^{t} \int_{t_0}^{t_1} \varphi(X_{t_2}) dt_2 dW_{t_1} \\
\ldots
\end{aligned}
$$

Analysis of higher order terms One may *not need all of the above terms* to construct higher order schemes. For simplicity we take

$$
\phi = 1,
$$

but the general case is similar.

First note that

$$
\mathbb{E}\left[\left(\int_0^{\delta t} dt\right)^2\right] = \delta t^2
$$

as well as

$$
\mathbb{E}\left[\left(\int_0^{\delta t} dW_t\right)^2\right] = \delta t
$$

.

Moreover:

$$
\mathbb{E}\left[\left(I_{(1,1)}[1]_{0,\delta t}\right)^2\right] = \mathbb{E}\left[\left(\int_0^{\delta t} W_{t_1} dW_{t_1}\right)^2\right] = \int_0^{\delta t} t_1 dt_1 = \delta t^2/2,
$$

the other satisfying:

$$
\begin{aligned}
\mathbb{E}\left[\left(I_{(0,0)}[1]_{0,\delta t}\right)^2\right] & = O(\delta t^4) \\
\mathbb{E}\left[\left(I_{(0,1)}[1]_{0,\delta t}\right)^2\right] & = O(\delta t^3) \\
\mathbb{E}\left[\left(I_{(1,0)}[1]_{0,\delta t}\right)^2\right] & = O(\delta t^3)
\end{aligned}
$$

Note that by induction one could check that:

$$\mathbb{E}\left[(I_\alpha[1]_{0,\delta t})^2\right] = O(\delta t^{l(\alpha)+n(\alpha)})$$

where $l(\alpha)$ is the size of the list α (list of 0 or 1), and $n(\alpha)$ the number of zeros.

As a consequence on a discretized full time interval $[0,t]$, on will get $O(1/\delta t)$ many rest terms associated with a square error of order $O(\delta t^q)$.

The cases $n(\alpha) \neq l(\alpha)$ (that is $(0,1)$ and $(1,0)$ and $(1,1)$) are however special because one could check that they satisfy the centering condition :

$$n(\alpha) \neq l(\alpha) \Rightarrow \mathbb{E}\left[I_\alpha[\varphi(X)]_{0,\delta t}\right] = 0$$

which implies that the time discrete process $k \mapsto \mathcal{M}_k^{\delta t}(\varphi)$ defined by

$$\mathcal{M}_{k+1}^{\delta t}(\varphi(X)) - \mathcal{M}_k^{\delta t}(\varphi(X)) \stackrel{\text{def}}{=} I_\alpha[\varphi(X)]_{k\delta t,(k+1)\delta t}$$

is a martingale if X is an adapted process. As a consequence the *square* of the $O(1/\delta t)$ many rest terms will be of order, on average, again $O(1/\delta t)$. This sums up to:

$$\begin{aligned}
&\alpha = (0) \Rightarrow O(1/\delta t^2) \times O(\delta t^2) \quad \text{order } 0\\
&\alpha = (1) \Rightarrow O(1/\delta t) \times O(\delta t) \quad \text{order } 0\\
&\alpha = (0,0) \Rightarrow O(1/\delta t^2) \times O(\delta t^4) \quad \text{square error}\\
&\alpha = (0,1) \Rightarrow O(1/\delta t) \times O(\delta t^3) \quad \text{square error}\\
&\alpha = (1,0) \Rightarrow O(1/\delta t) \times O(\delta t^3) \quad \text{square error}\\
&\alpha = (1,1) \Rightarrow \boxed{O(1/\delta t) \times O(\delta t^2)} \quad \text{square error}
\end{aligned}$$

The Milstein scheme The noise term associated with $\alpha = (1,1)$ can be modified using Itô formula:

$$W_{t_1} dW_{t_1} = \frac{1}{2} d\left[(W_{t_1})^2\right] - \frac{1}{2} dt_1$$

Definition 3.3.1 (Milstein scheme)**.** *The Milstein scheme is defined in dimension* 1 *by*

$$dX_t^{\delta t} = b(X^{\delta t}_{\delta t\lfloor t/\delta t\rfloor})dt + a(X^{\delta t}_{\delta t\lfloor t/\delta t\rfloor})dW_t + a'\, a(X^{\delta t}_{\delta t\lfloor t/\delta t\rfloor})\left(\frac{1}{2} d\left[(W_t)^2\right] - \frac{1}{2} dt\right).$$

Proposition 3.3.2. *Assume b, a, a' are bounded Lipschitz. Then the Milstein scheme is convergent of order* 1*:*

$$\mathbb{E}\left[\sup_{s\leq t} \left|X_s^{\delta t} - X_s^{\delta t'}\right|^2\right] \leq C_T \delta t^2.$$

Higher order schemes can be obtained by iterating the process. WARNING: quite intricate in higher dimension due to the required simulation of Levy area.

4 The weak approach

In this part I will avoid technicalities and be more informal.

There are two reasons one may want to study SDE schemes with a 'weak' approach:

- The SDE itself is not well-posed. This happens for instance when the coefficients a, b are no longer Lipschitz, but, for instance only continuous.
- One may want to obtain controls on large times. Indeed with the Gronwall technique of the last section, the constant C_T that encodes the discretization error typically become exponentially large with T. This is related with the problem of 'stability'.

5 Weak SDEs, Markov semi-groups, parabolic PDEs

If one seeks a solution X of

$$F(X, U) = 0$$

with U given. In most cases (e.g. our SDE) one has rather:

$$U = fn(X).$$

One may say that the solution X is 'weak' if it contains more random variables than just U. The associated notion of uniqueness is then uniqueness in law:

$$\exists!\, \mathrm{Law}(X) : \quad F(X, U) = 0.$$

For SDE this distribution is given by a *Markov semi-group solution of a parabolic PDE* !

Indeed if a weak solution of the SDE

$$dX_t = b(X_t) + a(X_t)dW_t$$

is unique in distribution, it implies the Markov property

$$\mathrm{Law}((X_s)_{s\geq t} \mid \mathcal{F}_t) = \mathrm{Law}((X_s)_{s\geq t} \mid X_t)$$

where in the above we consider the unique solution for $s \geq t$ with initial condition X_t. A (technical) theorem by Strook-Varhadan ensures the latter is well-posed:

Theorem 5.0.1. *Assume a, b are bounded continuous with $a \geq \delta > 0$. Then there exists a unique weak solution of the SDE.*

One can then define the semi-group P_t as:

$$P_t(\varphi)(x) = \mathbb{E}[\varphi(X_t) \mid X_0 = x].$$

Using Itô formula one has:

$$d\varphi(X_t) = L\varphi(X_t)dt + \varphi'(X_t)dW_t.$$

with

$$L\varphi = b\varphi' + \frac{1}{2}a\varphi''.$$

As a consequence the semi-group P_t is solution to the parabolic PDE:

$$\partial_t P_t = LP_t$$

Euler scheme The *formal* weak order of the Euler scheme cha thus be computed easily: if φ is C^4 then:

$$\mathbb{E}[\varphi(X_{\delta t}^{\delta t}) \mid X_0^{\delta t} = x] = L(\varphi)(x) + O(\delta t).$$

Obtaining weak convergence orders is nonetheless much mor difficult. We can state for example Theorem 14.1.5 of Kloeden and Platen [2013].

Theorem 5.0.2. *Assume a, b are l-Hölder bounded with $a \geq \delta > 0$. Let φ be smooth with bounded derivatives. Then the explicit Euler scheme satisfies:*

$$\left| \mathbb{E}[\varphi(X_t^{\delta t}) \mid X_0^{\delta t} = x] - \mathbb{E}[\varphi(X_t) \mid X_0 = x] \right| \leq C_t \delta t^{l/2}$$

6 Stability

The prototypical case of instability happens for the Explicit Euler scheme in the cas $b(x) = -cx$ is linear with $c > 0$ and $a = 1$:

$$X_{k+1} = X_k - cX_k\delta t + \sqrt{\delta t} N_{k+1}$$

Then:

$$\mathrm{Var}(X_{k+1}) = (1 - c\delta t)^2 \mathrm{Var}(X_k) + \delta t$$

We see that the scheme is unstable in the sense that

$$\lim_{k \to +\infty} \mathrm{Var}(X_k) = +\infty$$

iff:

$$c\delta t \geq 2.$$

This notion can be extended with the concept of Lyapounov stability.

Definition 6.0.1. *Let $L \geq 0$ be function with compact level sets. L is said to be a Lyapounov function for a Markov chain X_n iff there exists $\gamma < 1$ and c such that:*

$$\mathbb{E}[L(X_{n+1}) \mid X_n = x] \leq \gamma L(x) + c$$

By a Gronwall argument it yields:

$$\mathbb{E}[L(X_n)] \leq \mathbb{E}[L(X_0)]\gamma^n + c/(1-\gamma)$$

In particular the set $(\mathrm{Law}(X_n))_n$ is tight and up to extraction $\lim_n \mathrm{Law}(X_n)$ exists.

It is then tempting to think that if i) a scheme is stable and ii) it is weakly consistent with some given order, then then the distribution of the scheme even on large time, will converge to the solution of the SDE for smooth observables for the same order. *Not so simple there is no general result* ! More regularity on the coefficients are required see Talay-Tubaro theoy for a classical result.

7 Invariant probabilities and Metropolis

Consider the SDE:

$$dX_t = -\nabla V(X_t)dt + \sqrt{2}dW_t$$

The latter has the following structure:

Lemma 7.0.1. *Denote*

$$\mu(dx) = \mathrm{e}^{-V(x)}dx / \int \mathrm{e}^{-V(x')}dx'$$

and assume it is a probability. μ is a stationary probability of the SDE and the latter is time reversible when stationary:

$$\mathrm{Law}(X_0) = \mu \Rightarrow \mathrm{Law}(X_s, s \in [0,T]) = \mathrm{Law}(X_{T-s}, s \in [0,T])$$

Sketch of proof. The semi-group is symmetric, because the generator is. in fact the generator is self-adjoint ! This implies time reversibility.

□

Can we find a numerical scheme with the same property ? Yes, with the addition of a Metropolis rule !

Proposition 7.0.2 (Metropolis)**.** *Let X_n be a Markov chain with probability transition*

$$p(x, x')dx'.$$

Construct a modified Markov chain $\tilde{X}_n$ as follows:

- *Propose a new state $\tilde{X}_n'$ with $p(\tilde{X}_n, x')dx'$.*
- *With probability*

$$\min\left(1, \frac{p(\tilde{X}_n', \tilde{X}_n)\mathrm{e}^{-V(\tilde{X}_n')}}{p(\tilde{X}_n, \tilde{X}_n')\mathrm{e}^{-V(\tilde{X}_n)}}\right)$$

  accept and set $\tilde{X}_{n+1} = \tilde{X}_n'$. Other wise reject and set $\tilde{X}_{n+1} = \tilde{X}_n$.

The probability μ is invariant for the chain $\tilde{X}_n$ and the latter is reversible when stationary.

As a consequence, one can add to the Euler scheme (the proposal $p(x, x')dx'$ is Gaussian and explicitly computable) a Metropolis accept reject rule (the scheme is called Metropolis Adjusted Langevin Algorithm).

8 Hamiltonian Monte Carlo if time !!

References

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