Techniques and tOols for Programming (TOP)

Martin Quinson <martin.quinson@loria.fr>

Telecom Nancy – 1re année

2014-2015

About this document

License of this document:

Technical aspects

- LaTeX document (class latex-beamer), compiled with latex-make
- Figures: Some xfig, some tikz, some inkscape

URL: `http://www.loria.fr/~quinson/Teaching/TOP/`

- Sheets, Practicals, exams, projects (sources available to teachers that ask me)

About me

- Since Feb. 2005: Associate Professor at Université de Lorraine)
  Teaching: Télécom Nancy, Research: AlGorille team (LORIA = UL/INRIA/CNRS)

Research: Experimental methodologies

- Assess distributed applications (perfs, bugs)
- SimGrid project: Simulator of distributed systems
  Correct modeling, efficient simulation
- Formal verification (model-checking)

Teachings: Programming and Algorithms

- Introduction, Java/Scala, AlgoProg, C 2nd language
- System Prog; Ex-{Algo dist, P2P, Distributed Prog}
- PLM: Programmer's Learning Machine

Outreach: Unplugged Computer Science, etc.

- More info: `http://www.loria.fr/~quinson/` (`Martin.Quinson@loria.fr`)

About this module: Algorithmic and Programming

Programming? Let the computer do your work!

- How to explain what to do?
- How to make sure that it does what it is supposed to? That it is efficient?
- What if it does not?

Module content and goals:

- Introduction to Algorithmic
  - Master theoretical basements (computer science is a science)
  - Know some classical problem resolution techniques
  - Know how to evaluate solutions (correctness, performance)
- First steps in programming: learn-by-doing activity (you need to *practice*)

Other modules at Telecom Nancy

- Prerequisites
  - Tactical programming in Scala: `if`, `for`, methods
  - Sense of logic, intuition (good math background helps)
- Afterward: Object Oriented Programming; Object-Oriented Design

(please be patient, it's our second time with TOP before OOP)

Module organization

Time organization

- 6 two-hours lectures (CM, with Martin Quinson): Concepts introduction
- 10 two-hours exercise session (TD, with staff member[1]): Theoretical exercises
- 6 two-hours labs (TP, with staff member[1]): Coding exercises
- Homework: Systematically finish the in-class exercises

Evaluation

- Two hours table exam (closed book, only one sheet of notes allowed)
- Maybe some quiz at the beginning of labs

[1] Olivier Festor, Sébastien Da Silva, Martin Quinson.

Syllabus

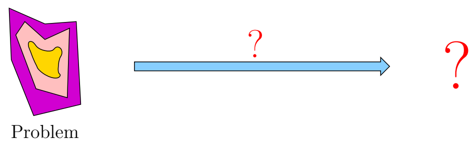
1. Practical and Theoretical Foundations of Programming
   - CS vs. SE; Abstraction for complex algorithms; Algorithmic efficiency.
2. Iterative Sorting Algorithms
   - Specification; Selection, Insertion and Bubble sorts.
3. Recursion
   - Principles; Practice; Recursive sorts; Non-recursive From; Backtracking.
4. Dynamic Programming
   - Introduction; Greedy algorithms, Dynamic Programming.
5. Software Correction
   - Introduction; Specifying Systems; Hoare Logic; Proving Recursive Functions.

This may change a bit to adapt and improve the class

First Chapter

Practical and Theoretical Foundations of Programming

Problems



Provided by clients (or teachers ;)

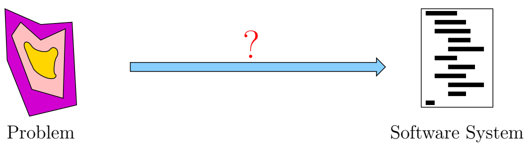
Problems

- Problems are generic
  Example: Determine the minimal value of a set of integers

Instances of a problem

- The problem for a given data set
  Example: Determine the minimal value of {17, 6, 42, 24}

Problems and Programs



Software systems (*ie.*, Programs)

- Describes a set of actions to be achieved in a given order
- Doable (tractable) by computers

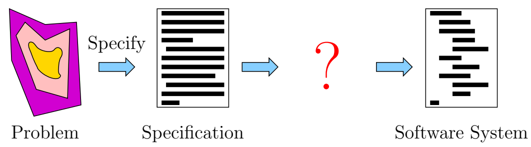
Problem Specification

- Must be clear, precise, complete, without ambiguities
  Bad example: find position of minimal element (two answers for {4, 2, 5, 2, 42})
  Good example: Let *L* be the set of positions for which the value is minimal.
  Find the minimum of *L*

Using the Right Models

- Need simple models to understand complex artifacts (ex: city map)

Methodological Principles



Abstraction think before coding (!)

- Describe how to solve the problem

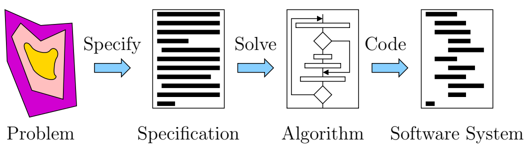
Divide, Conquer and Glue (top-down approach)

- Divide complex problem into simpler sub-problems (think of Descartes)
- Conquer each of them
- Glue (combine) partial solutions into the big one

Modularity

- Large systems built of components: **modules**
- Interface between modules allow to mix and match them

Algorithms



Precise description of the resolution process of a well specified problem

- Must be understandable (by human beings)
- Does not depend on target programming language, compiler or machine
- Can be an diagram (as pictured), but difficult for large problems
- Can be written in a simple language (called **pseudo-code**)

"Formal" definition

- Sequence of actions acting on problem data to induce the expected result

New to Algorithms?

Not quite, you use them since a long time

Lego bricks™	list of pictures →	Castle
Ikea™ desk	building instructions →	Desk
Home location	driving directions →	Party location
Eggs, Wheal, Milk	recipe →	Cake
Two 6-digits integers	arithmetic know-how →	sum

And now

List of students	sorting algorithm →	Sorted list
Maze map	appropriated algorithm →	Way out

Computer Science vs. Software Engineering

> *Computer science is a science of abstraction – creating the right model for a problem and devising the appropriate mechanizable technique to solve it.* – Aho and Ullman

NOT (only) Science of Computers

> *Computer science is not more related to computers than Astronomy to telescopes.* – Dijkstra

- Many concepts were framed and studied before the electronic computer
- To the logicians of the 20's, a *computer* was a person with pencil and paper

Science of Computing

- Automated problem solving
- Automated systems that produce solutions
- Methods to develop solution strategies for these systems
- Application areas for automatic problem solving

Foundations of Computing

Fundamental mathematical and logical structures

- To understand computing
- To analyze and verify the correctness of software and hardware

Main issues of interest in Computer Science

- Calculability
  - Given a problem, can we show whether there exist an algorithm solving it?
  - Which are the problems for which no algorithm exist? How to categorize them?
- Complexity
  - How long does my algorithm need to answer? (as function of input size)
  - How much memory does it take?
  - Is my algorithm optimal, or does a better one exist?
- Correctness
  - Can we be certain that a given algorithm always reaches a solution?
  - Can we be certain that a given algorithm always reaches the right solution?

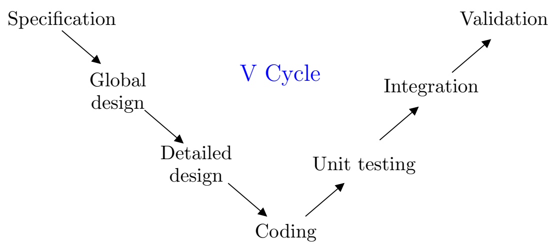
Software Engineering vs. Computer Science

Producing technical answers to consumers' needs

Software Engineering Definition

- Study of methods for producing and evaluating software

Life cycle of a software (*much* more details to come later)



- Global design: Identify application modules
- Detailed design: Specify within modules

As future IT engineers, you need both CS and SE

Without Software Engineering

- Your production will not match consumers' expectation
- You will induce more bugs and problems than solutions
- Each program will be a pain to develop and to maintain for you
- You won't be able to work in teams

Without Computer Science

- Your programs will run slowly, deal only with limited data sizes
- You won't be able to tackle difficult (and thus well paid) issues
- You won't be able to evaluate the difficulty of a task (and thus its price)
- You will reinvent the wheel (badly)

Two approaches of the same issues

- Correctness: CS ⇝ prove algorithms right; SE ⇝ chase (visible) bugs
- Efficiency: CS ⇝ theoretical bounds on performance, optimality proof;
  SE ⇝ optimize execution time and memory usage

First Chapter

Practical and Theoretical Foundations of Programming

There are always several ways to solve a problem

Choice criteria between algorithms

- Correctness: provides the right answer
- Simplicity: KISS! (jargon acronym for *keep it simple, silly*)
- Efficiency: fast, use little memory
- Stability: small change in input does not change output

Real problems ain't easy

- They are not fixed, but dynamic
  - Specification helps users understanding the problem better
    That is why they often add wanted functionalities after specification
  - My text editor is v23.2.1 (hundreds of versions for "just a text editor")
- They are complex (composed of several interacting entities)

> *In computing, turning the obvious into the useful is a living definition of the word "frustration".*
> – "Epigrams in Programming", by Alan J. Perlis.

Dealing with Complexity

Some classical design principles help

- Composition: split problem in simpler sub-problems and compose pieces
- Abstraction: forget about details and focus on important aspects

Object Oriented Programming

- Classical answer to specification complexity and dynamicity Encapsulation, polymorphism, heritage, ...
- That's one way to design applications in a modular manner
- Other approaches exists, but none have the same momentum currently

Rest of this module

- How to write each block / units / objects to be composed in OOP

Why algorithms before OOP and not the contrary?

- Coding at small before programming at large
- (that's an endless debate, pros and cons for both approaches)

Dealing with complexity: Composition

Composite structure

- Definition: a software system composed of manageable pieces
- ☺ The smaller the component, the easier it is to build and understand
- ☹ The more parts, the more possible interactions there are between parts
- ⇒ the more complex the resulting structure
- Need to balance between simplicity and interaction minimization

Good example: audio system

Easy to manage because:

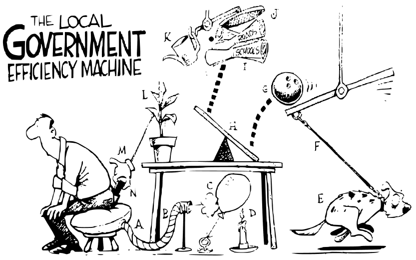
- each component has a carefully specified function
- components are easily integrated
- i.e. the speakers are easily connected to the amplifier

Composition counter-example (1/2)

Rube Goldberg machines

- Device not obvious, modification unthinkable
- Parts lack intrinsic relationship to the solved problem
- Utterly high complexity

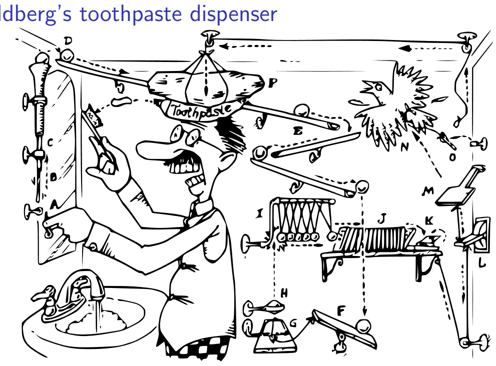
Example: Tax collection machine



A. Taxpayer sits on cushion
B. Forcing air through tube
C. Blowing balloon
D. Into candle
E. Explosion scares dog
F. Which pull leash
G. Dropping ball
H. On teeter totter
I. Launch plans
J. Which tilts lever
K. Then Pitcher
L. Pours water on plant
M. Which grows, pulling chain
N. Hand lifts the wallet

Composition counter-example (2/2)

Rube Goldberg's toothpaste dispenser

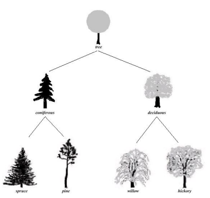


Such over engineered solutions should obviously remain jokes

Dealing with complexity: Abstraction

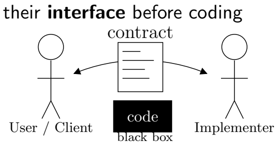
Abstraction

- Dealing with components and interactions without worrying about details
- Not "vague" or "imprecise", but focused on few relevant properties
- Elimination of the irrelevant and amplification of the essential
- Capturing commonality between different things



Abstraction in programming

- Think about what your components should do before
- Ie, abstract their **interface** before coding



- Show your interface, hide your implementation

First Chapter

Practical and Theoretical Foundations of Programming

Scala? Why??

Main reason for me: simplicity

- Most of you are absolute beginners to programming
- I want to talk about algorithms, not to bother you about syntax

Scala has much more to offer

- OOP (mixin+singleton), functional, properties, type inference, JVM-based ...
- But we don't care for now: see it as a simple language
- You'll learn its true beauty later on

Starting Scala

Installation: Get it from `http://scala-lang.org/` (version 2.10 at least)

Executing your code

myfile.scala

```
println("Hello, friends")
```

Run directly

```
$ scala myfile.scala
Hello, friends
$
```

Compile first

```
$ scalac -Xscript toto myfile.scala
$ scala toto
Hello, friends
$
```

myscript

```
#!/usr/bin/scala
!#
println("Hello, friends")
```

Turn it into a script

```
$ chmod +x myscript
$ ./myscript
Hello, friends
$
```

Run interactively

```
$ scala
Welcome to Scala [...]

scala> println("Hello, friends")
Hello, friends

scala> :load myfile.scala
Loading toto.scala...
Hello, friends
```


Getting Started in Scala

Declaring a variable: `var x:Int = 0`

`var` ⇝ because that's a **var**iable
`x` ⇝ name of that variable (its label)
`:Int` ⇝ type of this variable (what it can store)
`= 0` ⇝ initial value (mandatory)

- You can often omit the type (it's inferred): `var x = 0`

Some Scala data types

- Int: for integer values, Double: for dot numbers
- Boolean: `true/false`, String: `"some chars together"`

Declaring a value

- If your "variable" is constant, make it a value: `val answer:Int = 42`
- Seen as good style in Scala *mutable stateful objects are the new spaghetti code*
- Allows to detect errors, may produce faster code, easy multithreading.
- Do values unless you must use variables

The Scala Syntax

Looping

```
while (condition) {
  instructions
}
```

```
do {
  instructions
} while (condition)
```

```
for (i <- 0 to 10 by 2) {
  // i in 0,2,4,6,8,10
}
```

Methods and functions

```
def sayIt(msg:String) {
  print(msg)
}
```

```
def max3(x:Int, y:Int, z:Int):Int = {
  val m = if (x>y) x else y
  if (m>z) {
    return m
  } else {
    return z
  }
}
```


Pattern matching: cascading if / else if are over

```
name match {
  case "Martin" => println("Hey there")
  case "Gerald" => println("Hello")
  case _        => println("Gnii?")
}
```

- Veeery powerful construct
- Any expression can be filtered
- The default case is mandatory

```
name match {
  case "Martin" | "Gerald" => println("Hey there")
  case _                   => println("Gniii?")
}
```

```
age match {
  case i if i<20 => println("Hey dude!")
  case i if i<30 => println("Hello young man")
  case _         => println("Hello Sir")
}
```

```
(x,y) match {
  case (0,0) => println("Origin")
  case (_,0) => println("Abscissa")
  case (0,_) => println("Ordinate")
  case (_,_) => println("Random")
}
```


There is much more to Scala

But that's all you need to know for now...

First Chapter

Practical and Theoretical Foundations of Programming

Comparing Algorithms' Efficiency

There are always more than one way to solve a problem

Choice criteria between algorithms

- Correctness: provides the right answer
- Simplicity: *not* Rube Goldberg's machines
- Efficiency: fast, use little memory
- Stability: small change in input does not change output

Empirical efficiency measurements

- Code the algorithm, benchmark it and use runtime statistics
- ☹ Several factors impact performance:
  machine, language, programmer, compiler, compiler's options, operating system, ...
- ⇒ Performance not generic enough for comparison

Mathematical efficiency estimation

- Count amount of basic instruction as function of input size
- ☺ Simpler, more generic and often sufficient
  (true in theory; in practice, optimization necessary **in addition** to this)

Best case, worst case, average analysis

Algorithm running time depends on the data

Example: Linear search in an array

```
def linearSearch(val:Int, tab:Array[Int]): Boolean = {
  for (i <- 0 to tab.length-1)
    if (tab(i) == val)
      return true;
  return false
}
```

- Case 1: search whether 42 is in {42, 3, 2, 6, 7, 8, 12, 16, 17, 32, 55, 44, 12} answer found after one step
- Case 2: search whether 4 is in {42, 3, 2, 6, 7, 8, 12, 16, 17, 32, 55, 44, 12} need to traverse the whole array to decide (n steps)

Counting the instructions to run in each case

- t_{min}: #instructions for the best case inputs
- t_{max}: #instructions for the worst case inputs
- t_{avg}: #instructions on average (average of values coefficiented by probability) $t_{avg} = p_1 t_1 + p_2 t_2 + \ldots + p_n t_n$

Linear search runtime analysis

```
for (i <- 0 to tab.length-1)
  if (tab(i) == val)
    return true
return false
```

- For simplicity, let's assume the value is in the array, positions are equally likely
- Let's count tests (noted t), additions (noted a) and value changes (noted c)

Best case: searched data in first position

- 1 value change (i=0); 2 tests (loop boundary + equality)
- $t_{min} = c + 2t$

Worst case: searched data in last position

- 1 value change (i=0); {2 tests, 1 change, 1 addition (i+=1)} per loop
- $t_{max} = c + n \times (2t + 1c + 1a) = (n+1) \times c + 2n \times t + n \times a$

Average case: searched data in position p with probability $\frac{1}{n}$

- $t_{avg} = c + \sum_{p \in [1,n]} \frac{1}{n} \times (2t + c + a) \times p = c + \frac{1}{n} \times (2t + c + a) \times \sum_{p \in [1,n]} p$

  $t_{avg} = c + \frac{n(n-1)}{2n} \times (2t + c + a) = (n-1) \times t + \frac{n+1}{2} \times c + \frac{n-1}{2} \times a$

Simplifying equations

$t_{avg} = (n-1) \times t + \frac{n+1}{2} \times c + \frac{n-1}{2} \times a$ is too complicated

Reducing amount of variables

- To simplify, we only count the most expensive operations
- But which it is is not always clear...
- Let's take write accesses c (classical but arbitrary choice)

Focusing on dominant elements

- We can forget about constant parts if there is n operations
- We can forget about linear parts if there is n^2 operations
- ...
- Only consider the most dominant elements when n is very big

⇒ This is called **asymptotic complexity**

Asymptotic Complexity: Big-O notation

Mathematical definition

- Let $T(n)$ be a non-negative function
- $T(n) \in O(f(n))$ ⇔ ∃ constants c, n_0 so that $\forall n > n_0$, $T(n) \leq c \times f(n)$
- f(n) is an upper bound of T(n) ...

... after some point, and with a constant multiplier

Application to runtime evaluation

- $T(n) \in O(n^2) \Rightarrow$ when n is big enough, you need less than n^2 steps
- This gives a upper bound

Big-O examples

Example 1: Simplifying a formula

- Linear search: $t_{avg} = (n-1) \times t + \frac{n+1}{2} \times c + \frac{n-1}{2} \times a \Rightarrow$ $T(n) = O(n)$
- Imaginary example: $T(n) = 17n^2 + \frac{32}{17}n + \pi \Rightarrow$ $T(n) = O(n^2)$
- If T(n) is constant, we write T(n)=O(1)

Practical usage

- Since this is a upper bound, $T(n) = O(n^3)$ is also true when $T(n) = O(n^2)$
- But not as relevant

Example 2: Computing big-O values directly

array initialization

```
for (i <- 0 to tab.length-1)
  tab(i) = 0
```

- We have n steps, each of them doing a constant amount of work
- $T(n) = c \times n \quad \Rightarrow \quad T(n) = O(n)$

  (don't bother counting the constant elements)

Big-Omega notation

Mathematical definition

- Let $T(n)$ be a non-negative function
- $T(n) \in \Omega(f(n))$ ⇔ ∃ constants c, n_0 so that $\forall n > n_0$, $T(n) \geq c \times f(n)$
- Similar to Big-O, but gives a lower bound
- Note: similarly to before, we are interested in big lower bounds

Example: $T(n) = c_1 \times n^2 + c_2 \times n$

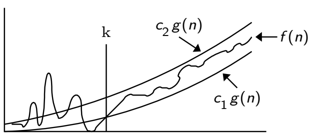
- $T(n) = c_1 \times n^2 + c_2 \times n \geq c_1 \times n^2 \quad \forall n > 1$

  $T(n) \geq c \times n^2$ for $c > c_1$
- Thus, $T(n) = \Omega(n^2)$

Theta notation

Mathematical definition

- $T(n) \in \Theta(g(n))$ if and only if $T(n) \in O(g(n))$ and $T(n) \in \Omega(g(n))$



Example

		n=10	n=1000	n=100000	
$\Theta(n)$	n	10	1000	10^5	seconds
	100n	1000	10^5	10^7	
$\Theta(n^2)$	n^2	100	10^6	10^{10}	minutes
	$100n^2$	10^4	10^8	10^{12}	
$\Theta(n^3)$	n^3	1000	10^9	10^{15}	hours
	$100n^2$	10^5	10^{11}	10^{17}	
$\Theta(2^n)$	2^n	1024	$> 10^{301}$	∞	...
	100×2^n	$> 10^5$	10^{305}	∞	
$\log(n)$	$\log(n)$	3.3	9.9	16.6	
	$100\log(n)$	332.2	996.5	1661	

Classical mistakes

Mistake notations

- Indeed, we have $O(\log(n)) = O(n) = O(n^2) = O(n^3) = O(2^n)$

  Because it's an upper bound; to be correct we should write ⊂ instead of =
- Likewise, we have $\Omega(\log(n)) = \Omega(n) = \Omega(n^2) = \Omega(n^3) = \Omega(2^n)$

  Because it's a lower bound; we should write ⊃ instead of =
- We only have $\Theta(\log(n)) \neq \Theta(n) \neq \Theta(n^2) \neq \Theta(n^3) \neq \Theta(2^n)$

  (but in practice, everybody use O() as if it were Θ() – although that's wrong)

Mistake worst case and upper bounds

- Worst case is the input data leading to the longest operation time
- Upper bound gives indications on increase rate when input size increases

  (same distinction between best case and lower bound)

Asymptotic Complexity in Practice

Rules to compute the complexity of an algorithm

Rule 1: Complexity of a sequence of instruction: Sum of complexity of each
Rule 2: Complexity of basic instructions (test, read/write memory): O(1)
Rule 3: Complexity of `if`/`switch` branching: Max of complexities of branches
Rule 4: Complexity of loops: Complexity of content × amount of loop
Rule 5: Complexity of methods: Complexity of content

Simplification rules

- Ignoring the constant:
  If f(n) = O($k \times g(n)$) and $k > 0$ is constant then f(n) = O(g(n))
- Transitivity
  If f(n) = O(g(n)) and g(n) = O(h(n)) then f(n) = O(h(n))
- Adding big-Os
  If A(n) = O(f(n)) and B(n) = O(g(n)) then A(n)+B(n) = O(max(f(n), g(n)))
  = O(f(n)+g(n))
- Multiplying big-Os
  If A(n) = O(f(n)) and B(n) = O(h(n)) then $A(n) \times B(n) = O(f(n) \times g(n))$

Some examples

Example 1: `a=b;` ⇒ Θ(1) (constant time)

Example 2

```
var sum=0;
for (i <- 1 to n)
  sum += n;
```

$\Theta(n)$

Example 3

```
var sum=0;
for (i <- 1 to n)
  for (j <- 1 to n)
    sum += 1
for (k <- 0 to n-1)
  A(k) = k;
```

$\Theta(1) + \Theta(n^2) + \Theta(n) = \Theta(n^2)$

Example 4

```
var sum=0;
for (i <- 1 to n)
  for (j <- 1 to i)
    sum += 1
```

$\Theta(1) + O(n^2) = O(n^2)$
one can also show $\Theta(n^2)$

Example 5

```
var sum=0; var i=0
while (i<n) {
  sum +=1
  i = i * 2
}
```

$\Theta(\log(n))$ log is due to the $i \times 2$

Going further on Algorithm Complexity

Problems' Classification

- Problems can also be sorted in class of complexities (not only algorithms)
  depending on the best existing algorithm to solve them
- Showing that no better algorithm exist for a given problem: Calculability
- Multi-million question: P=NP?
  P: polynomial algorithm to find the solution exists
  NP: candidate solution eval. in polynomial time, but no known polynomial algo
  NP-complete: set of NP problems for which if one P algorithm is found,
  it's applicable to every other NP-complete problems

Time is not the only metric of interest: Space too

- In computation, there is a sort of tradeoff between space and time
  Faster algorithms need to pre-compute elements . . . requiring more storage memory

So does Energy nowadays!

- Computational power of CPU grows linearly with frequency;
  Energy consumption grows (more than) quadratically with frequency
- To save energy (and money), split your task on several slower cores
  Parallel algorithms are the way to go (but it's *ways* harder)

First Chapter

Practical and Theoretical Foundations of Programming

Algorithmic stability

Computers use fixed precision numbers

- 10+1=11
- $10^{10} + 1 = 10000000001$
- $10^{16} + 1 = 10000000000000001$
- $10^{17} + 1 = 100000000000000000 = 10^{17}$

What is the value of $\sqrt{2}^2$?

- Old computers though it was 1.9999999

Other example

```
while (value < 2E9)
  value += 1E-8;
```

This is an infinite loop
(because when $value = 10^9$, $value + 10^{-8} = value$)

Numerical instabilities are to be killed to predict weather,
simulate a car crash or control a nuclear power plant

(but this is all *ways* beyond our goal this year ;)

Conclusion of this chapter

What tech guys tend to do when submitted a problem

- They code it directly, and rewrite everything once they understood
- And rewrite everything to improve performance
- And rewrite everything when the code needs to evolve

What managers tend to do when submitted a problem

- They write up a long and verbose specification
- They struggle with the compiler in vain
- Then they pay a tech guy (and pay too much since they don't get the grasp)

What theoreticians tend to do when submitted a problem

- They write a terse but formal specification
- They write an algorithm, and prove its optimality
  (the algorithm never gets coded)

What good programmers do when submitted a problem

- They write a clear specification
- They come up with a clean design
- They devise efficient data structures and algorithms
- Then (and only then), they write a clean and efficient code
- They ensure that the program does what it is supposed to do

Choice criteria between algorithms

Correctness

- Provides the right answer
- This crucial issue is delayed a bit further

Simplicity

- Keep it simple, silly
- Simple programs can evolve (problems and client's wishes often do)
- Rube Goldberg's machines cannot evolve

Efficiency

- Run fast, use little memory, dissipate little energy
- Asymptotic complexity must remain polynomial
- Note that you cannot have a decent complexity with the wrong data structure
- You still want to test the actual performance of your code in practice

Numerical stability

- Small change in input does not change output
- Advanced issue, critical for numerical simulations (but beyond our scope)

Second Chapter

Iterative Sorting Algorithms

Sorting Problem Specification

Input data

- A sequence of N comparable items $< a_1, a_2, a_3, \dots, a_N >$
- Items are *comparable* iff $\forall a, b$ in set, either $\underline{a < b}$ or $\underline{a > b}$ or $\underline{a = b}$

Result

- Permutation[2] $< a'_1, a'_2, a'_3, \dots, a'_N >$ so that: $a'_1 \leq a'_2 \leq a'_3 \leq \dots \leq a'_N$

Sorting complex items

- For example, if items represent students, they encompass name, class, grade
- Key: value used for the sort
- Extra data: other data associated to items, permuted along with the keys

Problem simplification

- We assume that items are chars or integers to be sorted in ascending order (no loss of generality)

Memory consideration

- Sort *in place*, without any auxiliary array. Memory complexity: O(1)

[2] reordering

Selection Sort

Big lines

- First get the smallest value, and put it in first position
- Then get the second smallest value, and put it in second position
- and so on for all values

Example:

U	N	S	O	R	T	E	D
D	N	S	O	R	T	E	U
D	E	S	O	R	T	N	U
D	E	N	O	R	T	S	U
D	E	N	O	R	T	S	U
D	E	N	O	R	T	S	U
D	E	N	O	R	S	T	U
D	E	N	O	R	S	T	U
D	E	N	O	R	S	T	U

PseudoScala code

```
/* for each element, do: */
for (i <- 0 to length-1) {

  /* (1) search min on [i;N] */
  var minpos=i
  for (j <- i to length-1) {/* ∀j ∈ [i; length − 1] */
    if (tab(j) < tab(minpos)) {
      minpos = j
    }
  }

  /* (2) put min first */
  temp=tab(i)
  tab(i)=tab(minpos)
  tab(minpos)=temp
}
```


Selection sort discussion

We apply a very generic approach here:

- Do right now what you can, delay the rest for later (put min first)
- Progressively converge to what you are looking for (sort the remaining)

```
for (i <- 0 to length-1) {
  var minpos=i
  for (j <- i to length-1) {
    if (tab(j) < tab(minpos)) {
      minpos = j
    }
  }
  temp=tab(i)
  tab(i)=tab(minpos)
  tab(minpos)=temp
}
```

Memory Analysis

- 2 extra variables (only one at the same time, actually)
- $\Rightarrow$ Constant amount of extra memory
- $\Rightarrow$ Space complexity is $O(1)$
- $O(1)$ is the smallest complexity $\rightsquigarrow \Theta(1)$

Time Analysis

- Forget about constant times, focus on loops!
- Two interleaved loops which length is *at most* N
- $\Rightarrow$ Time complexity is $O(N^2)$

Finer analysis of selection sort's time performance

```
for (i <- 0 to length-1) {
  var minpos=i
  for (j <- i to length-1)
    if (tab(j) < tab(minpos))
      minpos = j
  temp=tab(i)
  tab(i)=tab(minpos)
  tab(minpos)=temp
}
```

Best case, worst case, average case

- No matter the order of the data, 'selection sort' does the same
- $\Rightarrow t_{min} = t_{max} = t_{avg}$

Counting steps more precisely (but only dominant term)

- $T(N) = \sum_{i \in [1,N]} \left(\sum_{j \in [i,N[} 1 \right) = \sum_{i \in [1,N]} (N - i) = \sum_{i \in [1,N]} N - \sum_{i \in [1,N]} i$

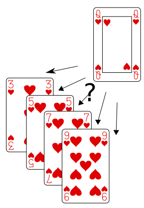
  $= N^2 - \frac{N \times (N+1)}{2} = N^2 - \frac{N^2+N}{2} = \frac{1}{2}N^2 - \frac{1}{2}N = \frac{1}{2}(N^2 - N)$
- Let's prove that $T(n) \in \Omega(n^2)$. For that, we want:
- $\exists c, n_0 / \forall N > n_0,\ \boxed{\frac{1}{2}(N^2 - N) \geq cN^2} \Leftarrow \boxed{N^2 - N \geq 2cN^2} \Leftarrow \boxed{N - 1 \geq 2cN}$
- So, we want $\exists c, n_0 / \forall N > n_0, N \geq \frac{1}{1-2c}$
- Let's take anything for c ($\neq \frac{1}{2}$), and $n_0 = \frac{1}{1-2c}$. Trivially gives what we want.

$T(n) \in \Theta(n^2)$

Insertion Sort

How do you sort your card deck?

- No human would apply *selection sort* to sort a deck!



Algorithm used most of the time to sort a card deck:

1. If the cards #1 and #2 need to be swapped, do it
2. Insert card #3 at its position in the [1,2] part of the deck
3. Insert card #4 at its position in the [1,3] part of the deck

...

Finding the common pattern

- Step n (≥ 2) is "insert card #(n+1) into [1,n]"
- Step 1 = insert the 2. card into [1,1]
- We may add a Step 0 to generalize the pattern (that's a no-op)

Algorithm big lines

```
For each element
  Find insertion position
  Move element to position
```

This is *Insertion Sort*

U	N	S	O	R	T	E	D
U	N	S	O	R	T	E	D
N	U	S	O	R	T	E	D
N	S	U	O	R	T	E	D
N	O	S	U	R	T	E	D
N	O	R	S	U	T	E	D
N	O	R	S	T	U	E	D
E	N	O	R	S	T	U	D
D	E	N	O	R	S	T	U

Writing the insertion sort algorithm

Fleshing the big lines

```
For each element
  Find insertion point
  Move element to position
```

- Finding the insertion point is easy (searching loop)
- Moving to position is a bit harder: "make room"
- We have to *shift* elements one after the other

Before:	N	S	U	O	R	T	E	D
After:	N	O	S	U	R	T	E	D

	0	1	2	3	4	5	6	7		
step 1.	N	S	U		R	T	E	D	tmp	O
step 2.	N	S		U	R	T	E	D	tmp	O
step 3.	N		S	U	R	T	E	D	tmp	O
step 4.	N	O	S	U	R	T	E	D	tmp	

- Shifting elements induce a loop also
- We can do both searching insertion point and shifting at the same time

```
for (i <- 1 to length-1) {/* i: boundary between unsorted/sorted areas*/
  /* save current value (it this case, that's O) */
  val tmp = tab(i)
  /* shift to right any element on the left being smaller than tmp */
  var j = i
  while (j>0 && tab(j-1)>tmp) {/* while previous cell exists and is bigger */
    tab(j) = tab(j-1) /* copy that element */
    j = j -1 /* consider the next element */
  }
  /* put tmp in cleared position */
  tab(j)=tmp
}
```


Bubble Sort

- All these sort algorithms are quite difficult to write. Can we do simpler?
- Like "while it's not sorted, sort it a bit"

Detecting that it's sorted

```
for (i <- 0 to length-2)
  /* if these two values are badly sorted */
  if (tab(i)>tab(i+1))
    return false
return true
```

How to "sort a bit?"

- We may just swap these two values

```
val tmp=tab(i)
tab(i)=tab(i+1)
tab(i+1)=tmp
```

All together

- Add boolean variable to check whether it sorted

```
var swapped = true
while (swapped) { /* until we do one traversal without swap */
  swapped = false
  for (i <- 0 to length-2)
    if (tab(i) > tab(i+1)) { /* if these 2 values are badly sorted */
      /* swap them */
      val tmp = tab(i)
      tab(i) = tab(i+1)
      tab(i+1) = tmp
      /* and remember we swapped something */
      swapped = true
    }
} /* repeat until a traversal without swapping */
```


Conclusion on Iterative Sorting Algorithms

Cost Theoretical Analysis

Amount of comparisons	Best Case	Average Case	Worst Case
Selection Sort	$O(n^2)$	$O(n^2)$	$O(n^2)$
Insertion Sort	$O(n)$	$O(n^2)$	$O(n^2)$
Bubble Sort	$O(n)$	$O(n^2)$	$O(n^2)$

Which is the best in practice?

- We will explore practical performance during the lab
- But in practice, bubble sort is awfully slow and should never be used

Is it optimal?

- The lower bound is $\Omega(n \log(n))$ – cf. TD lab
- Some other algorithms achieve it (Quick Sort, Merge Sort)
- We come back on these next week

(this ends the first lecture)

Third Chapter

Recursion

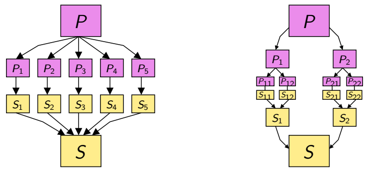
Divide & Conquer

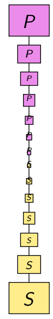
Classical Algorithmic Pattern

- When the problem is too complex to be solved directly, decompose it

When/How is it applicable?

1. Divide: Decompose problem into (simpler/smaller) sub-problems
2. Conquer: Solve sub-problems
3. Glue: Combine solutions of sub-problems to a solution as a whole





> *You don't have to see the whole staircase, just take the first step.*
> – Martin Luther King

Recursion

Divide & Conquer + sub-problems similar to big one

Recursive object

- Defined using itself
- Examples:
  - $U(n) = 3 \times U(n-1) + 1$; $U(0) = 1$
  - Char string = either a char followed by a string, or empty string
- Often possible to rewrite the object, in a non-recursive way (said *iterative way*)

Base case(s)

- Trivial cases that can be solved directly
- Avoids infinite loop

When the base case is missing...

There's a Hole in the Bucket (traditional)

There's a hole in the bucket, dear Liza, a hole.
So fix it dear Henry, dear Henry, fix it.
With what should I fix it, dear Liza, with what?
With straw, dear Henry, dear Henry, with straw.
The straw is too long, dear Liza, too long.
So cut it dear Henry, dear Henry, cut it!
With what should I cut it, dear Liza, with what?
Use the hatchet, dear Henry, the hatchet.
The hatchet's too dull, dear Liza, too dull.
So sharpen it dear Henry, dear Henry, sharpen it!
With what should I sharpen, dear Liza, with what?
Use the stone, dear Henry, dear Henry, the stone.
The stone is too dry, dear Liza, too dry.
So wet it dear Henry, dear Henry, wet it.
With what should I wet it, dear Liza, with what?
With water, dear Henry, dear Henry, water.
With what should I carry it dear Liza, with what?
Use the bucket, dear Henry, dear Henry, the bucket!
There's a hole in the bucket, dear Liza, a hole.

Classical Aphorism

To understand recursion,
you first have to understand recursion

Recursive Acronyms

- GNU is Not Unix
- PHP: Hypertext Preprocessor
- PNG's Not GIF
- Wine Is Not an Emulator
- Visa International Service Association
- HIRD of Unix-Replacing Daemons
  Hurd of Interfaces Representing Depth
- Your Own Personal YOPY

This is naturally to be avoided in algorithms

In Mathematics: Natural Numbers and Induction

Peano postulates (1880)

Defines the set of natural integers $\mathbb{N}$

1. 0 is a natural number
2. If n is natural, its successor (noted $n+1$) also
3. There is no number x so that $x+1=0$
4. Distinct numbers have distinct successors ($x \neq y \Leftrightarrow x+1 \neq y+1$)
5. If a property holds (i) for 0 (ii) for each number's successor,
   it then holds for any number

Proof by Induction

- One shows that the property holds for 0 (or other base case)
- One shows that when it holds for n, it then holds for $n+1$
- This shows that it holds for any number

In Computer Science

Two twin notions

- Functions and procedures defined recursively (generative recursion)
- Data structures defined recursively (structural recursion)

Naturally, recursive functions are well fitted to recursive data structures

This is an **algorithm** characteristic

- No problem is intrinsically recursive
- Some problems *easier* or more natural to solve recursively
- Every recursive algorithm can be *derecursived*

Third Chapter

Recursion

- Introduction
- Principles of Recursion
  - First Example: Factorial
  - Schemas of Recursion
  - Recursive Data Structures
- Recursion in Practice
  - Solving a Problem by Recursion: Hanoi Towers
  - Classical Recursive Functions
  - Recursive Sorting Algorithms
    - MergeSort
    - QuickSort

Recursive Functions and Procedures

Recursively Defined Function: its body contains calls to itself

The Scrabble™ word game

- Given 7 letter tiles, one should form existing English worlds
  T I R N E G S ⇝ RIG, SIRE, GRINS, INSERT, RESTING, ...
- How many permutation exist?
  - First position: pick one tile from 7
  - Second position: pick one tile from 6 remaining
  - Third position: pick one tile from 5 remaining
  - ...
  - Total: $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

This is the Factorial

- Mathematical definition of factorial: $\begin{cases} n! = n \times (n-1)! \\ 0! = 1 \end{cases}$
- Factorial : integer $\rightarrow$ integer
  Precondition: factorial(n) defined if and only if $n \geq 0$
  Postcondition: factorial(n)= $n!$

Recursive Algorithm for Factorial

Literal Translation of the Mathematical Definition

```
FACTORIAL(n):
  if n = 0 then r ← 1
          else  r ← n × factorial(n − 1)
  end
```

Remarks:

- $r \leftarrow 1$ is the base case: no recursive call
- $r \leftarrow n \times factorial(n-1)$ is the general case: Achieves a recursive call
- Reaching the base case is mandatory for the algorithm to finish

Factorial Computation Details

```
FACTORIAL(n):
  if n = 0 then r ← 1
           else  r ← n × factorial(n − 1)
  end
```

$factorial(4) = 4 \times factorial(3)$

$3 \times factorial(2)$

$2 \times factorial(1)$

$1 \times factorial(0)$ — Recursive Descent

$4 \times 3 \times 2 \times 1 \times 1$ — Base Case

$4 \times 3 \times 2 \times 1$

$4 \times 3 \times 2$

4×6

24 — Recursive Climb

$factorial(4) = 24$

Third Chapter

Recursion

General Recursion Schema

```
if COND then BASECASE
        else  GENCASE
end
```

- COND is a boolean expression
- If COND is true, execute the base case BASECASE (without recursive call)
- If COND is false, execute the general case GENCASE (with recursive calls)

The factorial(n) example

BASECASE: $r \leftarrow 1$

GENCASE: $r \leftarrow n \times factorial(n-1)$

Other Recursion Schema: Multiple Recursion

More than one recursive call

Example: Pascal's Rule and $\binom{n}{k}$

- Amount of k-long sets of n elements (order ignored)

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } n = k; \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{else } (1 \le k < n). \end{cases}$$

- $\binom{4}{2} = 6$ ⇝ 6 ways to build a pair of elements picked from 4 possibilities:
  {A;B},{A;C},{A;D},{B;C},{B;D},{C;D} (if order matters, 4×3 possibilities)

Corresponding Algorithm:

```
PASCAL (n, k)
  If k = 0 or k = n then r ← 1
               else  r ← PASCAL (n − 1, k) +
                         PASCAL (n − 1, k − 1)
```

First rows

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4  (6)  4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```


Other Recursion Schema: Mutual Recursion

Several functions calling each other

Example 1

$$A(n) = \begin{cases} 1 & \text{if } n \le 1 \\ B(n+2) & \text{if } n > 1 \end{cases} \qquad B(n) = \begin{cases} 1 & \text{if } n \le 1 \\ A(n-3)+4 & \text{if } n > 1 \end{cases}$$

Compute A(5):

Example 2: one definition of parity

$$even?(n) = \begin{cases} true & \text{if } n = 0 \\ odd(n-1) & \text{else} \end{cases} \quad \text{and} \quad odd?(n) = \begin{cases} false & \text{if } n = 0 \\ even(n-1) & \text{else} \end{cases}$$

Other examples

- Some Maze Traversal Algorithm also use Mutual Recursion (see lab)
- Mutual Recursion classical in Context-free Grammar (see compilation course)

Other Recursion Schema: Embedded Recursion

Recursive call as Parameter

Example: Ackerman function

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{else} \end{cases}$$

Thus the algorithm:

```
ACKERMAN(m, n)
  if m = 0 then n + 1
      else if n = 0 then ACKERMAN(m − 1, 1)
                    else ACKERMAN(m − 1, ACKERMAN(m, n − 1))
```

Warning, this function grows quickly:

$Ack(1,n) = n+2$ $\quad Ack(2,n) = 2n+3$

$Ack(3,n) = 8 \cdot 2^n - 3$ $\quad Ack(4,n) = 2^{2^{2^{\cdots^{2}}}}\Big\}n$

$Ack(4,4) > 2^{65536} > 10^{80}$ (estimated amount of particles in universe)

Recursive Data Structures

Definition

Recursive datatype: Datatype defined using itself

Classical examples

List: element followed by a list or empty list

Binary tree: {value; left son; right son} or empty tree

This is the subject of the module "Data Structures"

- After TOP and POO in track

Example: Strings as (linked) lists

Defined operations

[ ]		*The empty string object*
cons	Char × String ↦ String	*Adds the char in front of the list*
car	String ↦ Char	*Get the first char of the list (not defined if empty?(str))*
cdr	String ↦ String	*Get the list without first char*
empty?	String ↦ Boolean	*Tests if the string is empty*

- As you can see, strings are defined recursively using strings

Examples

- "bo" = cons('b',cons('o',[ ]))
- "hello" = cons('h',cons('e',cons('l',cons(cons('l',cons(cons('o',[ ])))))))
- cdr(cons('b',cons('o',[ ]))) = "o" = cons('o',[ ])

These are native constructs in LISP programing language

- But, these constructs are hard to remember (cdr vs. car)
- But, all these parenthesis are nasty (too much syntaxic sugar)

Doing the same in Java

Element Class representing a letter and the string following (ie, non-empty strings)
String Class representing a string (either empty or not)

```
public class Element {
  public char value;
  public Element tail;

  Element(char x, Element tail) {
      value = x;
      this.tail = tail;
  }
}
```

```
public class StringRec {
  private Element head = null;

  public boolean isEmpty() {
      return head == null;
  }
  public void cons(char x) {
    // Create new elem and connect it
    Element newElem = new Element(x, head);
    // This is new head
    head = newElem;
} }
```

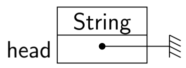
```
StringRec plop = new StringRec().cons('p').cons('o').cons('l').cons('p');
```

Object Orientation is helping (only) when programming at large

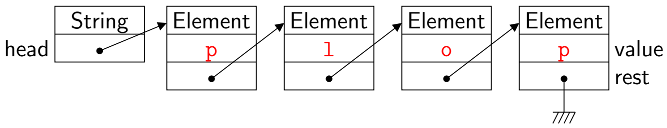
- It's "not really helping" when programming at small (both are orthogonal)
- Here, message lost under the syntaxic sugar
- Dotted notation not natural in this case (this could be improved? mail me!)

Some Memory Representation Examples in Java

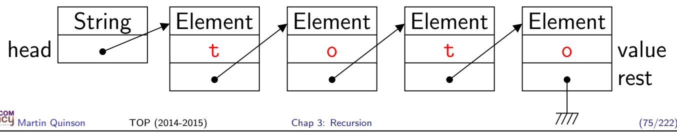
Empty String: `new StringRec();`



String "plop": new StringRec().cons('p').cons('o').cons('l').cons('p');



String "toto": : new StringRec().cons('o').cons('t').cons('o').cons('t');



Scala Lists

Nil		*The empty list object*
::	elm × List ↦ List	*Adds the element in front of the list (pronounced cons)*
head	List ↦ elm	*Get the first char of the list (not defined if lst.isEmpty)*
tail	List ↦ List	*Get the list without first char*
isEmpty	List ↦ Boolean	*Tests if the list is empty*

Example: "hello" ≡ 'h'::'e'::'l'::'l'::'o'::Nil

```
scala> val lst = 1::2::3::4::Nil
lst: List[Int] = List(1, 2, 3, 4)

scala> lst.head
res1: Int = 1

scala> lst.tail
res2: List[Int] = List(2, 3, 4)
```

```
scala> def sum(list:List[Int]): Int = list match {
     |   case Nil => 0
     |   case i::newlist => i + sum(newlist)
     | }
sum: (list: List[Int])Int

scala> lst.sum
res3: Int = 10
```

Functional orientation of Scala is a beauty

- Is is much more convenient than LISP, syntaxic-sugar-free compared to Java
- Plays very well with Scala's pattern-matching

Recursion in Practice

Recursion is a tremendously important tool in algorithmic

- Recursive algorithms often simple to understand, but hard to come up with
- Some learners even have a *trust issue* with regard to recursive algorithms

Holistic and Reductionist Points Of View

- Holism: *the whole is greater than the sum of its parts*
- Reductionism: *the whole can be understood completely if you understand its parts and the nature of their 'sum'.*

Writing a recursive algorithm

- Reductionism clearly induced since views problems as sum of parts
- But Holistic approach also mandatory:
  - When looking for general solution, assume that solution to subproblems given
  - Don't focus of every detail, keep a general point of view (not always natural, but)
    If you cannot see the forest out of trees, don't look at branches and leaves
- At the end, recursion is something that you can only learn through experience

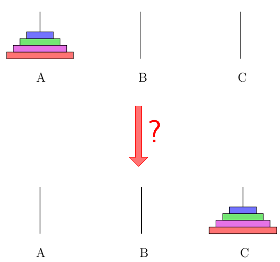
Third Chapter

Recursion

How to Solve a Problem Recursively?

1. Determine the parameter on which recursion will operate:
   Integer or Recursive datatype
2. Solve simple cases: the ones for which we get the answer directly
   They are the Base Cases
3. Setup Recursion:
   - Assume you know to solve the problem for one (or several) parameter value being strictly smaller (ordering to specify) than the value you got
   - How to solve the problem for the value you got with that knowledge?
4. Write the general case
   Express the searched solution as a function of the sub-solution you assume you know
5. Write Stopping Conditions (ie, base cases)
   Check that your recursion always reaches these values

A Classical Recursive Problem: Hanoï Towers



- Data: n disks of differing sizes
- Problem: change the stack location
  A third stick is available
- Constraint: no big disk over small one

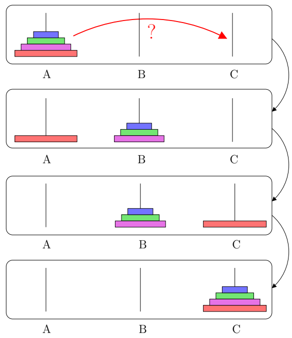
Problem Analysis

- Parameters :
  - Amount n of disks stacked on initial stick
  - The sticks
- ⇝ We recurse on integer n
- How to solve problem for n disks when we know how to do with $n-1$ disks?
- ⇝ Decomposition between bigger disk and (n-1) smaller ones

- We want to write procedure Hanoi(n, From, To).
  It moves the n disks from stick From to stick To
- ⇝ For simplicity sake, we introduce procedure Move(From,To)
  It moves the upper disk from stick From to stick To
  (also checks that we don't move a big one over a small one)

- Stopping Condition: when only one disk remains, use Move
  Hanoi(1,x,y)=Move(x,y)

Possible Decomposition of Hanoi(n, A, C)



Assume "someone" can move N-1 disks for you
Hanoi(n-1, A, B)

Move(A,C)

Ask "your friend" again
Hanoi(n-1, B, C)

Do you feel the *trust issue* against recursive algorithms?

> *To iterate is human, to recurse is divine.* — *L Peter Deutsch*

(Deutsch: ghostview; first JIT compiler (for SmallTalk) 15 yr ahead; wrote LISP interpreter for PDP-1 by 12yr)

Corresponding Algorithm

Function `hanoi` *(n, from, to, other)* **is**
  if *n = 1* **then**
    `move` (from, to)
  else
    `hanoi` (n-1, from, other)
    `move` (from, to)
    `hanoi` (n-1, other, to)

```
def hanoi(n:Int, from:Int,to:Int,other:Int)=
  if (n == 1) {
    move(from, to)
  } else {
    hanoi(n-1, from, other, to)
    move(from, to)
    hanoi(n-1, other, to, from)
  }
```

Variant with 0 as base case

Function `hanoi` *(n, from, to, other)* **is**
  if *n ≠ 0* **then**
    `hanoi` (n-1, from, other, to)
    `move` (from, to)
    `hanoi` (n-1, other, to, from)

```
def hanoi(n:Int, from:Int,to:Int,other:Int)=
  if (n != 0) {
    hanoi(n-1, from, other, to)
    move(from, to)
    hanoi(n-1, other, to, from)
  }
```


Back on the Hanoi Towers Problem

Problem first introduced in 1883 by Eduard Lucas, with a fake story

- Somewhere in India, Brahmane monks are doing this with 64 gold disks
- When they will be done, there will be the end of time

Anecdote Main Interest

- Amount of moves mandatory to move n disks: 1, 3, 7, 15, 31, 63, ...
- General term: $2^n - 1$
- The monks need $2^{64} - 1$ (ie 18 446 744 073 709 551 615) moves
- That's almost 600 000 000 000 years by playing one move per second

Other funny usage of the $2^n - 1$ suite

- Fibonacci searched the minimal amount of masses to weight any value up to N
- Tartaglia solution when masses are on the same arm:
  With n masses in the suite (1, 2, 4, 8, ...) you can weight any values up to $2^n - 1$
- *Mathematicians:* specialists of pointless stories leading to fundamental tools

[The Penguin Dictionary of Curious and Interesting Numbers, David Wells, 1997]

Third Chapter

Recursion

Classical Recursive Function: Fibonacci

Study of reproduction speed of rabbits (XII century)

- One pair at the beginning
- Each pair of fertile rabbits produces a new pair of offspring each month
- Rabbits become fertile in their second month of life
- Old rabbits never die
- $F_0 = 0$; $F_1 = 1$; $F_2 = 1$; $F_3 = 2$; $F_4 = 3$; $F_5 = 5$; $F_6 = 8$; $F_7 = 13$; ...

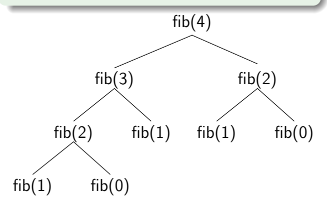
$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ \forall n, F_n = F_{n-1} + F_{n-2} \end{cases}$$

Exercice :
Compute amount of recursive calls

Corresponding Code

```
def fib(n:Int):Int =
  if (n <= 1)
    n // Base Case ('return' is optional)
  else
    fib(n-1) + fib(n-2)
```

(efficient implementations exist)



Classical Recursive Function: McCarthy 91

Definition

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M(M(n+11)) & \text{if } n \le 100 \end{cases}$$

Interesting Property:

$\forall n \le 101, M(n) = 91$
$\forall n > 101, M(n) = n - 10$

Proof

- When $90 \le k \le 100$, we have $f(k) = f(f(k+11)) = f(k+1)$
  In particular, $f(91) = f(92) = \ldots = f(101) = 91$
- When $k \le 90$: Let r be so that: $90 \le k + 11r \le 100$
  $f(k) = f(f(k+11)) = \ldots = f^{(r+1)}(k+11r) = f^{(r+1)}(91) = 91$

John McCarthy (1927-)

Turing Award 1971, Inventor of language LISP, of expression "Artificial Intelligence" and of the Service Provider idea (back in 1961).

Classical Recursive Function: Syracuse

Function `syracuse` *(n)*
  if *n = 0* **or** *n = 1* **then**
    1
  else if *n mod 2 = 0* **then**
    `syracuse`($n/2$)
  else
    `syracuse`($3 \times n + 1$)

- Question: Does this function always terminate?
  Hard to say: suite is not monotone
- Collatz's Conjecture: $\forall n \in \mathbb{N}$, $\text{SYRACUSE}(n) = 1$
- Checked on computer $\forall n < 5 \cdot 2^{60} \approx 6 \cdot 10^{18}$
  (but other conjectures were proved false for bigger values only)
- This is an open problem since 1937 (some rewards available)

> *Mathematics is not yet ready for such problems.*
> – Paul Erdös (1913–1996)

Third Chapter

Recursion

Back on Sorting Algorithms

Why don't CS profs ever stop talking about sorting?!

Sorting is the best studied problem in CS

- Variety of different algorithms (cf. PLM's lab for a small subset)
- Still some research on that topic (find best algorithm for a given workload kind)

Several Interesting ideas can be taught in that context

- Complexity: best case/worst case/average case as well as Big Oh notations
- Divide and Conquer and Recursion
- Randomized Algorithms

Sorting is a fundamental building block of algorithms

- Computers spend more time sorting than anything else (25% on mainframes)
- This is because a lot of problems come down to sorting elements

Applications of Sorting (1)

Searching

- Binary search algorithm: search item in dictionnary (sorted list) in $O(\log(n))$
- Speeding up searching perhaps the most important application of sorting

Closest pair

- Given n numbers, find the pair which are closest to each other
- Once the list is sorted, closest elements are next to each other

⇒ Linear scan is enough, thus $O(n\log(n)) + O(n) = O(n\log(n))$

Element uniqueness

- Given a list of n items, are they all unique or are there duplicates?
- Sort them, and do a linear scan of adjacent pairs
- (special case of closest pair, actually)

Applications of Sorting (2)

Frequency distribution

- Given a list of n items, which occures the largest number of times?
- Sort them, and do a linear scan to measure the length of adjacent runs

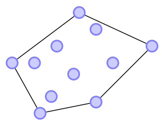
Median and Selection

- What is the kth largest item of a set?
- Sort keys, store them in an array (deal with dups)
- The kth larger can be found in constant time in kth pos of the array

Applications of Sorting (3)

Convex Hulls

- Given n points, find the smallest polygon containing them all
  (think of a elastic band stretched over the points)



- Sort points by x-coordinate, then y-coordinate
- Add them from left to right into the hull:
  - New rightmost point is on the boundary
  - Adding point to boundary may cause others to be deleted depending on whether the angle is convex or not

Huffman codes

- When storing a text, giving each letter's code the same length wastes space
- Example: e is more common than q, so give it a shorter code
- Huffman encoding: Sort letters by frequency, assign codes in order

Char	Freq.	Code
f	5	1100
e	6	1101
c	12	100

Char	Freq.	Code
b	13	101
d	16	111
a	45	0

- Simple & fast
- Not best compression
- Used in JPEG and MP3

Merge Sort

Recursive sorting

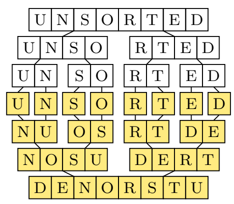
- Imagine the simpler way to sort recursively a list

1. Split your list in two sub-lists
   One idea is to split evenly, but not the only one
2. Sort each of them recursively
   (base case: size≤1)
3. Merge sorted sublists back
   at each step, pick smallest remaining elements of sublists, put it after already picked

Merge Sort

- List splited evenly
- Sub-list copied away
- Merge trivial

(invented by John von Neumann in 1945)



Merge Sort

Scala code

```
def mergeSort(m: List[Int]):List[Int] ={
  // short enought to be already sorted
  if (m.length <= 1)
    return m

  // Slice (=cut) the array in two parts
  val middle = m.length / 2
  val left  = m.slice(0,middle)
  val right = m.slice(middle,m.length)

  // Sort each parts
  val leftSorted  = mergeSort(left)
  val rightSorted = mergeSort(right)

  // Merge them back
  return merge(leftSorted, rightSorted)
}
```

```
def merge(xs:List[Int], ys:List[Int])
    :List[Int] = {

  (xs,ys) match {

    case ( _ , Nil) => xs
    case (Nil,  _ ) => ys

    case (x::x2, y::y2) =>
      if (x < y) {
        x :: merge(x2 , ys)
      } else {
        y :: merge(xs , y2)
      }
  }
}
```

Complexity Analysis

- Time: $\log(n)$ recursive calls, each of them being linear $\rightsquigarrow \Theta(n \times \log(n))$
- Space: Need to copy the array $\rightsquigarrow 2n$ (quite annoying) + $log(n)$ for the stack

QuickSort

Presentation

- Invented by C.A.R. Hoare in 1962
- Widely used (in C library for example)

Big lines

- Pick one element, called *pivot* (random is ok)
- Reorder elements so that:
  - elements smaller to the pivot are before it
  - elements larger to the pivot are after it
- Recursively sort the parts before and after the pivot

Questions to answer

- How to pick the pivot? (random is ok)
- How to reorder the elements?
  - First solution: build sub-list (but this requires extra space)
  - Other solution: invert in place (but hinders stability, see below)

Simple Quick Sort

It's easy with sub-lists:

- Create two empty list variables
- Iterate over the original list; copy elements in correct sublist
- Recurse
- Concatenate results

```
def quicksort(lst:List[Int]):List[Int] = {
  if (lst.length <= 1) // Base case
    return lst

  // Randomly pick a pivot value
  val pivot = lst(lst.length / 2)

  // split the list
  var lows: List[Int] = Nil
  var mids: List[Int] = Nil
  var highs: List[Int] = Nil
  for (item <- lst) { // classify the items
    if ( item == pivot)    { mids  = item :: mids }
    else if (item < pivot) { lows  = item :: lows }
    else                   { highs = item :: highs}
  }

  // return sorted list appending chunks
  quicksort(lows) ::: mids ::: quicksort(highs)
}
```

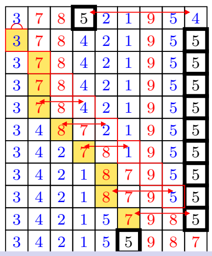
Problem

- Space complexity is about $2n + \log(n)$...
  (2n for array duplication, log(n) for the recursion stack)

In-place Quick Sort

Big lines of the list reordering

- Put the pivot at the end
- Traverse the list
  - If visited element is larger, do nothing
  - Else swap with "storage point"
    + shift storage right
    (storage point is on left initially)
- Swap pivot with storage point



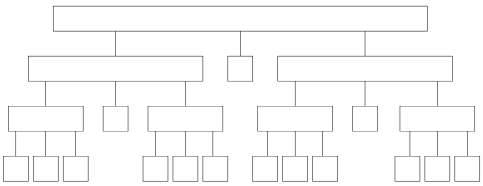
```
def quicksort(array:Array[Int]) {
  def lambda(array:Array[Int], left:Int, right:Int, pivotIndex:Int) {
    val pivotValue = array(pivotIndex)
    array.swap(pivotIndex, right) // Move pivot to end (swap() does not exist)
    val storeIndex = left
    for (i <- left to right-1) {
      if (array(i) <= pivotValue) {
        array.swap(i, storeIndex)
        storeIndex = storeIndex + 1
    } }
    array.swap(storeIndex, right) // Move pivot to its final place
    lambda(array, left, pivotIndex, (pivotIndex-left)/2)
    lambda(array, pivotIndex, right, (right-pivotIndex)/2)
  }
  lambda(array, 0, array.length-1, array.length/2) }
```


In-place QuickSort Complexity (1/2)

Best case for divide-and-conquer algorithms: Even Split

- Split the amount of work by 2 at each step (thus $\Theta(log(n))$ recursive calls)
- Work on each subproblem linear with its size (thus each call in $\Theta(n)$)

The recursion tree for best case:



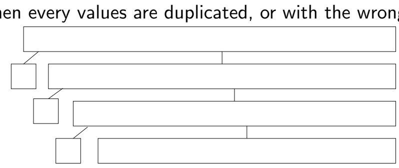
What if we split 1%/99% at each step (instead of 50%/50%)?

- We get $100 \times \log(n)$ steps $\rightsquigarrow$ whole algorithm in $\Theta(n\log(n))$

In-place QuickSort Complexity (2/2)

What if we have a fixed amount on one side?

- (happens when every values are duplicated, or with the wrong pivot)



- We get $O(n)$ steps $\rightsquigarrow$ whole algorithm in $O(n^2)$ in worst case

That's a fairly bad worst case time

- Worst than MergeSort, for example
- But called Quicksort anyway because faster *in practice* than MergeSort
- In-Place version of both algorithms are not stable
- Both can be quite easily parallelized
- Space complexity: $O(\log(n))$ (to store the recursion stack)

(this ends the second lecture)

Fourth Chapter

Correction of Software Systems

Sorting Algorithm Performance Discussion

We have shown that

	Amount of comparisons			Memory
	Best Case	Average Case	Worst Case	Complexity
Selection Sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$\Theta(1)$
Insertion Sort	$O(n)$	$O(n^2)$	$O(n^2)$	$\Theta(1)$
Bubble Sort	$O(n)$	$O(n^2)$	$O(n^2)$	$\Theta(1)$
Quick Sort	$O(n\log(n))$	$O(n\log(n))$	$O(n^2)$	$O(\log(n))$
Merge Sort	$O(n\log(n))$	$O(n\log(n))$	$O(n\log(n))$	$O(n)$

- Very accurate knowledge on achieved performance

But wait a second...

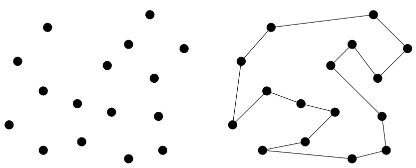
How do you know this code actually sorts the array?

- Because you see it, it's obvious (yeah, right...)
- Because the teacher / a friend says so
- Because it's written in a book / on the Internet
- Because you tested it

Because it's obvious you said?

Let's consider the following problem

- You have a robot arm equipped with a soldering iron (for example)
- You have several positions were the arm should do its soldering job
- You must decide the order in which the arm visit the positions
- You want to minimize the time (ie travel distance) it takes to visit all positions



Nearest Neighbor Tour

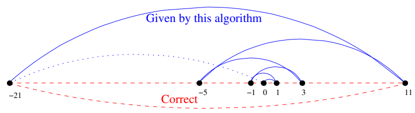
Here is a very popular algorithm to that problem

- Pick and visit an initial point p_0, and let $i = 0$
- While there are still unvisited points
  - $i = i + 1$
  - let p_i be the closest unvisited point to p_{i-1}
  - Visit p_i
- Return to p_0 from p_i

Advantage of that algorithm

- It is simple to understand and implement; It's very efficient: $O(n)$

But it is not correct!



Choosing carefully p_0
(left-most or whatever)
will **not** help

Closest Pair Tour

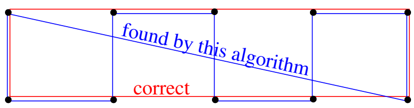
Let's try to fix our algorithm

- Walking to the closest point seems too restrictive: traps into unwanted moves
- Let's repeatedly connect the closest pairs (w/o forming cycles or 3ways branches)

The algorithm

- Let n be the number of points in the set
- For $i = 1$ to $n-1$ do
  - Let $d = \infty$
  - For each pair of endpoints (x, y) of partial paths
    - If $dist(x, y) \leq d$ then $x_m = x$, $y_m = y$, $d = dist(x, y)$
  - Connect (x_m, y_m) by an edge
- Connect the two endpoints by an edge

Works correctly for previous data, but still not correct



That's the Traveling Salesman Problem

A correct algorithm

- $d = \infty$
- For each permutation Π_i of the $n!$ existing ones
  - if $(cost(\Pi_i) \leq d)$ then
    - $d = cost(\Pi_i)$ and $P_{min} = \Pi_i$
- return P_i

Actually no known correct and polynomial algorithm

- This algorithm is very slow (exponential time)
- But that's the only correct known
- (this problem is one of the NP-Complete set, by the way)

Conclusion: never trust "obviously correct" algorithms

Other try to convince skeptics: you test it

Issues

- a *whole* load of arrays exists out there. Cannot test them all. . .
- How much should you test to get convincing? Which ones do you pick?

Let's look at another (simpler) problem

- Input: 3 integers values, representing the sides' length of a triangle
- Output: Tells whether the triangle is
  - Scalene: no two sides are equal
  - Isosceles: exactly two sides are equal
  - Equilateral: all sides are equal

Quiz: Create a set of Test Cases for this program

- Ie, the list of tests you need to write to ensure that the program is robust

Solutions – 1 point for each correct answer

- T1:
- T2:
- T3:
- T4:
- T5:
- T6:
- T7:
- T8:
- T9:
- T10:
- T11:
- T12:
- T13:

First Conclusions on Testing

About the Quiz

- All T1-T13 correspond to failures actually found in some implementations
- How many tests did you found yourself?
  < 5? $5-7$? $8-10$? > 10? All?
- Highly qualified, experienced programmers score 7.8 on average

Testing aint easy

- Finding good and sufficiently many test cases is difficult
- Even a good set of test cases cannot exclude all failures
- Without a specification, it is not clear even what a failure is

Stop academic examples, check Real Life!

Cost of Software Errors: some numbers

- \$60 billion: Estimated cost of software errors for US economy per year [2002]
- \$240 billion: Size of US software industry [2002]
  incl. profit, sales, marketing, development (50% maybe)
- 50%: estimated part of each software project spent on testing
  (spans from 30% to 80%)
- Rough estimate: money spent on testing ≈ cost of remaining errors
- That's 50% of size of software industry!

More on Testing in POO lecture, in january

- We need systematic, efficient, tool supported testing and debugging methods

To convince real skeptics, you have to **prove** correctness

- And you cannot do that without a proper specification (at least)

Fourth Chapter

Correction of Software Systems

How to prove that 'selection sort' sorts arrays?

Back to the roots: what exactly do you want to prove?

- Proper specification mandatory to proof: gives what we have, what we want
- We also need a mathematical logic to carry the proof

Hoare Logic [Hoare 1969]

- Set of logical rules to reason about the correctness of computer programs
- Central feature: description of state changes induced by code execution
- **Hoare triple:** $\{P\}\ C\ \{Q\}$
  - C is the code to be run
  - P is the **precondition** (assertion about previous state)
  - Q is the **postcondition** (assertion about next state)
  - This can be read as "If P is true, then when I run C, Q becomes true"
  - C is said to satisfy specification (P, Q)
- Such notation allows very precise algorithm specifications
- Axioms and Inference rules allow rigorous correctness demonstrations
- Note: other logics (temporal logic) proposed as replacement, but harder

Introducing (bad) joke about precise specification

While traversing Scotland, 3 people see a cow

The Economist says:

- Cows in Scotland are brown

The Logician says:

- No, no. There are cows in Scotland of which one is brown

The Computer Scientist says:

- No, no. There is at least one cow in Scotland of which one side is brown

Specification: Putting it into Practice

Back to our Example: A sorting program

```
def sort(a:Array[Int]):Array[Int] = { ... }
```

Specification V1

- Precondition: a is an array
- Post-condition: returns the sorted argument array
- Is it good enough? Not quite: `sort({2,1,2}` ⇝ `{1,2,2,17}` ☹

Specification V2

- Precondition: a is an array
- Post-condition: returns a sorted array with only elements of a

☹ `sort({3,2,1}` ⇝ `{2,2,2,2,2}`

Specification V3

- Precondition: a is an array
- Post-condition: returns a permutation of a that is sorted

☹ `sort({})` leads to unwanted behavior

Specification V4

- Precondition: a is a non-null array
- Post-condition: returns a permutation of a that is sorted

The Contract Metaphor

> *Contract is preferred specification metaphor for procedural and OO.*
> – B. Meyer, Computer 25(10)40-51, 1992

Same Principles as Legal Contract between a Client and Supplier

Supplier aka Implementer, in Java, a class or method

Client Mostly a caller object, or human user for main()

Contract One or more pairs of ensures/requires clauses defining mutual obligations of client and implementer

Meaning of a Contract: Specification of method `C@m()`

- "If a caller of `C@m()` fulfills the required Precondition, then the class `C` ensures that the Postcondition holds after `m()` finishes."

Wrong Interpretations:

- "Any caller of C@m() must fulfill the required Precondition."
- "Whenever the required Precondition holds, then C@m() is executed."

What is Design By Contract?

> *View the relationship between two classes as a formal agreement, expressing each party's rights and obligations.* – Bertrand Meyer

Example: Airline Reservation

	Obligations	Rights
Customer	▸ Be at Paris airport at least 3 hour before scheduled departure time ▸ Bring acceptable baggage ▸ Pay ticket price	▸ Reach Los Angeles
Airline	▸ Bring customer to Los Angeles	▸ No need to carry passenger who is late ▸ has unacceptable baggage ▸ or without paid ticket

- Each party expects benefits (rights) and accepts obligations
- Usually, one party's benefits are the other party's obligations
- Contract is declarative: it is described so that both parties can understand *what* service will be guaranteed without saying *how*

Testing vs. Verification

Testing

- Goal: find evidence for presence of failures
- Testing means to execute a program with the **intent of detecting failure**
- Related techniques: code reviews, program inspections
- Automatize the testing process is delayed until the POO module

Verification

- Goal: find evidence for absence of failures
- Testing cannot guarantee correctness, i.e., absence of failures
- Related techniques: code generation, program synthesis (from spec)
- This week: How can we prove the correctness of an algorithm?

Debugging

- Systematic process to find and eliminate defects leading to observed failures

Fourth Chapter

Correction of Software Systems

Back on Hoare Logic

Hoare Logic [Hoare 1969]

- Set of logical rules to reason about the correctness of computer programs
- Central feature: description of state changes induced by code execution
- **Hoare triple:** $\{P\}\ C\ \{Q\}$
  - C is the code to be run
  - P is the **precondition** (assertion about previous state)
  - Q is the **postcondition** (assertion about next state)
  - This can be read as "If P is true, then when I run C, Q becomes true"
  - C is said to satisfy specification (P, Q)
- Such notation allows very precise algorithm specifications
- Axioms and Inference rules allow rigorous correctness demonstrations
- Note: other logics (temporal logic) proposed as replacement, but harder

Game now

- See how we can **prove** that a Hoare triple holds

Assertions

What exactly is an assertion?

Definition

Formula of first order logic describing relationships between algorithm's variables

Constituted of:

- Variables of algorithm pseudo-code
- Logical connectors: $\wedge$ (and) $\vee$ (or) $\neg$ (not) $\Rightarrow$, $\Leftarrow$
- Quantifiers: $\exists$ (exists), $\forall$ (for all)
- Value-specific elements (describing integers, reals, booleans, arrays, sets, ...)

Example:

- $(x \times y = z)\ \wedge\ (x \leq 0)$
- $n^2 \geq x$

Examples of specification

Solving quadratic equations ($ax^2 + bx + c = 0$)

P: $a, b, c \in \mathbb{R}$ and $a \neq 0$

Q: $(solAmount \in \mathbb{N}) \wedge (s, t \in \mathbb{R}) \wedge$
$\quad((solAmount = 0) \vee$
$\quad\quad (solAmount = 1 \wedge as^2 + bs + c = 0) \vee$
$\quad\quad (solAmount = 2 \wedge as^2 + bs + c = 0 \wedge at^2 + bt + c = 0 \wedge s \neq t))$

Possible implementation

```
Δ = b² − 4ac
if (Δ > 0)
  s = (−b+√Δ)/2a ; t = (−b−√Δ)/2a ;
  solAmount = 2
else if (Δ = 0)
  s = −b/2a ; solAmount = 1
else (ie, Δ < 0)
  solAmount=0
```

- Here, the proof will be difficult...
- ...because it is trivial.
- Correctness comes from definitions!

Demonstration tool: inference rules

Definitions

- Inference: deducting new facts by combining existing facts correctly
- Inference rule: mechanism specifying how facts can be combined

Classical representation of each rule:

$$\frac{p_1, p_2, p_3, \ldots, p_n}{q}$$

- Can be read as "if all $p_1, p_2, p_3, \ldots, p_n$ are true, then q is also true"
- Or "in order to prove q, you have to prove $p_1, p_2, p_3, \ldots, p_n$"
- Or "q can be deduced from $p_1, p_2, p_3, \ldots, p_n$"

First axioms and rules

Empty statement axiom $\quad \overline{\{P\} skip \{P\}}$

Assignment axiom $\quad \overline{\{P[x/E]\} x := E \{P\}}$

- $P[x/E]$ is P with all free occurrences of variable x replaced with expression E
- Example:
  - P: $x = a \wedge y = b$
  - Q: $x = b \wedge y = a$
  - SWAP: algorithm achieving transition; For example: $t = x; x = y; y = t$
  - We should prove: $\{P\} SWAP \{Q\}$

Consequence rule $\quad \dfrac{P' \Rightarrow P, \{P\}\ C\ \{Q\}, Q \Rightarrow Q'}{\{P'\}\ C\ \{Q'\}}$

- P is said to be weaker than P'
- Q is said to be stronger than Q'

Rules for algorithmic constructs

Rule of composition $\quad \dfrac{\{P\} C_1 \{Q\},\ \ \{Q\} C_2 \{R\}}{\{P\} C_1; C_2 \{R\}}$

- $C_1; C_2$ means that both code are executed one after the other.
- Can naturally be generalized to more than two codes

Conditional Rule $\quad \dfrac{\{P \wedge Cond\}\ T\ \{Q\},\ \ P \wedge \neg Cond \Rightarrow Q}{\{P\}\ \textbf{if}\ Cond\ \textbf{then}\ T\ \textbf{endif}\ \{Q\}}$

Conditional Rule 2 $\quad \dfrac{\{P \wedge Cond\}\ T\ \{Q\},\ \ \{P \wedge \neg Cond\}\ E\ \{Q\}}{\{P\}\ \textbf{if}\ Cond\ \textbf{then}\ T\ \textbf{else}\ E\ \textbf{endif}\ \{Q\}}$

While Rule $\quad \dfrac{\{I \wedge Cond\}\ L\ \{I\}}{\{I\}\ \textbf{while}\ Cond\ \textbf{do}\ L\ \textbf{done}\ \{I \wedge \neg Cond\}}$

- $\{I\}$ is said to be the **loop invariant**

How to prove algorithms?

The two things to prove about algorithms

- Correction proof: when it terminates, the algorithm produce a valid result with regard to problem specification
- Termination proof: the algorithm always terminate

There is no perfect proof, only good ones

- Your main goal is to **convince** people that your code works
- If your friends are permissive, very sparse hints may be enough
- If your friends are picky, you need to provide more details
- Note that I'm gonna be very picky in exam ;)

Detailed proofs

- Most convinient way to prove an algorithm in practice: think backward
- Compute the weakest precondition you need to get the postcondition you want
- What must be the precondition of the given code to get the wanted postcondition?

Computing the Weakest Preconditions

Computing the WP to get the post-condition {Q} from C – WP(C,Q)

1. $\textbf{WP}(nop, Q) \equiv Q$
2. $\textbf{WP}(x := E, Q) \equiv Q[x := E]$
3. $\textbf{WP}(C; D, Q) \equiv \textbf{WP}(C, \textbf{WP}(D,Q))$
4. $\textbf{WP}(\texttt{if}\ Cond\ \texttt{then}\ C\ \texttt{else}\ D,\ Q)$
   $\equiv (Cond = \texttt{true} \Rightarrow \textbf{WP}(C, Q))\ \wedge\ (Cond = \texttt{false} \Rightarrow \textbf{WP}(D, Q))$
5. $\textbf{WP}(\texttt{while}\ E\ \texttt{do}\ C\ \texttt{done}\ ,Q) \equiv I$ (with I invariant, V variant)
   Plus the following proof obligations:
   - $(E = \texttt{true} \wedge I \wedge V = z) \Rightarrow \textbf{WP}(C, I \wedge V < z))$ (variant gets decremented)
   - $I \Rightarrow V \geq 0$ (variant remains valid)
   - $(E = \texttt{false} \wedge I) \Rightarrow Q$ (once done, Q is achieved)

Seems complicated, but isn't that much

- The process is automated enough to keep quite mecanical and simple
- We'll come back on this in lab (and exam ;)

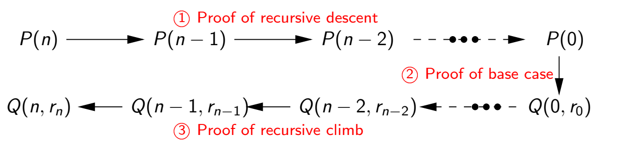
Fourth Chapter

Correction of Software Systems

Idea of the correction of recursive function

$P(n)$: Precondition at step n; $Q(n, r_n)$: Postcondition at step n with result r_n

We want to show $P(n)\ \{TREC\}\ Q(n, r_n)$



If $f(n)$ is expressed as function of $f(n-1)$, we need:

- In recursive case
  - Precondition of $f(n)$ implies precondition of $f(n-1)$ ①
    If not, the computation is impossible
  - Hyp: postcondition of $f(n-1)$ true. Proof postcondition of $f(n)$ ③
- In base case
  - precondition and computation allow to prove postcondition ②

Example of the factorial (how unexpected)

```
Function factorial(n) is
  if n == 0 then
    return 1                    /* base case */
  else
    return n × factorial(n-1)   /* recursive case */
```

$P(n)$: $n \geq 0$ cond(n): $n = 0$ $Q(n, r)$: $r = n!$ $n_{int} = n - 1$

- Descent: $P(n) \wedge \neg cond(n) \Rightarrow P(n_{int}) \equiv (n \geq 0) \wedge (n \neq 0) \Rightarrow (n - 1 \geq 0)$
  Trivial
- Base Case: $P(n) \wedge cond(n) \Rightarrow Q(n, r) \equiv n \geq 0 \wedge n = 0 \Rightarrow r = n!$
  True (since $1 = 0!$ no matter what happens – look at the base case's code)
- Recursive climb: $P(n) \wedge \neg cond(n) \wedge Q(n_{int}, r_{int}) \Rightarrow Q(n, r)$
  $\equiv (n \geq 0) \wedge (n \neq 0) \wedge (r_{int} = n_{int}!) \Rightarrow (r = n!)$

  True because:
  - $r = n \times r_{int}$ in general case (by *looking at the code*)
  - $r_{int} = n_{int}! = (n-1)!$ by induction hypothesis
  - $r = n \times (n-1)! = n!$

Proof of Termination

- Sufficient Conditions:
  - Successive values of parameter x: strictly monotonous suite (may need to specify the order)
  - Existence of an extrema x_0 verifying the stopping condition
- Remarque: that's no necessary condition
  The Syracuse suite seems to terminate without this
- Example: the factorial, of course
  - $n \geq 0$
  - n strictly decreasing
  - $0 =$ stopping condition

Fourth Chapter

Correction of Software Systems

The dark side of Software Correctness Proofs

Being picky can lead to long proofs

- Lot and lot of mathematical work to prove even simple algorithms
- Overly detailed proofs are done only when really needed:
  Aircraft, Nuclear power plant, Emergency room, ...
- But that's not impossible; One success story amongst hundreds:
  SACEM embedded system controling the train speed on the RER Line A in Paris.

Support from language / automated tools would be welcomed

- Unfortunately, Java/Scala is not Ada (or even better: Eiffel)
- Java solution (JML – Java Modeling Language): far from production ready
- Scala is better regarding algorithms' specification, but still a moving target

The bright side of Software Correctness Proofs

Sometimes you have to prove your code

- Cost/gain ratio: if you cannot afford to loose, prove it correct (nuclear plants)
- If your client wants proofs, the competitors disappear (competitive advantage)
- You're studying in Nancy, there is a local history of algorithm proofs
- There will be 1/4 of points on proofs at the exam ...

Proofs are useful even when it is not mandatory

- Expressing pre/post and loop invariant greatly helps understanding the code
- This understanding helps writing the right test cases
- And tests are not overly pleasant either (you'll see in POO!)

What is expected for the exam

- Well, that's similar to when you write code
- I don't bother a missing } in the code, as long as the idea is here
- I don't bother a partially wrong proof, as long as the method is here

(this ends the third lecture)

Fifth Chapter

Back on Recursion

Why do you want to avoid recursion

What gets done on Function Calls

1. Create a function frame on the stack
2. Push (copy) value of parameters
3. Execute function
4. Pop return value
5. Destruct stack frame

Recursion does not interfere with this schema

- Recursion can thus be less efficient than iterative solutions
- In time: function calling has a price
- In space: the call stack must be stored

Example: gcd of two natural integers

Greatest Common Divisor

gcd(a, b : Integer) = (r : Integer)

- Precondition: $a \geq b \geq 0$
- Postcondition: (a mod $r = 0$) and (b mod $r = 0$) and
  $\neg(\exists s, (s > r) \wedge (a \bmod s = 0) \wedge (b \bmod s = 0))$

Recursive Definition

if $b = 0$ **then** $r \leftarrow a$
else $r \leftarrow gcd(b, a \bmod b)$

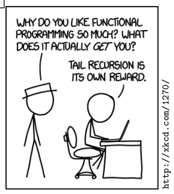
Computation of gcd(420,75)

b	0
a	15
b	15
a	30
b	30
a	45
b	45
a	75
b	75
a	420

Stack

if $b = 0$ **then** $r \leftarrow a$
else $r \leftarrow gcd(b, a \bmod b)$

- $gcd(420, 75) = gcd(75, 45) = \mathbf{15}$
- $gcd(75, 45) = gcd(45, 30) = \mathbf{15}$
- $gcd(45, 30) = gcd(30, 15) = \mathbf{15}$
- $gcd(30, 15) = gcd(15, 0) = \mathbf{15}$
- $gcd(15, 0) = 15$
  this is the Base Case



- Let's pop parameters
- $r \leftarrow r_{int}$ (no other computation: $G(x, y) = y$)
- The result of initial call is known as early as from Base Case
  This is known as **Tail Recursion**
- Factorial: multiplications during climb up
  $\Rightarrow$ **non-terminal** recursion

Transformation to Non-Recursive Form

Every recursive function can be changed to a non-recursive form

Several Methods depending on function:

- Tail Recursion: very simple transformation
- Non-Tail Recursion: two methods (only one is generic)

Compilers use these optimization techniques (amongst much others)

Non-Recursive Form of Tail Recursion

Cookbook to change Tail Recursion to Non-Recursive Form

- Generic recursive algorithm

$f(x)$:
if cond(x) **then** BaseCase(x)
else T(x); $r \leftarrow f(x_{int})$

Example: get last char of string

last(s):
if empty(s.tail) **then** $r \leftarrow s.head$
else $r \leftarrow last(s.tail)$

- Equivalent iterative algorithm

$f'(x)$:
$u \leftarrow x$
until cond(u) **do**
T(u)
$u \leftarrow h(u)$
end
BaseCase(u)

last'(s):
$l \leftarrow s$
until $empty(l.tail)$ **do**
// T(u) does nothing
l←l.tail
end
$r \leftarrow l.head$

Other Examples

nth(s,i): get char number *i* out of *s*

nth(s,i):
if n=0 **then** $r \leftarrow s.head$
else $r \leftarrow nth(s.tail, i-1)$

Two arguments, still no T(u)

nth'(s,i):
$l \leftarrow s$; $k \leftarrow i$
until k=0 **do**
l←l.tail; k←k-1
end
$r \leftarrow l.head$

is_member(s,c): assess whether *c* is member of *s*

is_member(s,c):
if empty(s) **then** $r \leftarrow FALSE$
if s.head=c **then** $r \leftarrow TRUE$
else $r \leftarrow is\_memb(s.tail)$

2 base cases, still no T(u)

is_memb'(s,c):
$l \leftarrow s$
until $empty(l)$ OR l.head=c **do**
l←l.tail
end
if empty(s) **then** $r \leftarrow FALSE$
$r \leftarrow TRUE$

Last Example

Non-Recursive Form of GCD

$gcd(a, b)$:
if $b = 0$ **then** $r \leftarrow a$
else $r \leftarrow gcd(b, a \bmod b)$

$gcd'(a, b)$:
$u \leftarrow a$; $v \leftarrow b$
until v=0 **do**
$temp \leftarrow v$
$v \leftarrow u \bmod v$
$u \leftarrow temp$
end
$r \leftarrow u$

- This is given by an immediate rewriting
- Computers are good at this kind of game (e.g., in compilers)
- Meta-programming troubling at first sight, but still fully mechanic

Non-Recursive form of Non-Tail functions

How to deal with non-tail functions?

- Previous method don't work because of those computations at recursive climb:
- Where should the ongoing computation be stored (they were stacked)?
  $fact(3) = 3 \times fact(2) = 3 \times 2 \times fact(1) = 3 \times 2 \times 1 = 3 \times 2 = 6$

what's done is no more to do

- Computing during descent $\leadsto$ nothing left at climb $\leadsto$ Tail Recursion
  $fact(3) = \boxed{3} \times fact(2) = 3 \times 2 \times fact(1) = \boxed{6} \times fact(1) = \boxed{6} \times 1 = \boxed{6} = 6$
- One extra variable is enough for the storage of "ongoing" computation
  - Since these computations are done, store their result not the stack of operations
  - Adding an extra parameter to my recursive function does the trick
  - Prototype change $\leadsto$ put recursion into a *helper* function with more parameters

Warning: this does not always work!

- Computations done out of order $\leadsto$ must be associative and commutative
- This (simple) method does not always work; another one comes afterward

Example: Changing Factorial into Tail Recursion

FACT(n):
if n = 0 **then** $r \leftarrow 1$
else $r \leftarrow n \times fact(n-1)$

FACT'(n): $\lambda(n, 1)$
$\lambda(n, acc)$:
if n = 0 **then** $r \leftarrow acc$
else $r \leftarrow \lambda(n-1, acc \times n)$

Step-by-step approach

0. (it works because the addition is commutative and associative)
1. Create a lambda function doing the recursion, with more parameters
   - Local copy of the parameters carrying the recursion
   - Add as many accumulators as operations done on climb up
2. Main function simply calls the lambda function
   - Copy of the parameters carrying the recursion
   - Initialize accumulators with base case's value (often identity element)
3. Body of the lambda function:
   - General treatment: as before, but do intermediate ops into the accumulators
   - Base case: get result directly from the accumulators

Let's take another example

Computation of a string's length

len(str):
if empty(s) **then** 0
else 1+len(cdr(s))

len'(str) = λ(str,0)

λ(str,acc):
if empty(str) **then** acc
else λ(cdr(str), acc+1)

0. (works because addition is commutative and associative)
1. Create a λ function doing the recursion, adding one accumulator per operation
2. Main function: calls the lambda function and initializes the parameters
3. Body of the λ function:
   - General treatment: as before, but do intermediate ops into the accumulator(s)
   - Base case: get result directly from the accumulator(s)

Closing the loop: Non-recursive form of Factorial

```
FACT(n):
  if n = 0 then r ← 1
         else  r ← n × fact(n − 1)
```

```
FACT'(n): λ(n, 1)
λ(n, acc) :
  if n = 0 then r ← acc
          else  r ← λ(n − 1, acc × n)
```

- This function uses Tail Recursion
- ⇝ We can turn the helper into non-recursion with the method seen before
- Then, we combine everything

```
λ'(n, acc) :
  td ← n; a ← acc
  until td = 0 do
    a ← a × td      // beware of the
    td ← td − 1     // updates' order
  end
  return a
```

```
FACT''(n):
  td ← n; a ← 1
  until td = 0 do
    a ← a × td
    td ← td − 1
  end
  return a
```

- These two transformations are simple, automatic and neat. . .
- . . . when applicable!!! ☹ If not, let's get angry and mean!

Generic Algorithm Using a Stack

Idea

- Processors are sequential and execute any recursive function
  ⇒ *Always possible to express without recursion*
- Principle: simulating the function stack of processors
  By using a stack explicitly

Example with only one recursive call

```
if cond(x) then r ← g(x)
        else  T(x); r ← G(x, f(x_int))
```

```
p ← emptyStack
a ← x (* a: locale variable *)
(* pushing on stack (descent) *)
until cond(a) do
  push(p, a)
  a ← h(a)
end
r ← g(a) (* Base Case *)
(* poping from stack (climb up) *)
until stackIsEmpty(p) do
  a ← top(p); pop(p); T(a)
  r ← G(a, r)
end
```

Remark:
If $h()$ is invertible, no need for a stack: parameter reconstructed by $h^{-1}()$

Stopping Condition = counting calls

Non-Recursive Form of Hanoï Towers (1/2)

```
HANOI(n,a,b,c): (a to b, with c as disposal)
  if n > 0 then hanoi(n-1, a, c)
                move(a, b)
                hanoi(n-1, c, b)
```

One should mimic processor behavior wrt stacking

- H(4,a,b,c) = H(3,a,c,b)+D(a,b)+H(3,c,b,a)
- Compute first unknown term: H(3,a,c,b) = H(2,a,b,c)+D(a,c)+H(2,b,c,a)
- Compute first unknown term: H(2,a,b,c) = H(1,a,c,b)+D(a,b)+H(1,c,b,a)
- Compute first unknown term: H(1,a,c,b) = D(a,c)
- Take on something casted aside: H(1,c,b,a) = D(c,b)
- and so on until everything casted aside is done (until stack is empty)

We get:

H(4,a,b,c) = D(a,c)+D(a,b)+D(c,b)+D(a,c)+H(2,b,c,a)+D(a,b)+H(3,c,b,a)

(D(a,c)+D(a,b)+D(c,b) = H(2,a,b,c); D(a,c)+D(a,b)+D(c,b)+D(a,c)+H(2,b,c,a) = H(3,a,c,b))

Non-Recursive Form of Hanoï Towers (2/2)

```
hanoi_derec(n, A, B) :
  push (n, A, B, 1) on stack
  while (stack non-empty)
    (n, A, B, CallKind) ← pop()
    if (n > 0)
      if (CallKind == 1)
        push (n, A, B, 2) on stack (* Cast something aside for later *)
        push (n-1, A, C, 1) (* Compute first unknown soon *)
      else /* ie, CallKind == 2 */
        move(A, B)
        push (n-1, C, B, 1) on stack
```

```
HANOI(n,a,b):
  if n > 0 then hanoi(n-1, a, c)
                move(a, b)
                hanoi(n-1, c, b)
```

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 8	Step 9	Step 11	Step 13
				0AB1					
			1AC1	1AC2	0BC1		0AB1		
		2AB1	2AB2	2AB2	2AB2	1CB1	1CB2	0AB1	
	3AC1	3AC2	3AC2	3AC2	3AC2	3AC2	3AC2	3AC2	2BC1
4AB1	4AB2	4AB2	4AB2	4AB2	4AB2	4AB2	4AB2	4AB2	4AB2

hanoi(4,a,b)=D(ac)+D(ab)+D(cb)+D(ac)+. . .

Rq: simpler iterative algorithms exist (they are not automatic transformations)

J.C. Fournier. *Pour en finir avec la dérécursivation du problème des tours de Hanoï*. 1990.
http://archive.numdam.org/article/ITA_1990__24_1_17_0.pdf

Fifth Chapter

Back on Recursion

Combinatorial Search and Optimization

Large class of Problems with similar algorithmic approach

- Solutions are really numerous; A set of constraints make some solution invalids
- Combinatorial Search ⇝ look for any valid solution
  Combinatorial Optimization ⇝ look for the solution maximizing a function

Examples

- Open the lock: Find the right 4-digits combination out of 10000
- Knapsac: Ali-Baba searches object set fitting in bag maximizing the value
- Minimum Spanning Tree of a given graph
- Traveling Salesman: visit n cities in order minimizing the total distance

First Resolution Approach: Exhaustive Search

- Study *every* solutions
  ⇝ Test all lock combinations
  ⇝ Enumerate all possible knapsack contents + get max value
- This often reveals to be exponential and thus infeasible

Better Approach?

Guessing the right number can become difficult that way

- 0001 ⇝ *no*; 0002 ⇝ *no*; 0003 ⇝ *no*; 0004 ⇝ *no*; 0005 ⇝ *no*; Booooring
- Let's more information: length of correct suffix instead of yes/no answers
  0001 ⇝ 0; 0002 ⇝ 0; 0004 ⇝ 1; 0024 ⇝ 2; 0424 ⇝ 3; 5424 ⇝ 4, *bingo*

This leads to a much more efficient algorithm:

- Guess each position by testing every digit in that pos until response increases
- That's even easy to write by mixing recursion with a for loop:

```
search(current,pos,len): // initial values: search({0,0,0,0}, 0, 0)
  for n ∈ [0; 9] do
    put n into current at position pos
    if try(current) > len then search(current,pos+1, try(current))
                         else // no luck. Let's test the next value of n
```

This is Backtracking

- Tentative choices + cut branches leading to invalid solutions (backtrack)
- Restrict study to valid solutions only ⇝ if bag is full, don't stuff something else
- Also factorize computations ⇝ only sum up once the N first objects' value

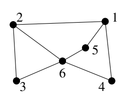
Back-tracking

Characterization

- Search for a solution in given space:
  - Choice of a (valid) partial solution
  - Recursive call for the rest of the solution
- Some built solutions are dead-ends
  (no way to build a valid solution with choices made so far)
- Backtracking then mandatory for *another* choice
- General Schema: **Recursive Call within an Iteration**

First example: Independent Sets

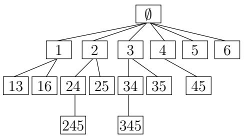
- Sets of vertices not interconnected by any graph edge
- Init: set of 1 element; Algo: increase size as much as possible then backtrack



- {1}, {1, 3}. Stuck. Remove 3. {1, 6}. Stuck.
  Removing 6 is not enough, remove everything.
- {2}, {2, 4}, {2, 4, 5} (Stuck; remove 5 then 4) {2, 5}
- {3}, {3, 4}, {3, 4, 5}, {3, 5}; {4}, {4, 5}; {5}, {6}

Algorithm Computation Time

Solution Tree of this Algorithm



- Traverse every nodes (without building it explicitly)
- Amount of algorithm steps = amount of solutions
- Let n be amount of nodes

Amount of solutions for a given graph?

- Empty Graph (no edge) $\rightsquigarrow I_n = 2^n$ independent sets
- Full Graph (every edges) $\rightsquigarrow I_n = n + 1$ independent sets
- On average $\rightsquigarrow I_n = \sum_{k=0}^{n} \binom{k}{n} 2^{-k(k-1)/2}$

n	2	3	4	5	10	15	20	30	40
I_n	3,5	5,6	8,5	12,3	52	149,8	350,6	1342,5	3862,9
2^n	4	8	16	32	1024	32768	1048576	1073741824	1099511627776

- Backtracking algorithm traverses I_n nodes on average
- An exhaustive search traverses 2^n nodes

Other example: n queens puzzle

Goal:

- Put n queens on a $n \times n$ board so than none of them can capture any other

Algorithm:

- Put a queen on first line
  There is n choices, any implying constraints for the following
- Recursive call for next line

Pseudo-code `put_queens(int line, board)`

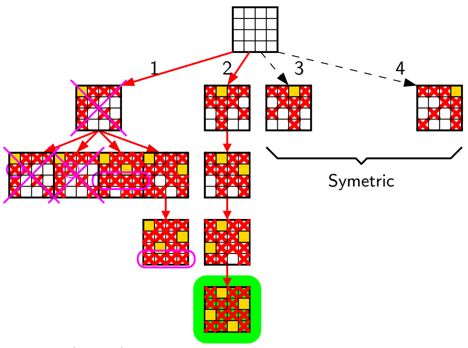
If *line* > *line_count*, return `board` (success)
∀ *cell* ∈ *line*,

- Put a queen at position *cell* × *line* of `board`
- If conflict, then return (stopping descent – failure)
- (else) call `put_queens(ligne+1, board` ∩ {*cell*, *line*})

⇒ Recursive Call within a Loop

Solving the 4 queens puzzle

- At each step of recursion, iterate on differing solutions
- Each choice induces impossibilities for the following
- For each iteration, one descent
- When stuck, climb back (and descent in following iteration)
- Until we find a solution (or not)



Scala implementation of n queens puzzle

```
def Solution(board:Array[Array[Boolean]], line:Int) {
  if (line >= board.length) // Base Case
     return true;

  for (col <- 0 to board.length - 1) {  // loop on possibilities
     if (validPlacement(board, line, col)) {
       putQueen(board, line, col);
       if (Solution(plateau, line + 1)) // Recursive Call
           return true; // Let solution climb back
       removeQueen(board, line, col);
     }
  }
  return false;
}
```


Some Principles on Backtracking

- Study “depth first” of solution tree
- On backtracking, restore state as before last choice
  Trivial here (parameters copied on recursive call), harder in iterative
- Strategy on branch ordering can improve things
- Progressive Construction of boolean function
- If function returns false, there is no solution
- Probable Combinatorial Explosion (4^4 boards)
  ⇒ Need for heuristics to limit amount of tries

Conclusion on Recursion

Essential Tool for Algorithms

- Recursion in Computer Science, induction in Mathematics
- Recursive Algorithms are frequent because easier to understand ...
  (and thus easier to maintain)
  ...but maybe slightly more difficult to write (that's a practice to get)
- Recursive programs maybe slightly less efficient...
  ...but always possible to transform a code to non-recursive form
  (and compilers do it)
- Classical Functions: Factorial, gcd, Fibonacci, Ackerman, Hanoï, Syracuse, ...
- Sorting Functions: MergeSort and QuickSort are amongst the most used (because efficient)
- BackTracking: exhaustive search in space of *valid* solutions
- Data Structure module: several recursive datatypes with associated algorithms
- Recursion is the root of computation since it trades description for time.
  – "Epigrams in Programming", by Alan J. Perlis of Yale University.

(this ends the fourth lecture)

Sixth Chapter

Testing

Back on Proofs

Most of you don't see the point of proofs. I know

- I even agree (to a given point: we need 2 legs to code – theory and practice)

> *Beware of bugs in the above code; I have only proved it correct, not tried it.* – D.E. Knuth.

- Cost/gain ratio: if you cannot afford to loose, prove it correct
  I hope that Nuclear Power Plants are [partially] proved
- Can give you a competitive advantage:
  With these, you may accept more complicated contracts
- You're studying in Nancy, there is a local history of algorithm proofs
- There will be 1/4 of points on proofs at the exam ...

What's expected at the exam

- Well, that's similar to when you write code
- I don't bother a missing } in the code, as long as the idea is here
- I don't bother a partially wrong proof, as long as the method is here

And now, let's see that tests that you guys prefer are even worst

Introduction

Bugs

- Bugs are inevitable in complex software system
- A bug can be very visible or can hide in your code until a much later date

Bugs can hide very well

Aug 2009 8 years old bug found in Linux (handling not implemented kernel fctions)
`http://www.theregister.co.uk/2009/08/14/critical_linux_bug/`
Jan 2010 Microsoft fixes a 17 years old bug (in code allowing NT to run 16bits apps)
`http://www.esecurityplanet.com/features/article.phpr/3860131/Microsoft-Warns-About-17-year-old-Windows-Bug.htm`
July 2008 25 years old bug found in BSD (`seekdir()` wrongly implemented)
`http://www.vnode.ch/fixing_seekdir`
July 2008 33 years old bug found in Unix (buffer overflow in YACC)
`http://www.computerworld.com/s/article/9108978/Developer_fixes_33_year_old_Unix_bug`

Chasing bugs

- Once identified, use print statements of IDE's debugger to hunt them down
- But how to discover all bugs in the system, even those with low visibility?

⇒ Testing and Quality Assurance practices

Why to test?

> *Testing can only prove the presence of bugs, not their absence.*
> – E. W. Dijkstra

Perfect Excuse

- Don't invest in testing: system will contain defects anyway

Counter Arguments

- The more you test, the less likely such defects will *cause harm*
- The more you test, the more *confidence* you will have in the system

Who should Test?

Fact: Programmers are not necessarily the best testers

- Programming is a constructive activity: try to make things work
- Testing is a destructive activity: try to make things fail

In practice

- Best case: Testing is part of quality assurance
  - done by developers when finishing a component (unit tests)
  - done by a specialized test team when finishing a subsystem (integration tests)
- Common case: done by rookies
  - testing seen as a beginner's job, assigned to least experienced team members
  - testing often done after completion (if at all)
  - but very difficult task; impossible to completely test a system
- Worst case (unfortunately very common too): no one does it
  - Not productive ⇒ not done [yet], postponed "by a while"
  - But without testing, productivity decreases, so less time, so less tests

> *Debugging is twice as hard as writing the code in the first place. Therefore, if you write the code as cleverly as possible, you are, by definition, not smart enough to debug it.* – Kernighan

What is "Correct"?

different meanings depending on the context

Correctness

- A system is correct if it behaves according to its specification

⇒ An absolute property (i.e., a system cannot be "almost correct")
⇒ ... undecideable in theory and practice

Reliability

- The user may rely on the system behaving properly
- Probability that the system will operate as expected over a specified interval

⇒ Relative property (system mean time between failure (MTTF): 3 weeks)

Robustness

- System behaves reasonably even in circumstances that were not specified

⇒ Vague property (specifying abnormal circumstances ⇝ part of the requirements)

Terminology

Avoid the term "Bug"

- Implies that mistakes somehow creep into the software from the outside
- Imprecise because mixes various "mistakes"

Error: incorrect software behavior

- A deviation between the specification and the running system
- A manifestation of a defect during system execution
- Inability to perform required function within specified limits
- *Example*: message box text said "Welcome null."
- Transient error: only with certain inputs; Permanent error: for any input

Fault: cause of error

- Design or coding mistake that may cause abnormal behavior
- *Example:* account name field is not set properly.
- A fault is not an error, but it can lead to them

Failure: particular instance of a general error, caused by a fault

Quality Control Techniques

Large systems bound to have faults. How to deal with that?

Fault Avoidance: Prevent errors by finding faults before the release

- Development methodologies:
  Use requirements and design to minimize introduction of faults
  Get clear requirements; Minimize coupling
- Configuration management: don't allow changes to subsystem interfaces
- [Formal] Verification: find faults in system execution
  Maturity issue; Assumes requirements, pre/postconditions are correct & adequate
- Review: manual inspection of system by team members
  shown effective at finding errors

Fault Detection: Find existing faults without recovering from the errors

- Manual tests: Use debugger to move through steps to reach erroneous state
- Automatic Testing: tries to expose errors in planned way ← **We are here**

Fault Tolerance: When system can recover from failure by itself

- Recovery from failure (*example:* DB rollbacks, FS logs)
- Sub-system redundancy (*example:* disk RAID-1)

Testing Concepts

Recapping generic terms

- Error: Incorrect software behavior
- Fault: Cause of the error (programming, design, etc)
- Failure: Particular instance of a general error, caused by a fault

Component

- A part of the system that can be isolated for testing (through *stub* and *driver*)

⇒ an object, a group of objects, one or more subsystems

Test Case

- {inputs; expected results} set exercising component to cause failures
- Boolean method: whether component's answer matches expected results
- "expected results" includes exceptions, error codes ...

Test Stub

- Partial implementation of components on which the tested compnt depends
- dummy code providing necessary input values and behavior to run test cases

Test Driver

- Partial implementation of a component that depends on the tested part
- a "main()" function that executes a number of test cases

Tests Campaign Planing

Goal

- Should *verify* the requirements (are we building the product right?)
- NOT *validate* the requirements (are we building the right product?)

Definitions

- **Testing**: activity of executing a program with the intent of finding a defect
  ⇒ A successful test is one that finds defects!
- **Testing Techniques**: Techniques to find yet undiscovered mistakes
  ⇒ Criterion: Coverage of the system
- **Testing Strategies**: Plans telling *when* to perform *what* testing technique
  ⇒ Criterion: Confidence that you can safely proceed with the next activity

Sixth Chapter

Testing

White Box Testing

Focuses on internal states of objects

Use internal knowledge of the component to craft input data

- *Example:* internal data structure = array of size 256
  ⇒ test for size = 255 and 257 (near boundary)
- Internal structure include design specs (like diagram sequence)
- Derive test cases to maximize structure coverage, yet minimize # of test cases

Coverage criteria: Path testing

- every statement at least once
- all portions of control flow (= branches) at least once
- all possible values of compound conditions at least once (condition coverage)
  Multiple condition coverage ⇝ all true/false combinations for all simple conditions
  Domain testing ⇝ {a < b; a == b; a > b}
- all portions of data flow at least once
- all loops, iterated at least 0, once, and N times (loop testing)

Main issue: white box testing negates object encapsulation

Black Box Testing

Component ≡ "black box"

Test cases derived from external specification

- Behavior only determined by studying inputs and outputs
- Derive tests to maximize coverage of spec elements yet minimizing # of tests

Coverage criteria

- All exceptions
- All data ranges (incl. invalid input) generating different classes of output
- All boundary values

Equivalence Partitioning

- For each input value, divide value domain in classes of equivalences:
  - Expects value within [0, 12] ⇝ negative value, within range, above range
  - Expects fixed value ⇝ below that value, expected, above
  - Expects value boolean ⇝ {true, false}
- Pick a value in each equivalence class (randomly or at boundary)
- Predict output, derive test case

Sixth Chapter

Testing

Testing Strategies

Unit testing

⇝ Looks for errors in objects or subsystems

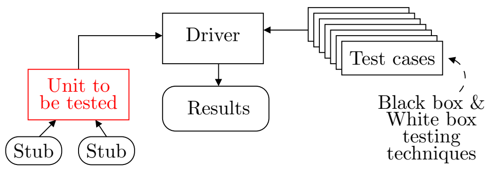
Integration testing

⇝ Find errors with connecting subsystems together
- System structure testing: integration testing all parts of system together

System testing

⇝ Test entire system behavior as a whole, wrt use cases and requirements
- functional testing: test whether system meets requirements
- performance testing: nonfunctional requirements, design goals
- acceptance testing: done by client

Unit Testing



Why?

- Locate small errors (= within a unit) fast

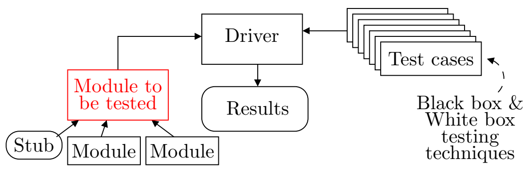
Who?

- Person developing the unit writes the tests

When?

- At the latest when a unit is delivered to the rest of the team
  - No test ⇒ no unit
- Write the test first, i.e. before writing the unit
  ⇒ help to design the interface right

Integration Testing



Why?

- Sum is more than parts, interface may contain faults too

Who?

- Person developing the module writes the tests

When?

- Top-down: main module before constituting modules
- Bottom-up: constituting modules before main module
- In practice: a bit of both

Remark: Distinction between unit testing and integration testing not that sharp

Regression Testing

Ensure that things that used to work still work after changes

Regression test

- Re-execution of tests to ensure that changes have no unintended side effects
- Tests must avoid regression (= degradation of results)
- Regression tests must be repeated *often*
  (after every change, every night, with each new unit, with each fix,...)
- Regression tests may be conducted manually
  - Execution of crucial scenarios with verification of results
  - Manual test process is slow and cumbersome
    ⇒ preferably completely automated

Advantages

- Helps during iterative and incremental development + during maintenance

Disadvantage

- Up front investment in maintainability is difficult to sell to the customer
- Takes a lot of work: more test code than production code

Acceptance Testing

Acceptance Tests

- conducted by the end-user (representatives)
- check whether requirements are correctly implemented
  borderline between verification ("Are we building the system right?")
  and validation ("Are we building the right system?")

Alpha- & Beta Tests

- Acceptance tests for "off-the-shelves" software (many unidentified users)
- Alpha Testing
  - end-users are invited at the developer's site
  - testing is done in a controlled environment
- Beta Testing
  - software is released to selected customers
  - testing is done in "real world" setting, without developers present

Other Testing Strategies

Recovery Testing

- Forces system to fail and checks whether it recovers properly
- ⇝ For fault tolerant systems

Stress Testing (Overload Testing): Tests extreme conditions

- e.g., supply input data twice as fast and check whether system fails

Performance Testing: Tests run-time performance of system

- e.g., time consumption, memory consumption
- first do it, then do it right, then do it fast

Back-to-Back Testing

- Compare test results from two different versions of the system
- ⇝ requires N-version programming or prototypes
  `git` version control system does so to isolate regressions (`bisect` command)

Sixth Chapter

Testing

Tool support

Test Harness

- Framework merging all test code in environment
- Main example for Java is called JUnit
- It inspired CppUnit, PyUnit, ...

Test Verifiers

- Measure test coverage for a set of test cases
- JCov for Java, gcov for gcc, ...

Test Data Generators

- Assist in selecting test data
- Based on the formal specification such as JML

Sixth Chapter

Testing

- Introduction
- Testing Techniques
  - White Box Testing
  - Black Box Testing
- Testing Strategies
  - Unit Testing
  - Integration Testing
  - Regression Testing
  - Acceptance Testing
- Testing in Practice
  - JUnit
  - Right BICEP + CORRECT
- Other Techniques for Practical Software Correctness
  - Design By Contract
  - Fuzzing
  - Formal Methods
- Conclusion

Introduction

What is JUnit?

- It is a unit testing framework for Java.
- It provides tools for easy implementation of unit test plans
- It eases execution of tests
- It provides reports of test executions

What is NOT JUnit?

- It cannot design your test plan
- It does only what you tell it to
- It does not fix bugs for you

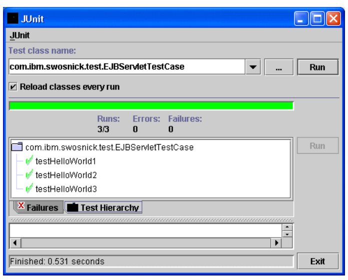
JUnit has two major versions

- JUnit 3.x: uses convention on method naming
- JUnit 4.x: uses Java 5 annotations

Structure of JUnit tests

Running a test suite consists of

- Setting up test environment
- For each test
  - Setting test up
  - Invoking test function
  - Tearing test down
- Tearing down everything
- Report result



Setting up test environment

Purpose

- Get things ready for testing.
- Create common instances, variables and data to use in tests.

Two kinds may co-exist

- Setting up before each test function
  - Named `public void setUp()` in JUnit 3.x
  - Annotated `@Before` in JUnit 4.x
- Setting up once for all
  - Placed in constructor in JUnit 3.x
  - Annotated `@BeforeClass` in JUnit 4.x

Cleaning test environment: Tearing down methods

Purpose

- Clean up after testing
- I.e., closing any files or connexions, etc
- Not used as often as setup methods

Two kinds may co-exist

- Tearing down after each test function
  - Named `public void tearDown()` in JUnit 3.x
  - Annotated `@After` in JUnit 4.x
- Tearing down once for all (JUnit 4.x only)
  - Annotated `@AfterClass`

Actually doing the tests

Test functions

- It is where the tests are performed
- Need one function per test case (which may call helper functions)
- Name must start with `test` in JUnit 3.x
- Annotated `@Test` (in JUnit 4.x)

Verifying results

- All tests are verified with assertions.
- JUnit comes with an Assert class for this purpose
  - public void assertTrue(String message, boolean condition)
  - public void assertNotNull(String message, Object obj)
  - public void assertEquals(String message, Object expected, Object actual)
  - public void assertSame(String message, Object expected, Object actual)
    uses ==, not .equals()
  - public void assertFalse(String message, boolean condition)
  - public void assertNotEquals(String message, Object expected, Object actual)
  - public void assertNotSame(String message, Object expected, Object actual)
  - public void fail(String message)

Example: Combination Lock (1/2)

Data and setting up

```
public class CombinationLockTest {
    // Locks with the specified combinations
    private CombinationLock lock00; // comb. 00
    private CombinationLock lock03; // comb. 03
    private CombinationLock lock12; // comb. 12
    private CombinationLock lock99; // comb. 99
    @Before
    public void setUp () {
        lock00 = new CombinationLock(0);
        lock03 = new CombinationLock(3);
        lock12 = new CombinationLock(12);
        lock99 = new CombinationLock(99);
    }
    ...
}
```

Tear down not necessary here

- object data will be deallocated automatically
- setup method overwrites instance variables

Example: Combination Lock (2/2)

Simple test method

```
@Test
public void testOpenLock () {
    lock12.enter(3);
    lock12.enter(4);
    assertTrue(lock12.isOpen());
}
```

Test method with helper

```
@Test
public void testFirstDigitTwice () {
    closeLocks();
    firstDigitTwice(lock03,0,3);
    firstDigitTwice(lock12,1,2);
}
private void firstDigitTwice(CombinationLock lock, int first, int second) {
    lock.enter(first);
    lock.enter(first);
    assertFalse(lock.isOpen());
    lock.enter(second);
    assertTrue(lock.isOpen());
}
```


Going further with JUnit: TDD

Test-Driven Development

- That's a methodology to write code
- Aims at ease/productivity + code quality

Principle: Write the Test Cases First (before the code)

- Ensures that the codes actually get written
- Improves the interface: you're user of your own code before coding it

That's easy and pleasant to do

- It's one of the "agile development methodologies", very light-weighted
  More than just TDD in agile methods (but too long to say it all here)
- Eclipse correction suggestion and ability to generate stubs very helpful
- Try it for your next project

Right BICEPS

Thinking of all mandatory test cases is difficult

- I.e., challenging to discover all the ways a code might fail
- Good news: Experience quickly gives a feel for what is likely to fail

Beginners need a bit of help (until they get experienced)

- Guidelines on what can fail
- Reminders of areas that are important to test
- These guidelines are not very complex, but quite useful/powerful
- See Software Systems and Architecture [Scott Miller] for details

Right-BICEP

Guidelines in a Nutshell

- Right: Are the results right?
- B: Are all the boundary conditions CORRECT?
- I: Can you check inverse relationships?
- C: Can you cross-check results using other means?
- E: Can you force error conditions to happen?
- P: Are performance characteristics within bounds?

Right?

- We need to compute what the correct result should be to test
- Quite often these can be inferred from the specification
- If the "right" results cannot be determined... you shouldn't be writing code!
- If spec not completed [by client], assume what's correct, and fix afterward

B: Boundary Tests (1/3)

Discovering boundary conditions is crucial!

- This is where most of the bugs generally live
- These are also the "edges" of our code

Remember our little experience

- We had to refine it several time our specifications
  - Triangle with negative length
  - Sort an empty array
- The algorithm in exercise 3 of proof lab were false
  - Failed to find smallest value if at the end of the array

B: Boundary Tests (2/3)

Example of boundary conditions

- Totally bogus, inconsistent input values: filename of "#()*%)Q*#%&@"
- Badly formatted data: e-mail address without TLD (zastre@foo)
- Empty or missing values: 0, 0.0, "", null
- Values above some reasonable expectation: age of 10,000; #children == 30
- Duplicates in lists meant to be free of duplicates
- Ordered lists that are not ordered
  Also: Presorted lists passed to sort algorithms? reverse-sorted?
- Things which arrive out of order? or out of expected order?

B: Boundary Tests (3/3)

Another guideline for boundaries: CORRECT

- Conformance: Does the value conform to an expected value?
- Ordering: Is the set of values ordered or unordered as appropriate?
- Range: Is the value within reasonable minimum and maximum values?
- Reference: Does the code reference anything external that isn't under control?
- Existence: Does the value exist? (e.g., is non-null, non- zero, present in a set)
- Cardinality: Are there exactly enough values?
- Time: Is everything happening in order? At the right time? In time?

I: Check Inverse Relationships

Some methods can be checked almost trivially

- Data inserted in table should appear in a search immediately afterwards
- Lossless compression algorithm $\rightsquigarrow$ data uncompressed to the original value
- Check square-root calculation by squaring result (ensure it is "close enough")

Inverse Gotchas

- You usually write the function/method and its inverse
- What if both are buggy? errors gets be masked
- Ideally, the inverse function is written by somebody else
  - Square root example: use built-in multiplication
  - Database insert: vendor-provided search routine to test insert

C: Cross-check Using Other Means

Idea 1: For methods without an inverse

- Use a different algorithm to compute the result, and compare to yours
- For example, use a $O(n^2)$ sorting algorithm to check your $O(n \times log(n)$ one

Idea 2: Ensure that different pieces of the class's data "add up"

- Example: library system with book, copies
- "books out" + "books in library" should equal "total copies"

E: Force Error Conditions

Production code defensive to system failures

- Disks: fill up
- Networks: go down
- E-mail: gets lost
- Programs: crash

This also should to be tested

- Easy: invalid parameters
- Harder: environmental errors

Environmental errors/constraints

- Out of memory; Out of disk space
- Network availability and errors
- System load
- Limited colour palette; Very high or very low video resolution
- ...

P: Performance Characteristics

What is the time performance as:

- Input size grows?
- Problem sets become more complex

Idea: "regression test" on performance characteristics

- Ensure that version N+1 is not awfully slower than version N

Very hard to ensure

- Bad performance can come from external factors
- Performance not portable from machine to machine
- (even harder to ensure *automatically*)

When to stop writing tests?

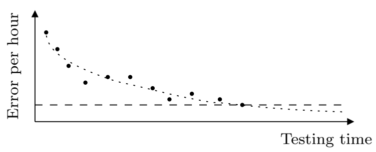
> *Testing can only prove the presence of defects, not their absence.*
> – E. W. Dijkstra

Cynical answer (sad but true)

- You're never done: each run of the system is a new test
  $\Rightarrow$ Each bug-fix should be accompanied by a new regression test
- You're done when you are out of time/money
  - Include test in project plan and **do not give in to pressure**
  - ... in the long run, tests **save** time

Statistical testing

- Test until you've reduced failure rate under risk threshold



Why to test? (continued)

Because it helps ensuring that the system matches its specification

But not only (more good reason to test)

- Traceability
  - Tests helps tracing back from components to the requirements that caused their presence
- Maintainability
  - Regression tests verify that post-delivery changes do not break anything
- Understandability
  - Newcomers to the system can read the test code to understand what it does
  - Writing tests first encourage to make the interface really useable

Sixth Chapter

Testing

Introduction

Design by Contract

- Programming methodology trying to prevent code to diverge from specs
- Mistakes are possible
  - while transforming requirements into a system
  - while system is changed during maintenance

What's the difference with Testing?

- Testing tries to diagnose (and cure) errors after the facts
- Design by Contract tries to prevent certain types of errors

Design by Contract is particularly useful in an Object-Oriented context

- preventing errors in interfaces between classes
  incl. subclass and superclass via subcontracting
- preventing errors while reusing classes
  incl. evolving systems, thus incremental and iterative development
  Example of the Ariane 5 crash

Use Design by Contract in combination with Testing!

What is Design By Contract?

> *View the relationship between two classes as a formal agreement, expressing each party's rights and obligations.* – Bertrand Meyer

Example: Airline Reservation

	Obligations	Rights
Customer	▸ Be at Paris airport at least 3 hour before scheduled departure time ▸ Bring acceptable baggage ▸ Pay ticket price	▸ Reach Los Angeles
Airline	▸ Bring customer to Los Angeles	▸ No need to carry passenger who is late ▸ has unacceptable baggage ▸ or has not paid ticket

- Each party expects benefits (rights) and accepts obligations
- Usually, one party's benefits are the other party's obligations
- Contract is declarative: it is described so that both parties can understand *what* service will be guaranteed without saying *how*

Connecting back to Hoare logic

Pre- and Post-conditions + Invariants

- Obligations are expressed via pre- and post-conditions

> *If you promise to call me with the precondition satisfied, then I, in return promise to deliver a final state in which the postcondition is satisfied.*

pre-condition: $x >= 9$ post-condition: $x >= 13$
component: x := x + 5

- and invariants

> *For all calls you make to me, I will make sure the invariant remains satisfied.*

Isn't this pure documentation?

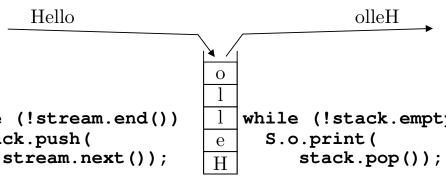
(a) Who will register these contracts for later reference (the notary)?
The source code

(b) Who will verify that the parties satisfy their contracts (the lawyers)?
The running system

Example: Stack

Specification

- Given
  - A stream of characters, length unknown
- Requested
  - Produce a stream containing the same characters but in reverse order
  - Specify the necessary intermediate abstract data structure




```
while (!stream.end())
  stack.push(
     stream.next());
```

```
while (!stack.empty())
  S.o.print(
     stack.pop());
```


Example: Stack Specification

```
class stack
 invariant: (isEmpty (this)) or (! isEmpty (this))
 /* Implementors promise that invariant holds after all methods return
    (incl. constructors)*/
 public char pop ()
    require: !isEmpty(this)          /* Clients' promise (precondition) */
    ensure: true                     /* Implementors' promise (postcondition)
                                        Here: nothing */
 public void push(char)
    require: true
    ensure: (!isEmpty(this))         /* Implementors' promise:
            and (top(this)==char)       Matches specification */
 public void top (char) : char
    require: ...                     /* left as an exercise */
    ensure: ...
 public void isEmpty() : boolean
    require: ...
    ensure: ...
```


Defensive Programming

Redundant checks

- Redundant checks are the naive way for including contracts in the source code

```
public char pop () {
  if (isEmpty (this)) {
    ... //Error-handling
  } else {
    ...}
```

This is redundant code: it is the responsibility of the client to ensure the pre-condition!

Redundant Checks Considered Harmful

- Extra complexity
  due to extra (possibly duplicated) code ... which must be verified as well
- Performance penalty
  Redundant checks cost extra execution time
- Wrong context
  - How severe is the fault? How to rectify the situation?
  - A service provider cannot asses the situation, only the consumer can.
  - Again: What happens if the precondition is not satisfied?

Assertions

Any boolean expression we expect to be true at some point

Advantages

- Help in writing correct software (formalizing invariants, and pre/post-conditions)
- Aid in maintenance of documentation (specifying contracts in the source code)
  ⇒ tools to extract interfaces and contracts from source code
- Serve as test coverage criterion (Generate test cases that falsify assertions)
- Should be configured at compile-time (to avoid performance penalties in prod)

What happens if the precondition is not satisfied?

- When an assertion does not hold, throw an exception

Assertions in Programming Languages

Eiffel
- Eiffel is designed as such ... but only used for correction (not documentation)

C++
- assert.h does not throw an exception, but close program
- Possible to mimick. Documentation extraction rather difficult

Smalltalk
- Easy to mimic, but compilation option requires some language idioms
- Documentation extraction is possible (style JavaDoc)

Java
- Assert is standard since Java 1.4 ... very limited
- JML provide a mechanism ... but not ported to Java 5 (damn genericity)
- Modern Jass seems very promising, but needs more polishing

Design by Contract vs. Testing

They serve the same purpose
- Design by contract *prevents* errors; Testing *detect* errors

⇝ One of them should be sufficient!

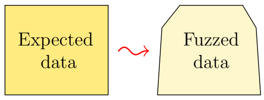
They are complementary
None of the two guarantee correctness ... but the sum is more than the parts
- Testing detects wide range of coding mistakes
  ... design by contract prevents specific mistakes due to incorrect assumption
- Design by contract ease black box testing by formalizing spec
- Condition testing verify whether all assertions are satisfied
  (whether parties satisfy their obligations)

Sixth Chapter

Testing

Fuzzing big picture

1. Intercept the data an application gets from its environment
2. Fuzz it, i.e. provide data vaguely resembling to expected one
   (sort of model-checking, exploring only execution paths close to the usual one)



Why? Motivation
- It's a security assessment method:
  if you can get the application to segfault, it must be a buffer overflow to exploit
- Easy to write some tests w/o system knowledge
  ⇒ launch a fuzzer when arriving in company, it may find something interesting
- One of the best price/quality ratio

Target applications
- Classically, network protocols; Recently used on media files

How to actually fuzz the data?

Brute force: random
- Pick a byte in the stream, and change its value
- ☺ Really easy to do
- ☹ Get easily caught by checksums
- ☹ "Interesting changes" are rather improbable

Refined manner: templating
- Understand the logic of the stream
- ☺ Pass checksums and easy validity protection levels
- ☹ Rather hard to do
- ☹ Fuzzing process becomes stream-dependent
  no longer first tool to use: you need to understand stream first

Some research leads
- Stateful fuzzing: Build a protocol automaton, test invalid transitions
- Automatic templating: Extend `file` format metagrammar,use almost valid data

Sixth Chapter

Testing

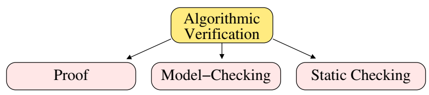
- Introduction
- Testing Techniques
  - White Box Testing
  - Black Box Testing
- Testing Strategies
  - Unit Testing
  - Integration Testing
  - Regression Testing
  - Acceptance Testing
- Testing in Practice
  - JUnit
  - Right BICEP + CORRECT
- Other Techniques for Practical Software Correctness
  - Design By Contract
  - Fuzzing
  - Formal Methods
- Conclusion

Formal Methods

Goal: Develop safe software using automated methods
- Strong mathematical background
- Safe ≡ respect some given properties

Kind of properties shown
- Safety: the car does not start without the key
- Liveness: if I push the break paddle, the car will eventually stop

Existing Formal Methods



Proof of programs
- In theory, applicable to any class of program
- In practice, quite tedious to use
  often limited to help a specialist doing the actual work (system state explosion)

Model-checking
- Goal: Shows that a system:

(safety) never evolves to a faulty state from a given initial state
(liveness) always evolve to the wanted state (stopping) from a given state (breaking)

- ☹ Less generic than proof: lack of faulty states for all initial state?
- ☺ Usable by non-specialists (at least, by *less-specialists*)

Static Checking
- Automatic analyse of the source code (data flow analysis; theorem proving)
- Completely automated, you should use these tools, even if partial coverage

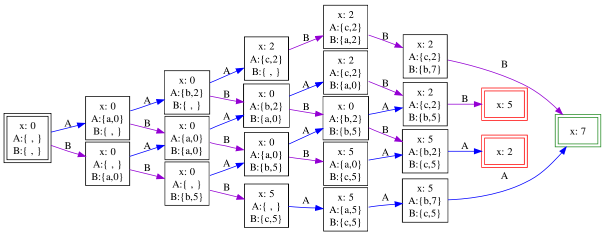
Example of problem to detect: Race Condition

x is a shared variable; *Alice* adds 2, *Bob* adds 5; Correct result : $x = 7$

a. Read the value of shared variable x and store it locally
b. Modify the local value (add 5 or 2)
c. Propagate the local variable into the shared one

- Execution of *Alice* **then** *Bob* or opposite: result = 7
- Interleaved execution: result = 2 or 5 (depending on last propagator)

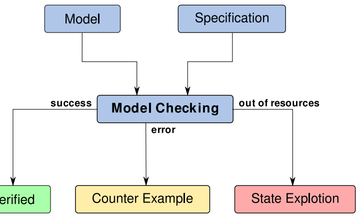
Model-checking: traverse graph of executions checking for properties



- Safety: assertions on each node
- Liveness by studying graph (cycle?)

Model-Checking Big Picture

1. User writes Model (formal writing of algorithm) and Specification (set of properties)
2. Each decision point in model (if, input data) $\rightsquigarrow$ a branch in model state space
3. Check safety properties on each encountered node (state)
4. Store encountered nodes (to avoid looping) and transitions (to check liveness)
5. Process until:
   - State space completely traversed ($\Rightarrow$ model verified against this specification)
   - One of the property does not hold (the path until here is a counter-example)
   - We run out of resource ("state space explosion")



Sixth Chapter

Testing

A Comparison of Bug Finding Tools for Java

Rutar, Almazan, Foster, U. Maryland – ISSRE04

Example code

```
1 import java.io.*;
2 public class Foo{
3  private byte[] b;
4  private int length;
5  Foo(){ length = 40;
6   b = new byte[length]; }
7  public void bar(){
8   int y;
9   try {
10   FileInputStream x =
11    new FileInputStream("z");
12   x.read(b,0,length);
13   x.close();}
14  catch(Exception e){
15   System.out.println("Oopsie");}
16  for(int i=1; i<=length; i++){
17   if (Integer.toString(50) ==
18             Byte.toString(b[i]))
19    System.out.print(b[i] + " ");
20  }
21 }
22 }
```

Defects of this code

- l8 W: Unused variable
- l12 W: Return of read() ignored
- l14 W: Stream may not be closed
- l17 E: == to compare strings
- l18 E: may acces out of bound
- l18 FP: possible null-dereference to b

Some tools for Java

Name	Input	Interface	Technology
Bandera	Source	CL, GUI	Model checking
ESC/Java	Source+spec	CL, GUI	Theorem proving
FindBugs	Bytecode	CL,GUI,Ant,IDE	Syntax, data-flow
JLint	Bytecode	CL	Syntax, data-flow
PMD	Source	CL,GUI,Ant,IDE	Syntax

No tool is perfect/sufficient

- All detect something
- Some give false positive (b inited in ctor)
- All have false negative

Conclusion

Failure is not an option. It comes bundled with software.

- Testing searches for defects
  But not that easy and endless?
- Proof tries to ensure that there is no defect
  But quite heavy-weighted
- Automatic tools (static checking, theorem provers) may help
  But none is enough/sufficient (false negatives); all have false positives
- Design by Contract constitutes a global (methodological) answer
  Too bad that nobody use it / that Java offers no tool support for it (yet?)

Optimistic Last Note

- That's a hot research topic, things move fast
- Tools improve quickly, you really should learn to use them
- Methodologies exist (DBC, TDD), you should try to follow them

Bibliography for this chapter (and previous one)

Lectures

- Testing, Debugging, and Verification (W. Ahrendt, R. Hähnle – U. Göteborgs)
  `www.cse.chalmers.se/edu/year/2009/course/TDA566`
- Software Engineering (Serge Demeyer – U. Antwerpen)
  `http://www.lore.ua.ac.be/Teaching/SE3BAC/`
- Software Systems and Architecture (Scott Miller – U. of Victoria)
  `http://samiller.ece.uvic.ca/courses/SENG271/2009/05/`
- JUnit (Dirk Hasselbalch – U. Copenhagen)
  `http://isis.ku.dk/kurser/blob.aspx?feltid=217458`

Other

- Test Infected: Programmers Love Writing Tests (Tutorial on JUnit)
  `http://www.cril.univ-artois.fr/~leberre/MI32001/TESTING/junit3.7/doc/testinfected/ntesting_fr.htm`