Epistemic reasoning in Lineland

Philippe Balbiani  Olivier Gasquet  François Schwarzentruber

IRIT - Université Paul Sabatier
118 Route de Narbonne F-31062 TOULOUSE CEDEX 9

Abstract

In this paper we investigate a concrete situation of epistemic spatial reasoning: there are agents (humans, cameras,...) which are located on a line, and according to their respective positions and to where agents are looking at, the epistemic situation will vary. Using standard epistemic logic language, we aim at expressing various properties ("Agent A knows that agent B sees agent C", ...) For the case of the line, we provide here a qualitative geometry of what agents see, and give an axiomatic theory for both this geometry and for the corresponding epistemic reasoning. We also provide algorithms for the model checking and satisfiability problems and show that they are PSPACE-complete.

Keywords: Epistemic reasoning, modal logic, axiomatization, complexity.

1 Introduction

This work is concerned with epistemic geometrical reasoning. Initially, it was motivated by multi-agents applications where agents need to reason about what they see or not, and about what they know that other agents see or not. One may think of multi-players games for example, where we aim at being able to formalize that some agent, just by seeing where are her partners, knows that no enemy could sneak upon her from behind without being seen by the partners.

While many authors in Artificial Intelligence and Computer Science [5] developed epistemic logic and others have studied qualitative spatial reasoning [8] [4], fewer works concern their combination (but we can cite [7] and [6] which combine a spatial modal operator dealing with topology and an epistemic modal operator). For sure, one must then ask question about how knowledge is founded; in this article, we choose to investigate the case where factual knowledge is based on what agents see. More precisely, we consider a framework where agents can see both other agents and where they are looking at. We do not provide operators in the language to deal with space but only an epistemic operator for each agent in the language.

Of course, our aim is to tackle concrete situations in the plane or in the space, but in this paper we will focus on one dimension: agents are disposed along a line, looking right or left (see example of fig. 1). We will see that this simple case is already hard from the computational point of view (both model checking and
satisfiability are PSPACE-complete). Interestingly, the obvious semantics induced by such situations can be axiomatized as shown in Section 4, thus providing a basis for a theory of knowledge about some qualitative geometry, which, we believe is the necessary condition for tackling the problems of model checking and the satisfiability problems in dimensions 2 and 3 and provide reasonable algorithms.

Fig. 1. Example of a lineworld

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Let $AGT$ be a countable set of agents with typical members denoted $a$, $b$, .... In this paper, the language $\mathcal{L}_{PK}$ of our epistemic theory is defined by the rule:

$$\varphi ::= a \triangleright b \mid \bot \mid \neg \varphi \mid (\varphi \lor \varphi) \mid K_a \varphi$$

where $a, b \in AGT$. The formula $a \triangleright b$ is read “agent $a$ sees agent $b$” and is called a perception literal. The formula $K_a \varphi$ is read “agent $a$ knows that $\varphi$ is true”. As usual $\top = \text{def} \neg \bot$, $(\varphi \land \psi) = \text{def} \neg(\neg \varphi \lor \neg \psi)$, $\hat{K}_a \varphi = \text{def} \neg K_a \neg \varphi$, $(\varphi \rightarrow \psi) = \text{def} (\neg \varphi \lor \psi)$ and $(\varphi \leftrightarrow \psi) = \text{def} ((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi))$. We will also be interested by the perception fragment $\mathcal{L}_P \subseteq \mathcal{L}_{PK}$ without epistemic operators:

$$\varphi ::= a \triangleright b \mid \bot \mid \neg \varphi \mid (\varphi \lor \varphi)$$

where $a, b \in AGT$. Formulas in $\mathcal{L}_P$ are called perception formulas.

Definition 2.1 A lineworld $w$ is a tuple $<, \text{dir}>$ where:

- $<$ is a strict total order over $AGT$, that is to say:
  - $<$ is irreflexive: for all $a \in AGT$, $a \not< a$;
  - $<$ is transitive: for all $a, b, c \in AGT$, if $a < b$ and $b < c$ then $a < c$;
  - $<$ is trichotomous: for all $a, b \in AGT$, we have $a < b$, $b < a$ or $a = b$.
- $\text{dir} : AGT \rightarrow \{\text{Left}, \text{Right}\}$.

The set of all lineworlds is noted $W$. From the relation $<$ and the function $\text{dir}$, we can define if an agent $a$ sees another agent $b$. The semantics of $w \models a \triangleright b$ is intuitive: agent $a$ sees agent $b$ iff either $b$ is on the left of $a$ and $a$’s direction is left or $b$ is on the right of $a$ and $a$’s direction is right. Formally: $w \models a \triangleright b$ iff either $(\text{dir}(a) = \text{Left}$ and $b < a)$ or $(\text{dir}(a) = \text{Right}$ and $a < b)$. Truth conditions for Boolean connectives are standard.

Example 2.2 Let us reconsider the lineworld $w$ depicted in Figure 1. We have $w \models a_1 \triangleright a_3$ because $\text{dir}(a_1) = \text{Right}$ and $a_1 < a_3$. Note that agents are transparent: here agent $a_2$ is transparent and agent $a_1$ sees beyond agent $a_2$. 

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Given \( w \in W \) and \( a \in AGT \), we note \( V(a)_w = \{ b \in AGT \mid w \models a \triangleright b \} \). It is the set of all agents that agent \( a \) sees in the lineworld \( w \).

**Definition 2.3** Let \( a \in AGT \). We define the epistemic relation \( R_a \) on the set of worlds \( W \). For all \( w = \langle <_w, dir_w \rangle \in W \) and \( v \in W \), we have \( wR_av \) iff there exists \( u = \langle <_u, dir_u \rangle \) such that:

- \( V(a)_w = V(a)_u \);
- for all \( b \in V(a)_w \), \( dir_w(b) = dir_u(b) \);
- for all \( b, c \in V(a)_w \cup \{ a \} \), \( b <_w c \) iff \( b <_u c \).

Two worlds \( w \) and \( v \) are epistemically indistinguishable (\( wR_av \)) for agent \( a \) iff agent \( a \) sees exactly the same things in both worlds. Note that \( R_a \) is an equivalence relation on \( W \). For all agents \( a \) and for all lineworlds \( w \), \( R_a(w) \) denotes the set of all lineworlds \( u \) such that \( wR_u \). We define \( w \models K_a \psi \) iff for all \( u \in R_a(w) \), \( u \models \psi \).

The Figure 2 shows the Kripke structure when \( AGT = \{ a, b \} \). Nodes (rectangles) represent worlds, that is to say lineworlds where agents are settled in Lineland. Edges represent relations \( R_a \) and \( R_b \).

### 3 Model checking and satisfiability

**Definition 3.1** We call model checking in lineland the following problem:

- Input: a formula \( \varphi \), a lineworld \( w \) (where only agents occurring in \( \varphi \) are taken into account);
- Output: Yes if we have \( w \models \varphi \). No, otherwise.

**Definition 3.2** We call satisfiability problem in lineland the following problem:

- Input: a formula \( \varphi \);
- Output: Yes if there exists a lineworld \( w \in W \) such that \( w \models \varphi \). No, otherwise.

**Remark 3.3** As for the standard propositional logic, the model checking problem of a given formula from \( \mathcal{L}_P \) in a given lineworld is easily proved to be in \( \text{P} \).

**Theorem 3.4** The satisfiability problem of a formula in \( \mathcal{L}_P \) is NP-complete. If we restrict the language to a fixed finite number of agents then it is in \( \text{P} \).
The model checking and the satisfiability problems in the epistemic language $L_{PK}$ are PSPACE-complete.

4 Axiomatization

Here is an axiomatization of the valid formulas in $L_{PK}$:

Boolean tautologies and modus ponens;

$(Ax_1)$ $\neg a \triangleright a$;

$(Ax_2)$ $a \triangleright b, c \land (c \triangleright a \leftrightarrow c \triangleright b) \rightarrow (\neg b \triangleright a \leftrightarrow b \triangleright c)$;

$(Ax_3)$ $\neg a \triangleright b \land \neg a \triangleright c \land (c \triangleright a \leftrightarrow \neg c \triangleright b) \rightarrow (b \triangleright c \leftrightarrow b \triangleright a)$;

$(Ax_4)$ $a \triangleright b \land a \triangleright c \rightarrow (b \triangleright a \leftrightarrow b \triangleright c)$;

$(Ax_5)$ $a \triangleright b \wedge \neg a \triangleright c \land a \triangleright d \land (b \triangleright a \leftrightarrow b \triangleright c) \land (c \triangleright a \leftrightarrow c \triangleright d) \rightarrow (b \triangleright a \leftrightarrow b \triangleright d)$;

$(Ax_6)$ $a \triangleright b \wedge a \triangleright c \wedge a \triangleright d \land (b \triangleright a \leftrightarrow \neg b \triangleright c) \land (c \triangleright a \leftrightarrow \neg c \triangleright d) \rightarrow (b \triangleright a \leftrightarrow \neg b \triangleright d)$

for all $a, b, c, d \in AGT$.

**Definition 4.1** A $G$-describing conjunction is a maximal satisfiable conjunction of literals of the form $b \triangleright c$ or $\neg b \triangleright c$ where $b, c \in G$.

**Definition 4.2** Let $G \subseteq AGT$ be such that $G$ is finite and non-empty. Let $a \in G$. We say that $\varphi$ is a $G$-what-$a$-perceives-mc iff $\varphi$ is a conjunction of literals such that there exists a subset $V \subseteq G$ such that:

- for all $b \in V$, $a \triangleright b$ appears in $\varphi$;
- for all $b \in G \setminus V$, $\neg a \triangleright b$ appears in $\varphi$;
- for all $b \in V$, for all $c \in G$, either $b \triangleright c$ or $\neg b \triangleright c$ appears in $\varphi$;
- $\varphi$ is satisfiable.

In the previous Definition, the set $V$ represents the set of agents seen by agent $a$. The first and second items correspond to the information about agents that agent $a$ sees and does not see. The third item corresponds to what agents visible to $a$ see. The fourth item implies that $a \notin V$. Now we give an axiomatization describing the interaction between knowledge and perception:

all the principles of the axiomatizations given for formulas in $L_{PK}$;

necessitation rules;

$(Ax_{K})$ $K_a(\varphi \rightarrow \psi) \rightarrow (K_a\varphi \rightarrow K_a\psi)$;

$(Ax_{7})$ $a \triangleright b \rightarrow K_a a \triangleright b$;

$(Ax_{8})$ $a \triangleright b \rightarrow K_a \neg a \triangleright b$;

$(Ax_{9})$ $a \triangleright b \land b \triangleright c \rightarrow K_a b \triangleright c$;

$(Ax_{10})$ $a \triangleright b \land \neg b \triangleright c \rightarrow K_a \neg b \triangleright c$;

$(Ax_{11})$ $\varphi \rightarrow \hat{K}_a \Phi$ where $\varphi$ is $G$-what-$a$-perceives-mc and $\Phi$ is any $G$-describing conjunction containing $\varphi$.
5 Conclusion and perspectives

We have studied an epistemic logic interpreted over lineworlds where knowledge of agents is based on what they can see. We have given a complete axiomatization and tight decision procedures for model checking and satisfiability problems.

We do not know if our epistemic logic is finitely axiomatizable. In other respects, it is unknown whether PSPACE-hardness of the model checking and the satisfiability problems still hold when we only consider construction of the form $K_a \varphi$ where $a$ is in a fixed finite set of agents $AGT' \subseteq AGT$.

We aim at extending our work to Flatland [1], i.e. interpreting formulas of $L_{PK}$ in flatworlds. A flatworld is specified by giving to any agent a position in the plane and a direction the agent is looking in. For example, in Figure 3 agent $a$ sees $b$ and $d$ but cannot see $c$. We do not know how to axiomatize valid formulas in this 2-dimensional setting. For the satisfiability problem, in [2], we have been able to polynomially reduce it to the satisfiability problem in elementary geometry[9], thus providing a decision procedure of high computational cost, since according to [3] the latter is EXPSPACE-complete.

References