CVFP (Software Design and Formal Verification)
TD 3*: Modal Logic $\mathcal{L}_K$

Exercise 1 `Let's begin with a small proof` ̈⌣

We recall that a proof (in HILBERT style) is a sequence of formulae such that each formula is an instance of a tautology, of axiom $K : \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ or obtained by previous formulae of the sequence by necessitation or modus ponens. A proof of ϕ is such a sequence whose last formula is ϕ.

Give a proof of $(\Box A \wedge \Box B) \to \Box(A \wedge B)$.

Exercise 2 `Size matters not.`

Let $\phi \in \mathcal{L}_K$ be a formula.

2.1. Define the set $SF(\phi)$ of all subformulae of ϕ.

2.2. Show card $(SF(\phi)) = |\phi|$.

2.3. Let $\mathcal{M}$ be a model and $\mathcal{M}^f$ its filtration. Prove that $\mathcal{M}, w \vDash \phi$ iff $\mathcal{M}^f, [w] \vDash \phi$.

Exercise 3 `To be or not to be SAT`

Are each of the following formulae satisfiable? Prove it formally by giving a model or a proof of unsatisfiability.

More formally, for each ϕ, does a $\mathcal{M} = (W, R, V)$ and a $w \in W$ exist such that $\mathcal{M}, w \vDash \phi$?

- $\Diamond(p \to \Box q)$
- $\Box\bot$
- $\Diamond p \wedge \Box\bot$
- $\Box\Diamond p \wedge \Diamond\bot\neg p$

*The real one this time!

Exercise 4 `Let's give a nice frame to this whole story.`

A *frame* $\mathcal{F} = (W, R)$ is a set of worlds W and a binary relation $R \subseteq W^2$. Intuitively, it's a modal logic model without valuation for atomic propositions.

Let $\mathcal{F} = (W, R)$ be a frame, $w \in W$ a world and $\phi \in \mathcal{L}_K$ a formula. We define $\mathcal{F}, w \models \phi$ iff $(\mathcal{F}, V), w \models \phi$ for all V. Furthermore, $\mathcal{F} \models \phi$ iff $\mathcal{F}, w \models \phi$ for all w.

4.1. Show that $\mathcal{F} \models p \to \Box\Diamond p$ iff R is symetric.

4.2. What property of frames is characterized by the formula $p \to \Diamond p$?

4.3. What property of frames is characterized by the formula $\Box p \to \Diamond p$?

4.4. Give a formula that expresses the transitivity of R.

Exercise 5

5.1. Design an algorithm that given a dotted model $\mathcal{M}, w$ and a formula $\phi \in \mathcal{L}_K$, decides whether $\mathcal{M}, w \models \phi$.

5.2. Show that model checking of $\mathcal{L}_K$ is in P.