MAD

Models & **A**lgorithms for **D**istributed systems

-- 5/5 --

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Today…

- A new model for distributed systems: Petri nets

- Main features
  - concurrency naturally (graphically) encoded
  - runs easily encoded as partial orders of events
  - languages encoded as branching processes and unfoldings (tightly related to the formal notion of event structure)

What do we have so far ?

Model

- network of automata $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_N$
- language = set of runs, a run = a sequence of events
- factorization $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \times \ldots \times \mathcal{L}(\mathcal{A}_N) \subseteq \Sigma^*$
- a Mazurkiewicz trace :
  - one way to recover concurrency, a run becomes a partial order of events
  - encoding of traces as tuples of local words $w \in w_1 \times \ldots \times w_N$

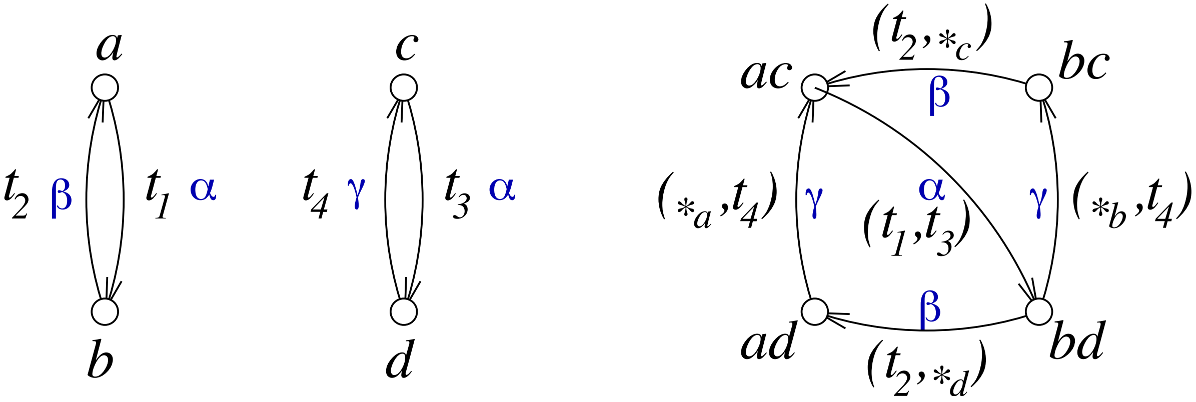
Algebra

- projection & product on (networks of) automata and languages
- rich properties $\Rightarrow$ distributed/modular computations in this algebra
- working with factorized forms is like working with traces
- application: distributed diagnosis, distributed planning

Limitations

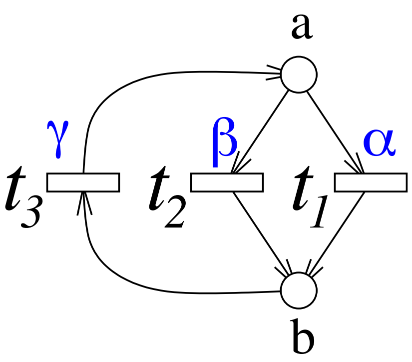
- the product of automata does not make concurrency visible (creates concurrency diamonds), and leads to state explosion
- the natural sequential semantics (runs as sequences of events) does not capture well concurrency
- traces are an indirect way to recover a true concurrency semantics from sequences, where “$a \prec b$ and $b \prec a$” is made equivalent to “$b \perp a$”; one may need to distinguish these situations :

“I can go first” $\wedge$ “you can go first” $\not\Rightarrow$ “we can go at the same time”



Petri nets

change of notation



- automaton $\mathcal{A} = (S, T, \Sigma, s_0, S_F)$
    - transitions set $T \subseteq S \times \Sigma \times S$
    - one transition $t = (s, \alpha, s') = ({}^\bullet t, \sigma(t), t^\bullet)$

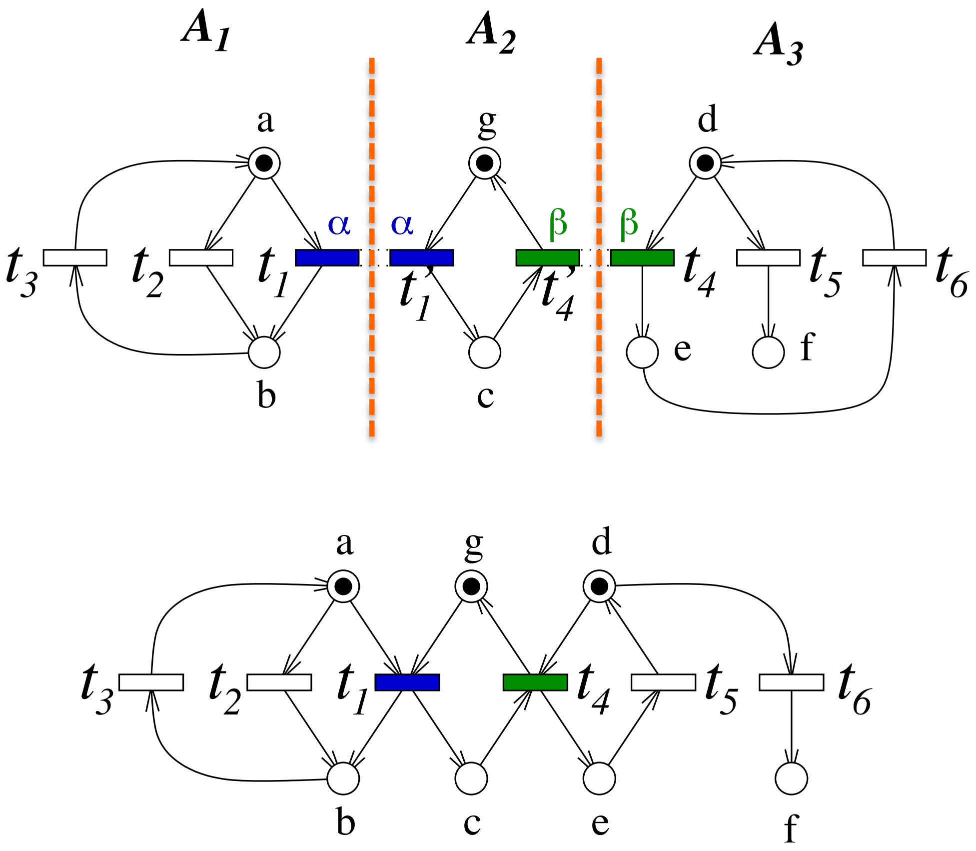
- new notation (Petri Net inspired) $\mathcal{A} = (S, T, \rightarrow, s_0, \lambda, \Lambda)$
    - S, T, Λ are finite sets of states (places), transitions, labels
    - flow connects transitions and states $\rightarrow \subseteq (S \times T) \cup (T \times S)$
    - preset $\forall x \in S \cup T, \; {}^\bullet x = \{y \in S \cup T \; : \; y \rightarrow x\}$ and sym. for postset $x^\bullet$
    - labeling of transitions $\lambda : T \rightarrow \Lambda$

Product

$$\mathcal{N} = \mathcal{A}_1 \times \mathcal{A}_2 = (P, T, \rightarrow, P_0, \lambda, \Lambda) \text{ where } \mathcal{A}_i = (S_i, T_i, \rightarrow_i, s_{0,i}, \lambda_i, \Lambda_i)$$

- **Places:**
  - disjoint union (not the product !) $P = S_1 \uplus S_2$
  - initial places $P_0 = \{s_{0,1}, s_{0,2}\}$
- **Transitions**: a single copy of each private transition
  - synchro on common labels $T = \{(t_1, t_2) \ : \ \lambda_1(t_1) = \lambda_2(t_2)\}$
  - private transitions in 1st comp. $\cup \ \{(t_1, \star) \ : \ \lambda_1(t_1) \in \Lambda_1 \setminus \Lambda_2\}$
  - private transitions in 2nd comp. $\cup \ \{(\star, t_2) \ : \ \lambda_2(t_2) \in \Lambda_2 \setminus \Lambda_1\}$
- **Flow**:
  - $\rightarrow$ is defined by ${}^\bullet(t_1, t_2) = {}^\bullet t_1 \uplus {}^\bullet t_2$ and $(t_1, t_2)^\bullet = t_1^\bullet \uplus t_2^\bullet$
  - where $\star^\bullet = \emptyset$ and ${}^\bullet\star = \emptyset$

Example



Remarks

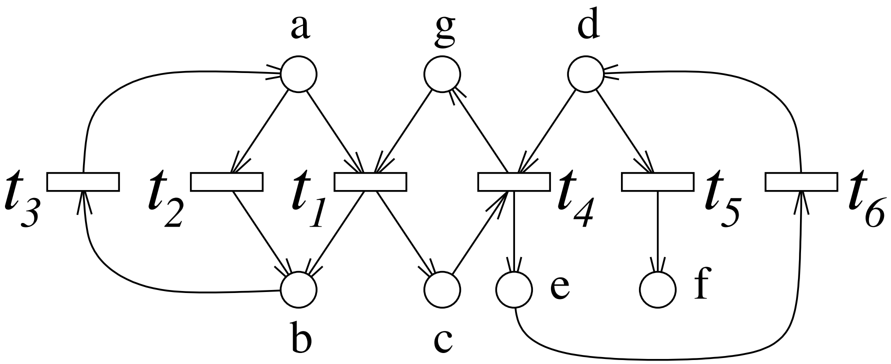
- in general, as for the product of automata, the association of transitions is not one to one
- this definition of product extends to (safe) Petri nets…
- …and makes the product associative

Dynamics

in a Petri net $\mathcal{N} = \mathcal{A}_1 \times \mathcal{A}_2 = (P, T, \rightarrow, P_0, \lambda, \Lambda)$

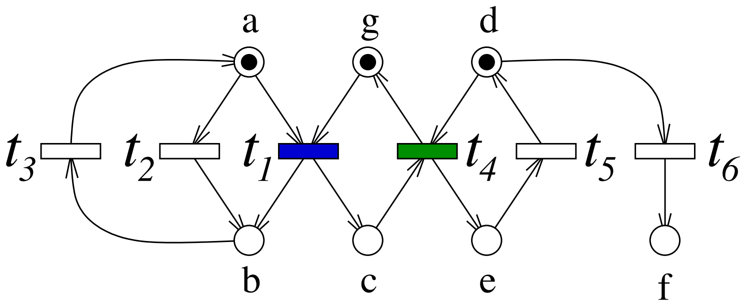
- **Marking:**
  - a function $m : P \rightarrow \mathbb{N}$
  - assigns a number of tokens to each place
  - notation : $m \subseteq P$ if places contain at most one token (safe net)

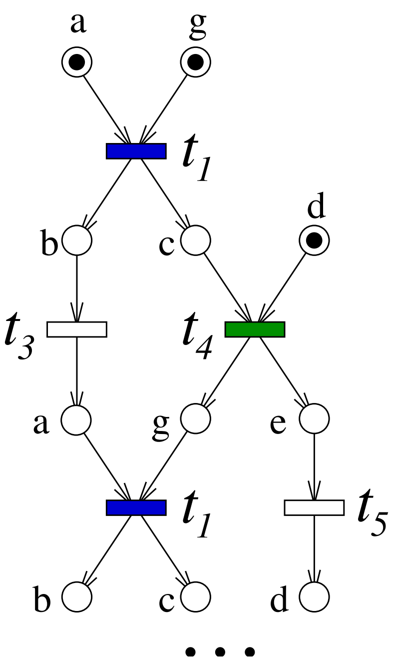
- **Enabling** of a transition
  - transition $t \in T$ is enabled at marking $m \subseteq P$ iff $^\bullet t \subseteq m$
  - the resources/tokens needed by t are present in the current marking
- **Firing** of a transition
  - it changes the current marking m into m' with $m' = m - {}^\bullet t + t^\bullet$
  - t consumes tokens in its present, and produces some in its postset



True concurrency semantics

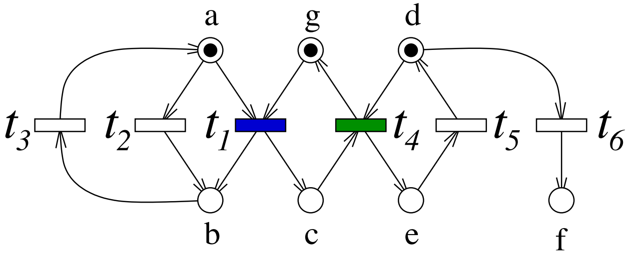
- **sequential semantics**
  - a run = a sequence of transition firings, rooted at $m_0 = P_0$
  - imposes the interleaving of concurrent events
  - different interleavings = different runs
- **true concurrency semantics**
  - a run is a partial order of events
  - encoded as another Petri net, without circuits, called a **configuration**



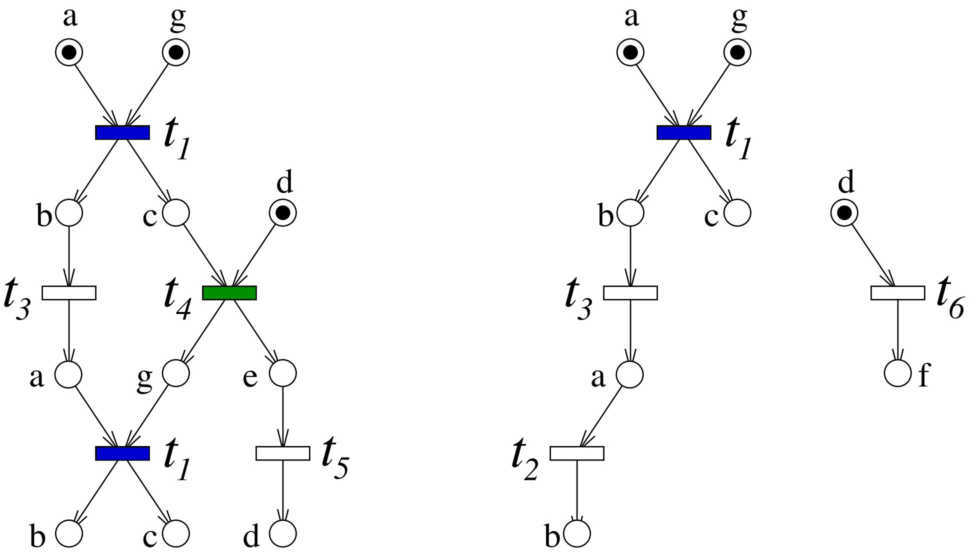


Unfoldings

A safe Petri net…



…and two of its configurations (runs), as partially ordered events



Unfoldings

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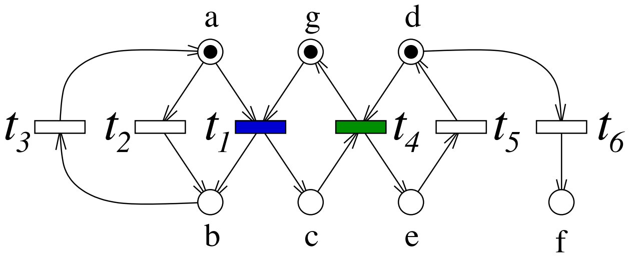
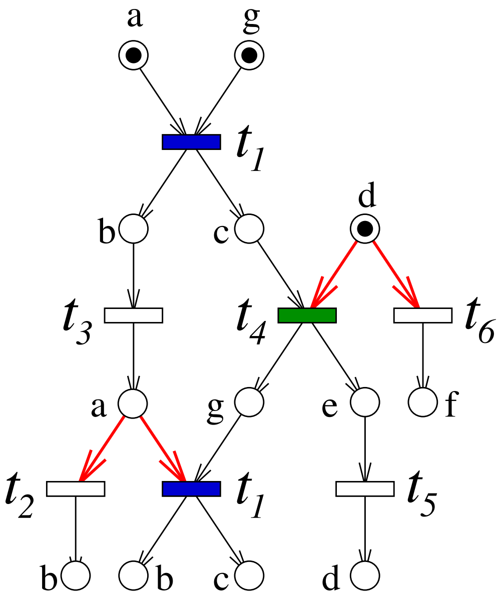


merging common prefixes yields an **occurrence net**



Occurrence net

- a special Petri net $\mathcal{O} = (C, E, \rightarrow, C_0, \lambda, \Lambda)$
- places are called conditions, transitions are called events
- the flow $\rightarrow$ is acyclic (partial ordering)
- and this partial order is well founded

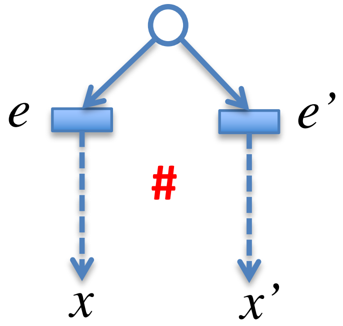
$$\forall x \in C \cup E, \quad |\{y \in C \cup E \, : y \rightarrow^* x\}| < \infty$$

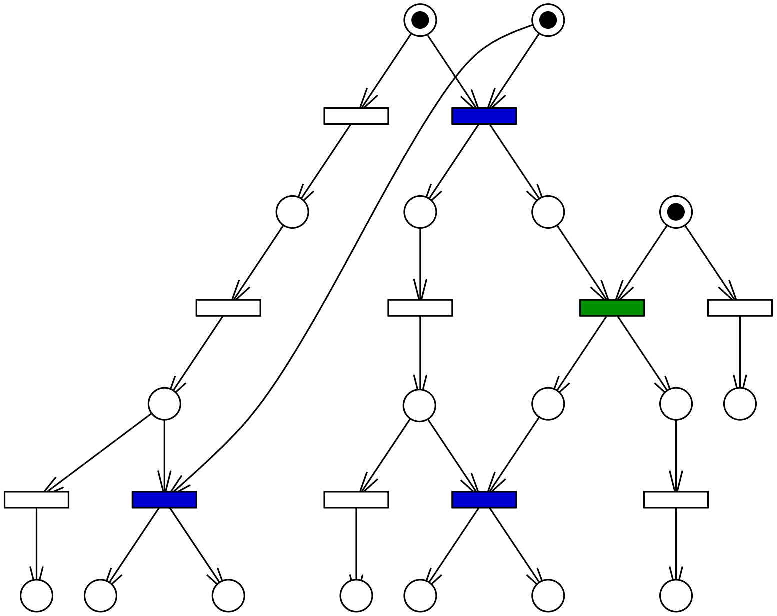
- every condition has a unique cause or is minimal

$$\forall c \in C, \quad |{}^\bullet c| \leq 1 \qquad \text{and} \qquad C_0 = \{c \in C, {}^\bullet c = \emptyset\}$$

- no event is in self-conflict

$$x \# x' \quad \Leftrightarrow \quad \exists e \neq e' \in E, \;\; {}^\bullet e \cap {}^\bullet e' \neq \emptyset, \;\; e \rightarrow^* x, \;\; e' \rightarrow^* x'$$





concurrency $x \perp y \;\Leftrightarrow\; \neg(x \rightarrow^* y) \wedge \neg(y \rightarrow^* x) \wedge \neg(x \# y)$

represents nodes that can lie in the same configuration

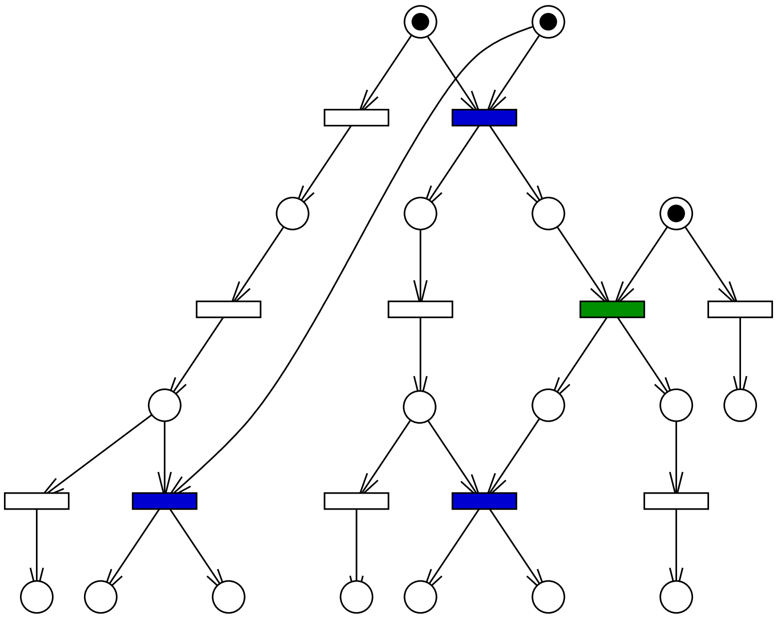
co-set : $X \subseteq C$ such that $\forall c, c' \in X, \;\; c \perp c'$

represents resources (tokens) that are available at the same time in some run/configuration

cut : a maximal co-set for $\subseteq$

prefix : $\mathcal{O}' = (C', E', \rightarrow', C_0, \lambda', \Lambda) \sqsubseteq \mathcal{O}$

iff $\mathcal{O}'$ is a causally closed sub-net of $\mathcal{O}$, containing C_0 and $E'^{\bullet}$



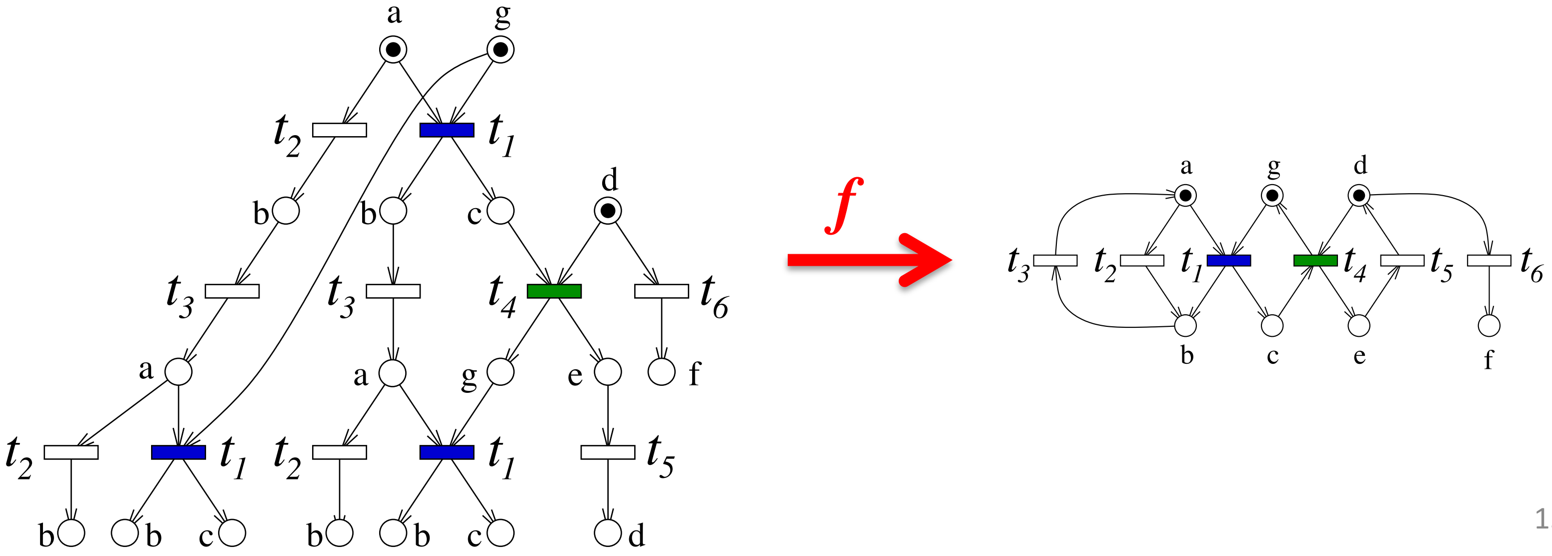
configuration : denoted κ , a conflict-free prefix of $\mathcal{O}$

local configuration : [e] = smallest configuration containing event e,
= causal past of e

Lem : relating cuts and configurations
X is a cut of $\mathcal{O}$ $\Leftrightarrow$ $\exists \kappa = (C', E', \ldots)$ such that $X = max(C')$

Branching process

- a branching process of net $\mathcal{N}$ is a pair $(\mathcal{O}, f)$
  where $\mathcal{O}$ is an occurrence net,
  and $f : \mathcal{O} \to \mathcal{N}$ a morphism of nets (a total function)
- f "labels" conditions/events of $\mathcal{O}$ by places/transitions of $\mathcal{N}$
  it turns a configuration of $\mathcal{O}$ into a run of $\mathcal{N}$
- parsimony : $\forall e, e' \in E, \quad {}^\bullet e = {}^\bullet e' \ \wedge \ f(e) = f(e') \ \Rightarrow \ e = e'$
- if X = maximal conditions in configuration κ (X forms a cut)
  then $f(X)$ is the marking of $\mathcal{N}$ produced by run κ



Unfolding

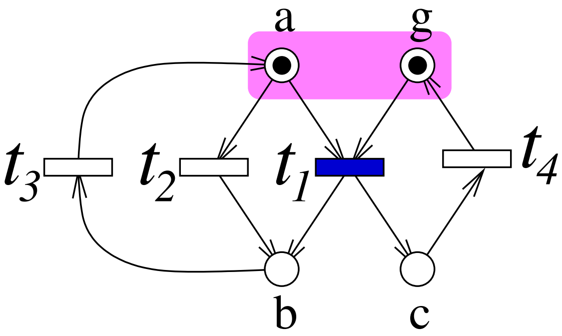
Thm : there exists a unique branching process $(\mathcal{U}_\mathcal{N}, f_\mathcal{N})$ of $\mathcal{N}$ maximal for prefix inclusion $\sqsubseteq$, it is called the unfolding of $\mathcal{N}$

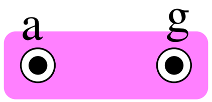
Proof : main idea is to define the union of branching processes, a little technical, but not difficult (see refs. at the end of the lesson).

Algorithm (unfolding)

- init
  - $C = C_0$, isomorphic to P_0 through f
  - $E = \emptyset, \quad \rightarrow = \emptyset$
- repeat until stability (extension with a new event)
  - for a coset $X \subseteq C$ and transition t such that $f(X) = {}^\bullet t$
  - create event $e \in E$ (if it does not already exist) such that ${}^\bullet e = X,\ f(e) = t$
  - create new conditions $X' = e^\bullet \subset C$ and extend f so that $f(X') = t^\bullet$

Example





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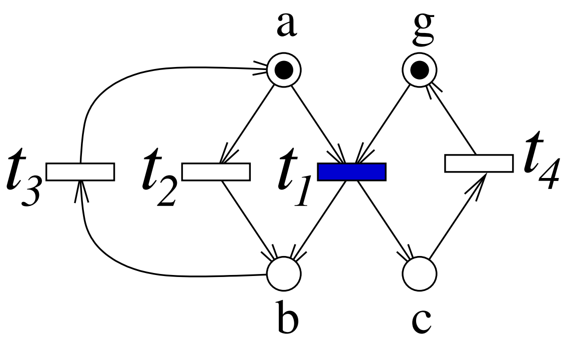

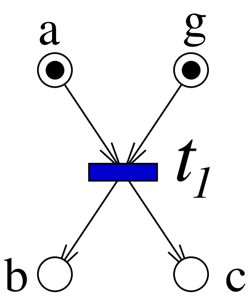

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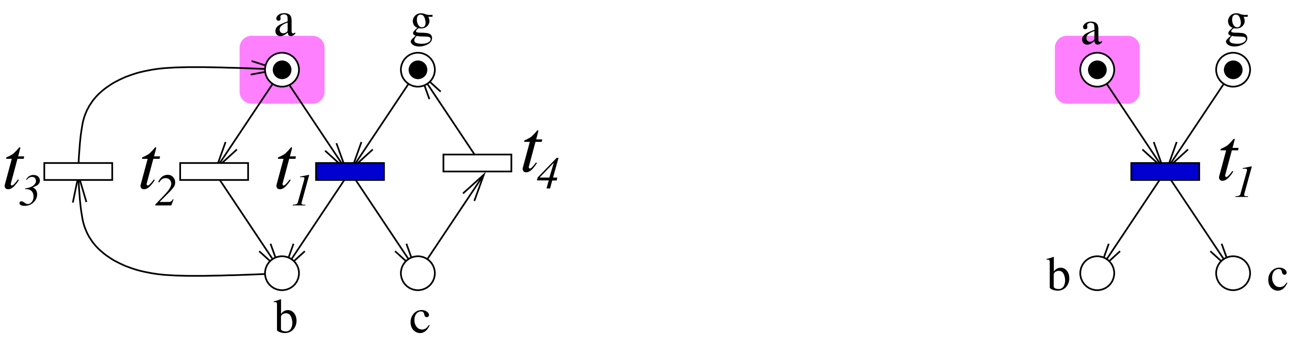

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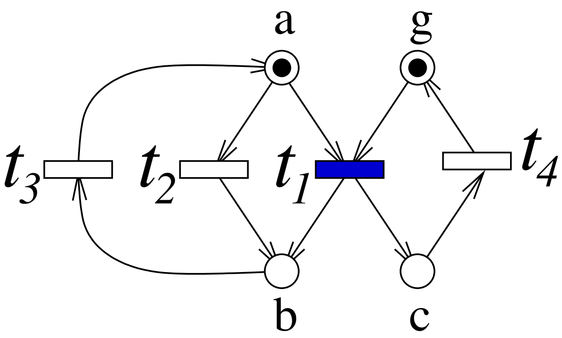

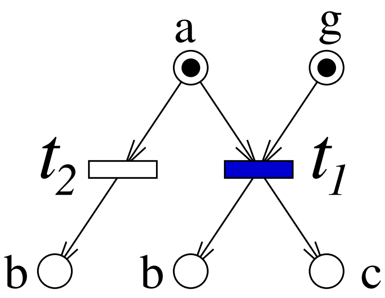

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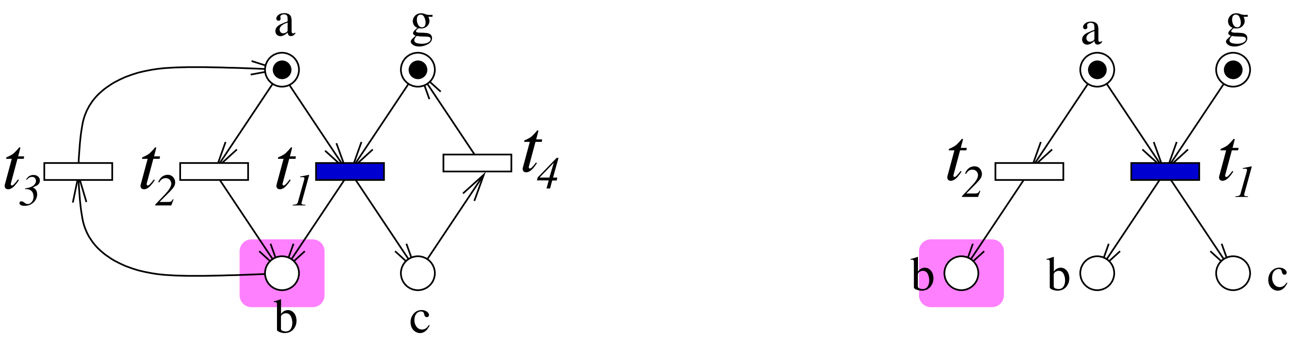

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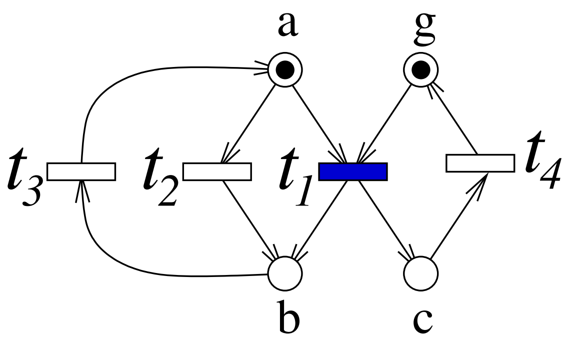

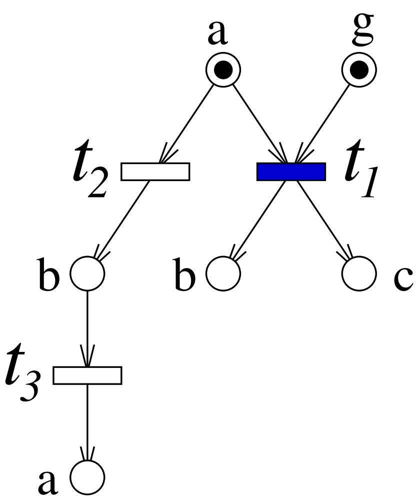

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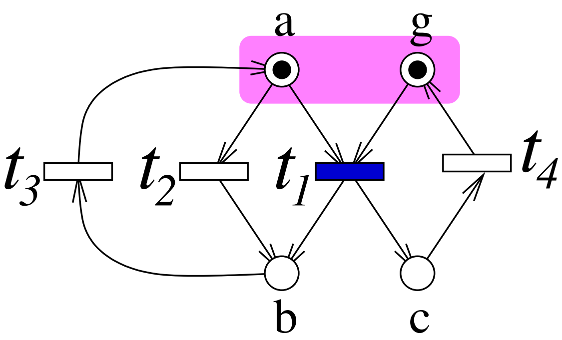

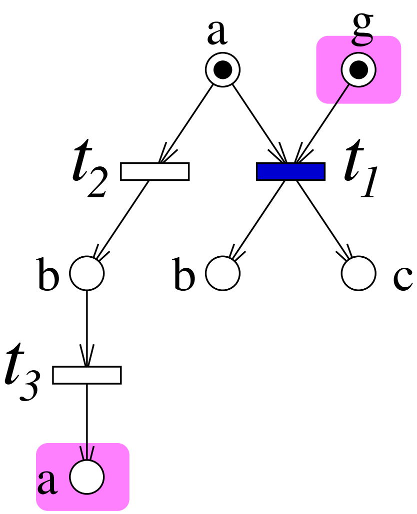

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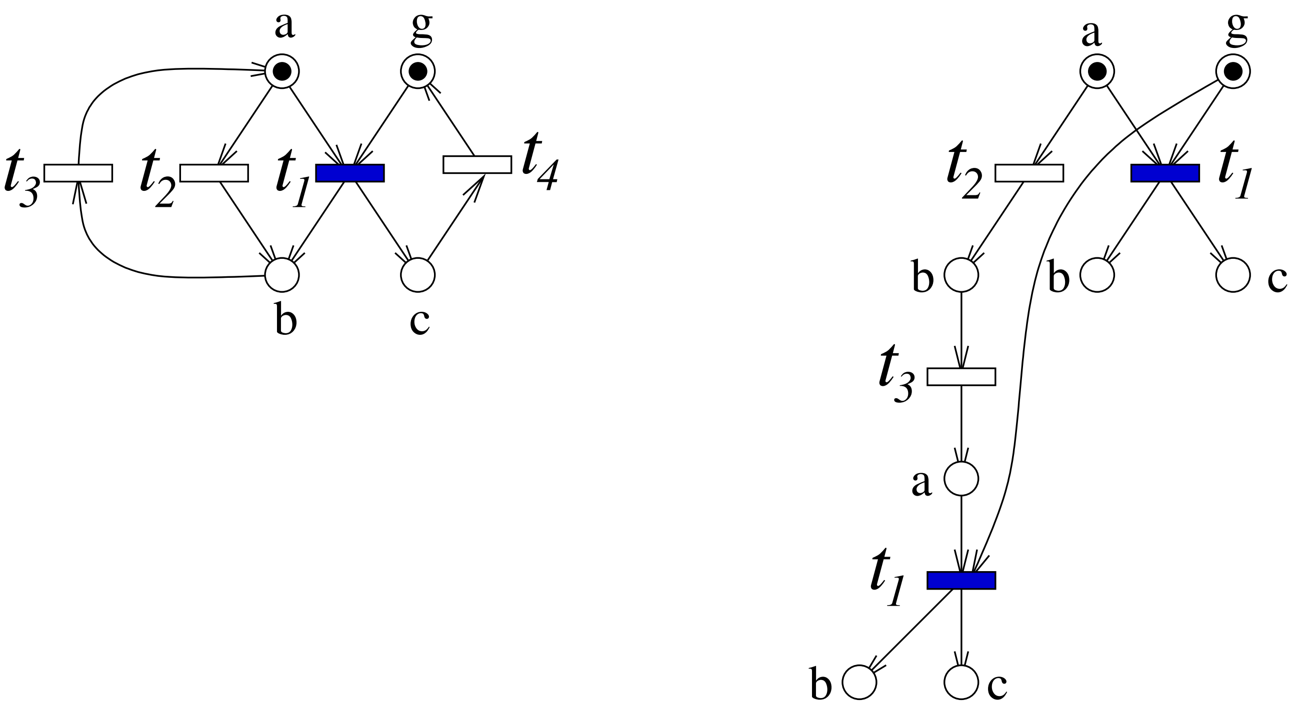

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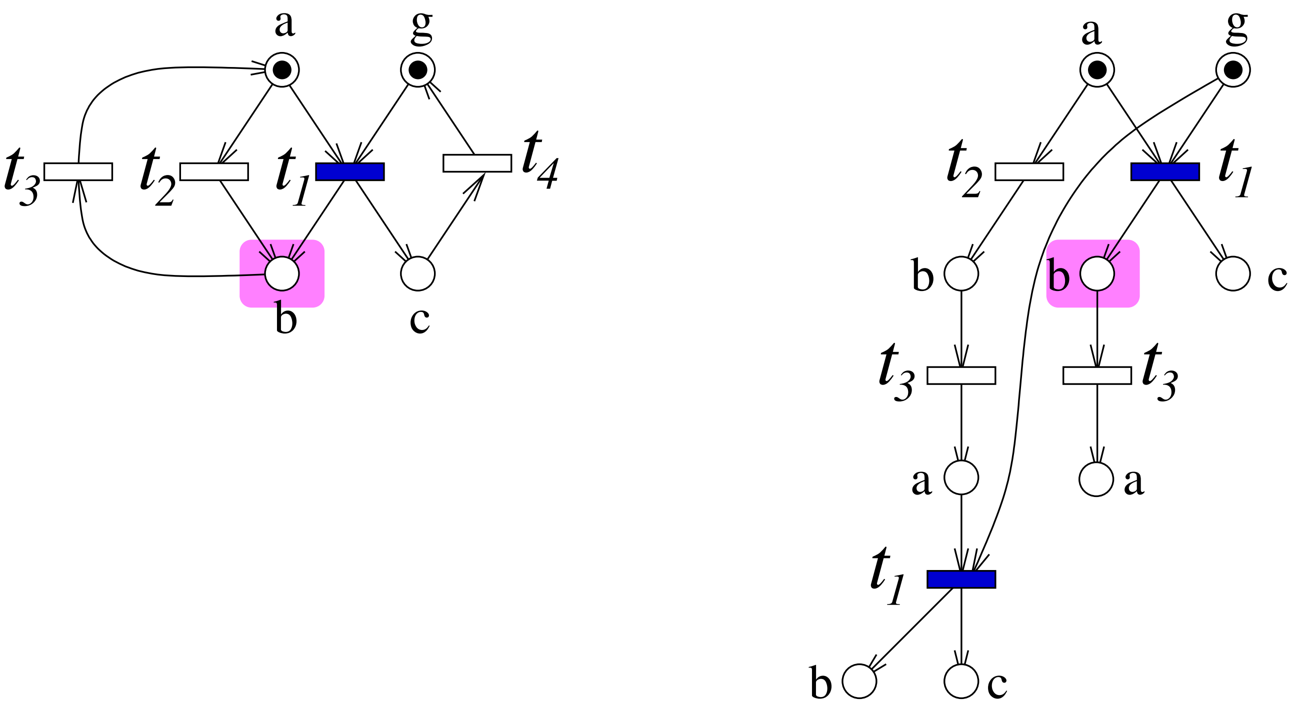

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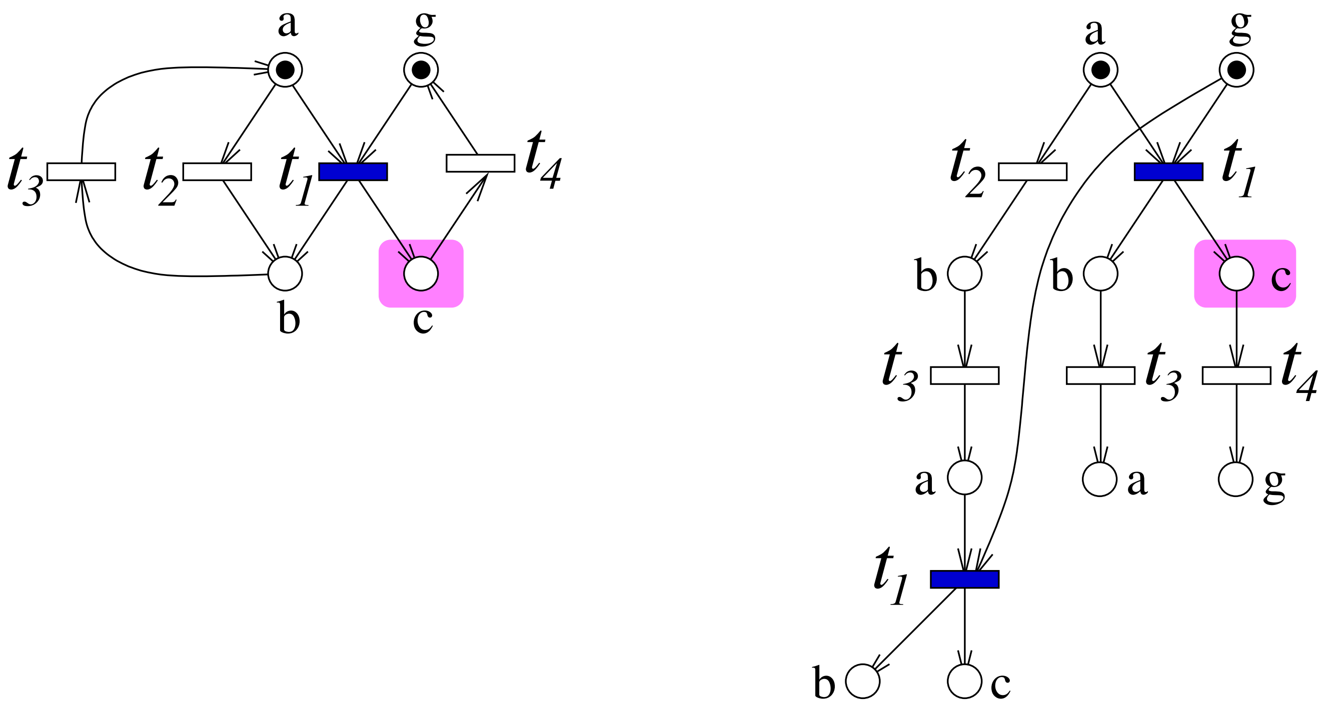

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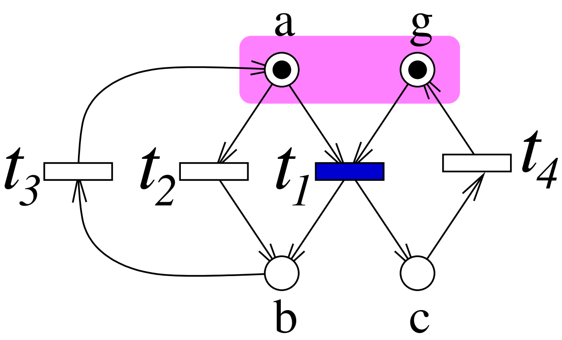
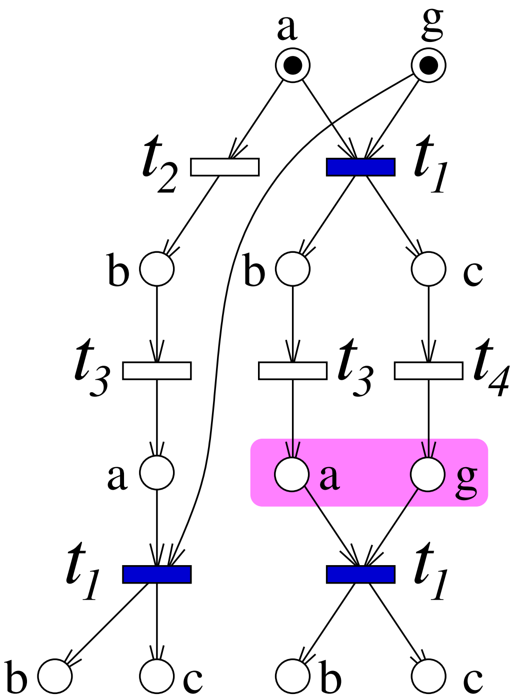




Application of the unfolding

Reachability/coverability test

- one wishes to know if there exists an accessible marking m in net $\mathcal{N}$ where each place of $Q \subseteq P$ holds a token, i.e. $Q \subseteq m$
- by creating in $\mathcal{N}$ a new transition t with $Q = {}^\bullet t$ this amounts to checking if t is accessible

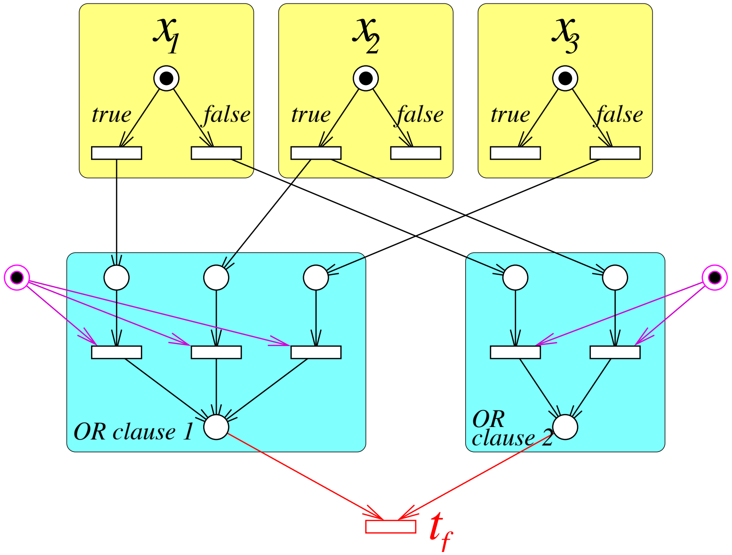
Questions

1. what is the complexity of this test ?
2. how far should one go in the computation of the unfolding ?

Thm : the reachability/coverability test (co-set construction) is NP-complete.

Proof: by reduction of SAT problems (at least 3-SAT)

example : encoding SAT problem $(x_1 \vee x_2 \vee \bar{x_3}) \wedge (\bar{x_1} \vee x_2)$



- The complexity of unfolding this net (before t_f) is polynomial, so the complexity of finding a co-set where t_f is firable is NP-hard.
- As building an unfolding requires finding co-sets, one must rely on SAT solvers (which modern unfolders do).

Finite complete prefix

Idea: A prefix $\mathcal{O} \sqsubseteq \mathcal{U}_{\mathcal{N}}$ is said to be complete if all reachable markings in $\mathcal{N}$ are represented as (the image of) a cut in $\mathcal{O}$.
(One wishes to avoid useless repetitions of similar patterns in $\mathcal{O}$)

More formally: $\mathcal{O} \sqsubseteq \mathcal{U}_{\mathcal{N}}$ is complete iff

- $\forall m$ reachable marking in $\mathcal{N}$, it appears in the prefix

$$\exists \kappa \in \mathcal{O} \ : \ m = Mark(\kappa) = f_{\mathcal{N}}(\max(\kappa))$$

- $\forall t \in T, \ m[t\rangle m'$, i.e. t firable from m, it appears as an event on top of marking m

$$\exists \kappa, \kappa' \in \mathcal{O} \ : \ m = Mark(\kappa), \ \kappa' = \kappa \oplus \{e\}, \ f_{\mathcal{N}}(e) = t$$

How to build a finite complete prefix ?

Naive idea:

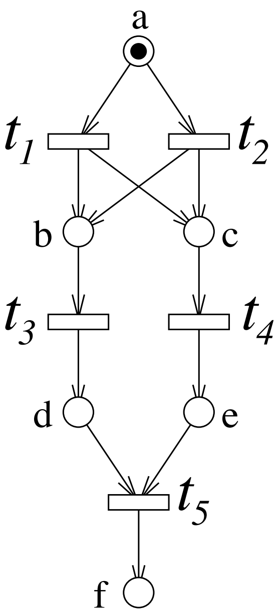
- apply the unfolding algorithm, and stop at event e when the marking produced by *[e]* is already present in the prefix:

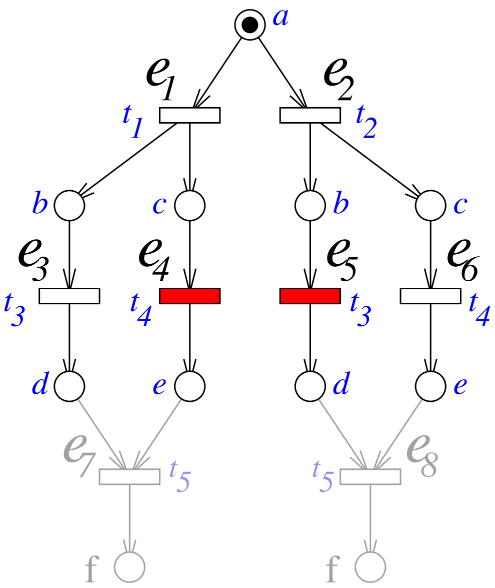
$$\exists \kappa \in \mathcal{O} \ : \ \mathrm{Mark}(\kappa) = \mathrm{Mark}([e])$$

- this makes e a cut-off event, on top of which no more event will be added

Problem: it generally yields an incomplete prefix…

<u>example</u> : stop events in red, firing of t_5 not seen





Solution: break the symmetry, by favoring some configurations for extension

Adequate order : $\prec$ on (local) configurations *[e]*

- well founded partial order (finite number of predecessors)
- refines prefix inclusion : $\kappa \sqsubset \kappa' \Rightarrow \kappa \prec \kappa'$
- preserved by isomorphic extensions :

  $$\kappa \prec \kappa' \wedge \mathrm{Mark}(\kappa) = \mathrm{Mark}(\kappa') \Rightarrow \kappa \oplus e \prec \kappa' \oplus e' \text{ where } f_{\mathcal{N}}(e) = f_{\mathcal{N}}(e')$$

Examples

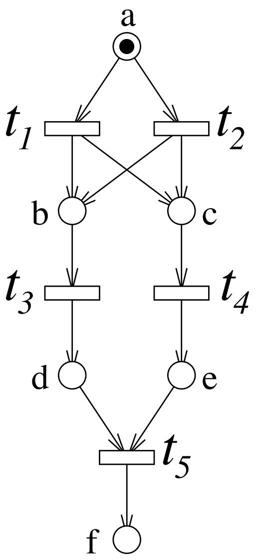
1. take for $\prec$ the prefix inclusion $\sqsubset$
2. $\prec$ defined by the number of events (total order, proposed by McMillan)
3. take for $\prec$ the lexicographic order, when net $\mathcal{N}$ is made of several components,
   by ordering components, and counting events in each component,
   as in Mattern's vector clocks (partial order, proposed by Esparza)

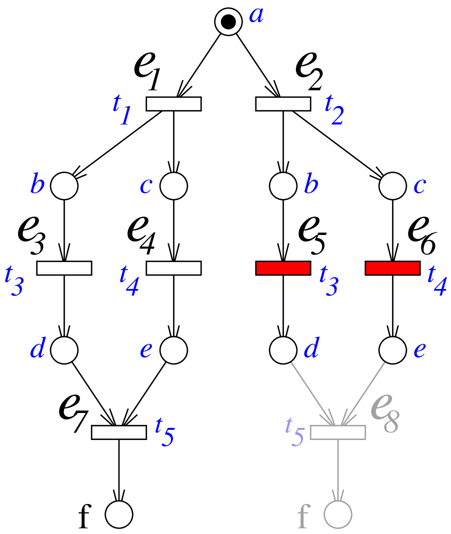
Cut-off event : event e is a cut-off event in the BP $(\mathcal{O}, f)$ of $\mathcal{N}$ iff there exists another event e' in $(\mathcal{O}, f)$ such that

$$\mathrm{Mark}([e']) = \mathrm{Mark}([e]) \quad \wedge \quad [e'] \prec [e]$$

Example (continued)

- assume $[e_3] \prec [e_5]$, which makes e_5 a cut-off event
- this entails $[e_3, e_4] \prec [e_5, e_6]$, by isomorphic extension
- $[e_6] \prec [e_4]$, which would make e_4 a cut-off event, is false, as this would entail $[e_6, e_5] \prec [e_4, e_3]$ by isomorphic extension, which is false





Thm the prefix $\mathcal{O} \sqsubseteq \mathcal{U}_{\mathcal{N}}$ obtained by stopping the unfolding algorithm at cut-off events is finite and complete [McMillan, Esparza].

<u>Proof</u> : see references at the end of the lesson ;
finiteness and completeness are proved separately,
and heavily rely on properties of adequate orders.

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Thm if the adequate order $\prec$ used to buid the FCP $\mathcal{O} \sqsubseteq \mathcal{U}_{\mathcal{N}}$ is a total order, then the number of non-cut-off events in $\mathcal{O}$ is bounded by the number of reachable markings in $\mathcal{N}$.

Proof : for two events $e, e' \in \mathcal{O}$ such that $\mathrm{Mark}([e]) = \mathrm{Mark}([e'])$
either $[e] \prec [e']$ or $[e'] \prec [e]$ holds,
so one of these events is a cut-off

Application to deadlock checking

Deadlock : a marking of $\mathcal{N}$ where no more transition can be fired

Thm Let $\mathcal{O} \sqsubseteq \mathcal{U}_{\mathcal{N}}$ be a finite complete prefix.
There is no deadlock in $\mathcal{N}$ iff every configuraion $\kappa \sqsubseteq \mathcal{O}$ can be extended into a configuration $\kappa \sqsubseteq \kappa' \sqsubseteq \mathcal{O}$ that contains a cut-off event. [McMillan]

Proof : (sketch of)

- a maximal configuration with no cut-off can't be extended :
  the terminal marking is a dead-end
- conversely, at a cut-off, one reaches a marking that is present and extended elsewhere in the prefix, which means that a continuation is possible

Take home messages

(Safe) Petri nets

- are a natural model for concurrent systems
- can be built by product, as networks of automata
- admit natural (built in) true concurrency semantics for their runs
- sets of runs can be handled by branching processes (unfoldings) instead of languages

Extra results

- by restricting branching processes to events, one gets (prime) event structures $\mathcal{E} = (E, \rightarrow, \#)$ a complete theory of event structures exists
- one can define a product on unfoldings, and

$$\mathcal{U}(\mathcal{N}_1 \times ... \times \mathcal{N}_K) \;=\; \mathcal{U}(\mathcal{N}_1) \times ... \times \mathcal{U}(\mathcal{N}_K)$$

projections exists as well, which enables distributed computations based branching processes, or on other structures (e.g. event structures).

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About prefix construction

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