# A Syntactic Approach to Type Soundness

#### Andrew K. Wright, Matthias Felleisen

#### Summary by: Joshua Peignier

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## 1 Introduction

2 Functional-ML and the associated type system

**3** Proving Type Soundness



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  - $\rightarrow$  For instance: preventing a program from trying to evaluate 1 + true, or (1 f).
- Such errors are usually detected during the compilation, and stop the compilation if they occur.
- Soundness properties: ensure that if a program is well-typed, then no such error can happen.

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#### Strong Soundness

If  $\triangleright e : \tau$  and eval(e) = v, then  $v \in V^{\tau}$ .

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  - Different techniques are required depending on whether the semantics is specified as operational or denotational.
- Contribution: Provide a new approach for proofs of type soundness for Hindley/Milner-style type systems.
  - $\rightarrow$  Based on subject reduction result and rewriting system as semantics.

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(a)

Expressions and Values in Functional-ML:

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## Functional-ML Syntax

$$e ::= v | e_1 e_2 | \text{let } x \text{ be } e_1 \text{ in } e_2$$
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Consider a partial function  $\delta$  : Const imes ClosedVal  $\rightarrow$  ClosedVal.

Reduction relation

 $c \ v \ o \ \delta(c, v)$  when defined  $(\lambda x.e) \ v \ o \ e[x \mapsto v]$ let x be v in  $e \ o \ e[x \mapsto v]$  $Y \ v \ o \ v \ (\lambda x.(Y \ v) \ x)$ 

Semantics based on the reduction  $\rightarrow$  and on evaluation contexts:

Evaluation contexts

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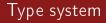
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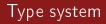
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- Reduction have the form:

$$E[e_1] \longmapsto \cdots \longmapsto E[v] = E'[e'_1] \longmapsto \dots$$
$$\longmapsto E'[v'] = E''[e''_1] \longmapsto \cdots \longmapsto v_0$$

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### Types in Functional-ML have the following form



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$$\tau ::= \iota_1 \mid \ldots \mid \iota_n \mid \alpha \mid \tau_1 \to \tau_2$$

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- Generalization relation  $\succ$  over type schemes:  $\sigma \succ \tau$  when  $\sigma'$  is obtained by substitution of bound type variables in  $\sigma$ .

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- Type scheme:  $\sigma = \forall \alpha_1 \dots \alpha_n . \tau$   $\rightarrow$  Represents the set of types obtained by substitution of  $\alpha_1, \dots, \alpha_n$ .
- Generalization relation  $\succ$  over type schemes:  $\sigma \succ \tau$  when  $\sigma'$  is obtained by substitution of bound type variables in  $\sigma$ .
- Type environment Γ: map from free variables to type schemes.

## $\delta$ -typability

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If  $TypeOf(c) \succ \tau' \rightarrow \tau$  and  $\triangleright v : \tau'$ , then  $\delta(c, v)$  is defined and  $\triangleright \delta(c, v) : \tau$ .

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This condition is required for soundness to hold.

## Typing rules in Functional-ML

 $\Gamma \triangleright x : \tau \text{ if } \Gamma(x) \succ \tau$ 

 $\Gamma \triangleright c : \tau \text{ if } TypeOf(c) \succ \tau$ 

 $\mathsf{\Gamma} \triangleright \mathrm{Y} : ((\tau_1 \to \tau_2) \to \tau_1 \to \tau_2) \to \tau_1 \to \tau_2$ 

$$\frac{\Gamma[x \mapsto \tau_1] \triangleright e : \tau_2}{\Gamma \triangleright \lambda x.e : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \triangleright e_1 : \tau_1 \qquad \Gamma[x \mapsto Close(\tau_1, \Gamma)] \triangleright e_2 : \tau_2}{\Gamma \triangleright \det x \ be \ e_1 \ in \ e_2 : \tau_2}$$

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## Well-typed program

If e is closed and if there exists  $\tau$  such that  $\triangleright e : \tau$ , then e is a well-typed program.

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If  $\triangleright e : \tau$ , then either  $e \Uparrow$ , or  $e \longmapsto v$  and  $\triangleright v : \tau$ .

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Layout of the proof:

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Showing that faulty expressions are untypable.

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If  $\Gamma \triangleright e_1 : \tau$  and  $e_1 \rightarrow e_2$ , then  $\Gamma \triangleright e_2 : \tau$ .

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### Corollary

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Proof by case analysis over  $e_1 \rightarrow e_2$  (one step of reduction).

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Proof by case analysis over  $e_1 \rightarrow e_2$  (one step of reduction). Abstraction and let cases relie on the following lemma:

#### Lemma

If  $\Gamma[x \mapsto \forall \alpha_1 \dots \alpha_n . \tau] \triangleright e : \tau'$  and  $x \notin Dom(\Gamma)$  and  $\Gamma \triangleright v : \tau$  and  $\alpha_1, \dots, \alpha_n$  are not free in  $\Gamma$ , then  $\Gamma \triangleright e[x \mapsto v] : \tau'$ .

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#### Definition

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Example:  $((\lambda y.2) (\lambda x.(+ 1 \text{ true})).$ This expression reduces to 2, but we can expect it to cause a type error.

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Lemma (Uniform Evaluation)

For closed *e*, if *e* cannot be reduced to a faulty expression, then either  $e \uparrow$ or  $e \vdash v$ .

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### Corollary: Syntactic Soundness

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Let *eval* denote the following function:  $eval(e) = \begin{cases} wrong & \text{if } e \longmapsto w' \text{ and } e' \text{ is faulty} \\ v & \text{if } e \longmapsto v \end{cases}$ 

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- In this article, the authors designed a new approach, based on subject reduction, defining faulty expressions and showing that they are not typable.
- This approach was used to prove type soundness for Functional ML.
- Type soundness was also proven for extensions of Functional ML including references, exceptions, and first-class continuation.