Revisiting and Improving Algorithms for the 3XOR Problem

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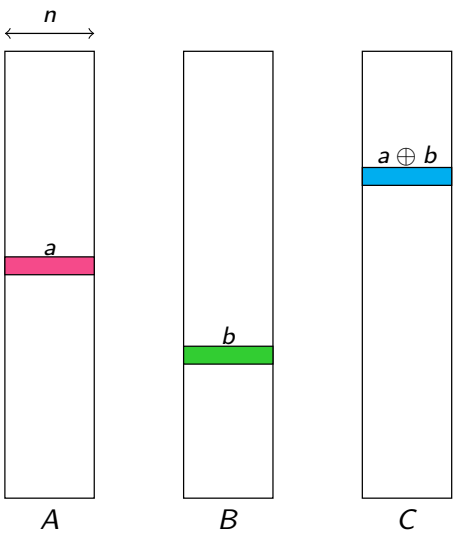
3XOR Problem

Problem

Given three lists A, B, and C of uniformly random elements of $\{0,1\}^n$, find $(a,b,c) \in A \times B \times C$, such that $a \oplus b \oplus c = 0$.



- Difficult case of Generalised Birthday Problem
- Application in cryptanalysis of some authenticated encryption scheme
- Lists formed by querying oracles $\Rightarrow$ can be as big as we want
- $|A| \cdot |B| \cdot |C| \geq 2^n \Rightarrow$ solution w.h.p.

Our Contributions

- Discuss practical issues arising from the computation of a 3XOR
- Propose a new efficient algorithm
- Present an adaptation of an [BDP05] to the 3XOR problem
- Compute a 96-bit 3XOR of SHA-256

A Naive Quadratic Algorithm

Problem: find $(a, b, c) \in A \times B \times C$ s.t. $a \oplus b \oplus c = 0$

Idea

- Create all $v = a \oplus b$
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- Time complexity: $\mathcal{O}(|A| \cdot |B| + |C|)$
- Space: $\mathcal{O}(|A| + |B| + |C|)$
- $|A| = |B| = |C| = 2^{n/3} \Rightarrow$ Time: $\mathcal{O}\left(2^{2n/3}\right)$, Space: $\mathcal{O}\left(2^{n/3}\right)$
- $|A| = |B| = 2^{n/4}$, $|C| = 2^{n/2} \Rightarrow$ Time: $\mathcal{O}\left(2^{n/2}\right)$, Space: $\mathcal{O}\left(2^{n/2}\right)$

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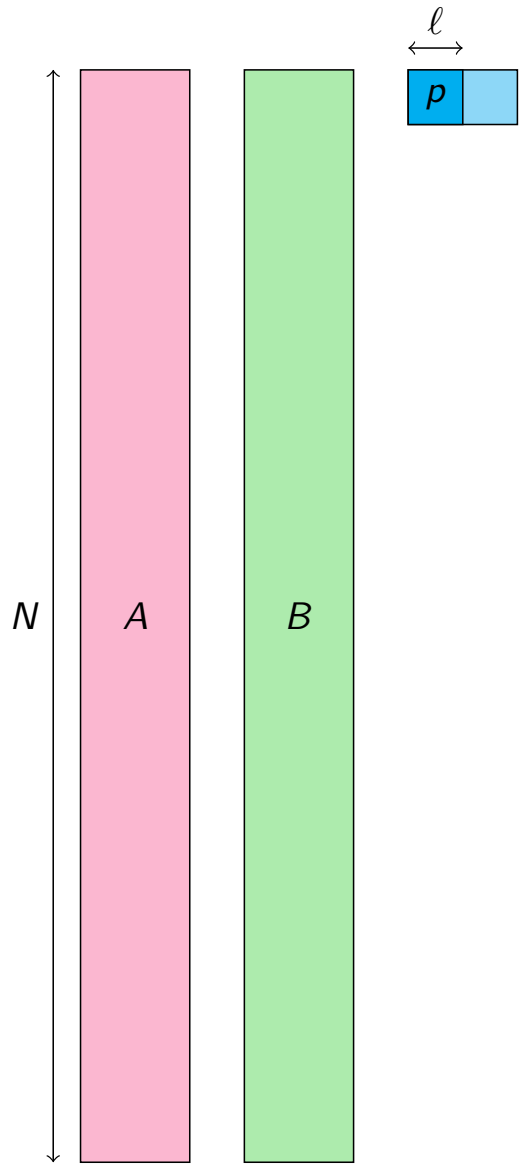
Tradeoffs are to be made (e.g. [Wagner02], [Bernstein07]).

About Wagner's Algorithm

- Designed for the kXOR problem
- Works best when k is a power of 2
- Not better than quadratic Algorithm in case $k = 3$
- Has known improvement since ([Joux09, NS14])

Wagner's idea



Description

- Number of queries: increased up to $N \simeq 2^{n/2}$
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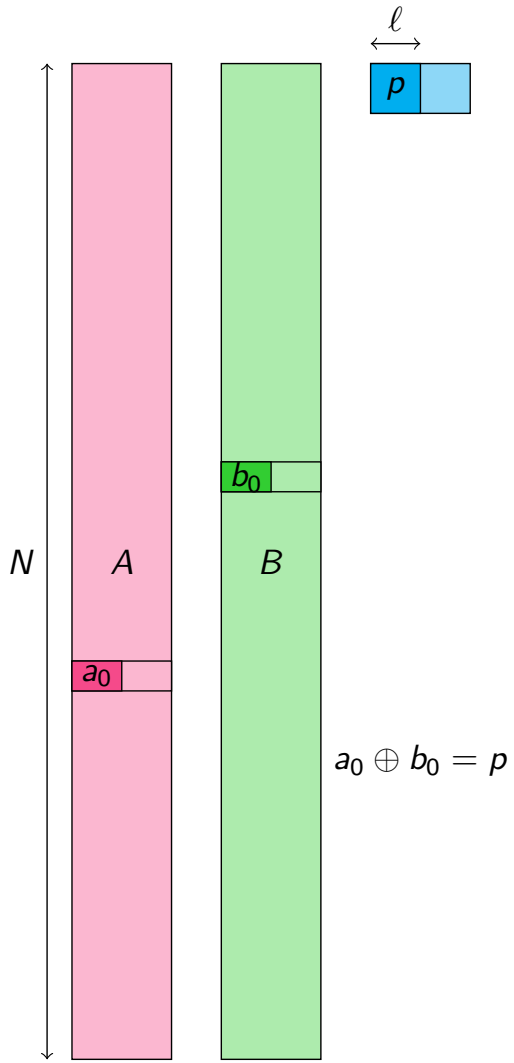

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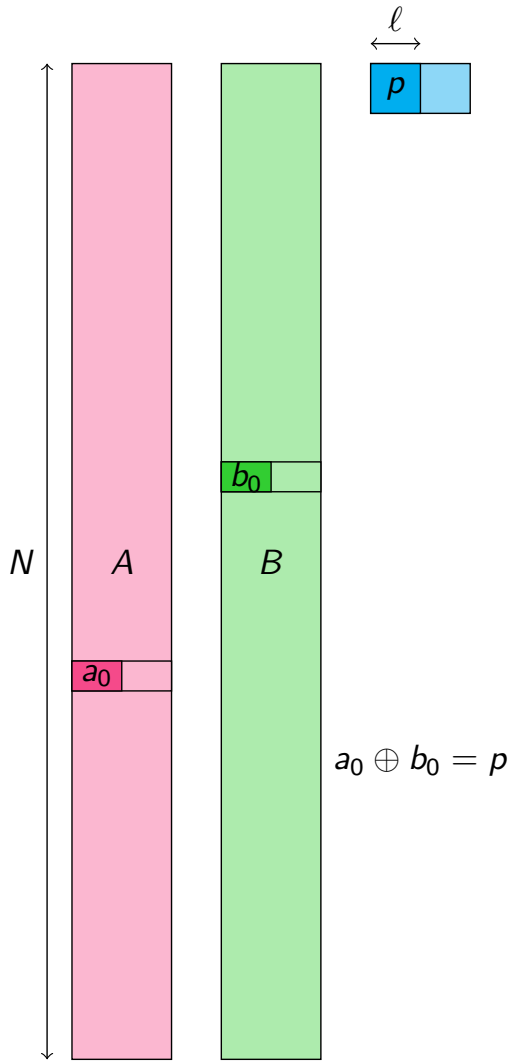

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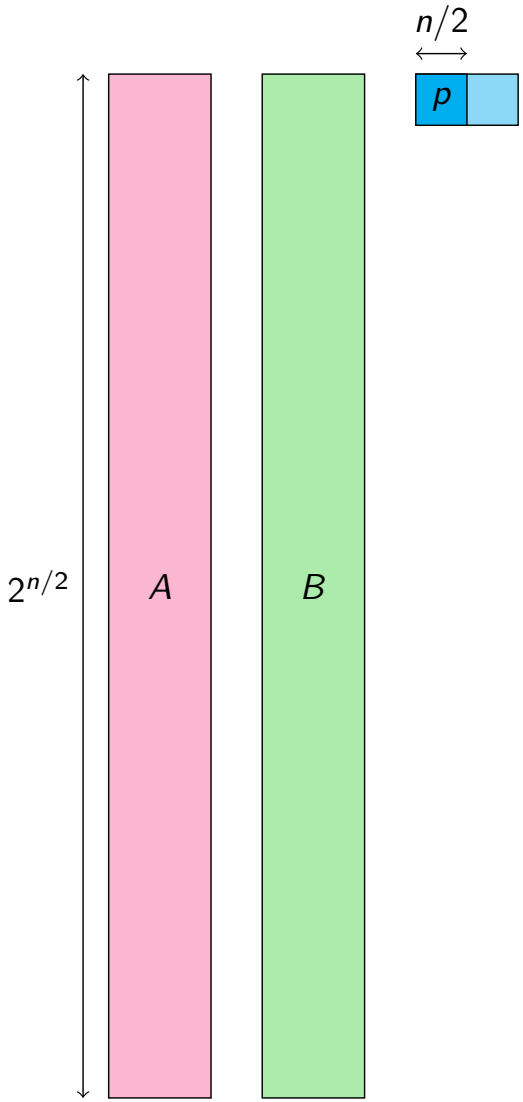
Complexity

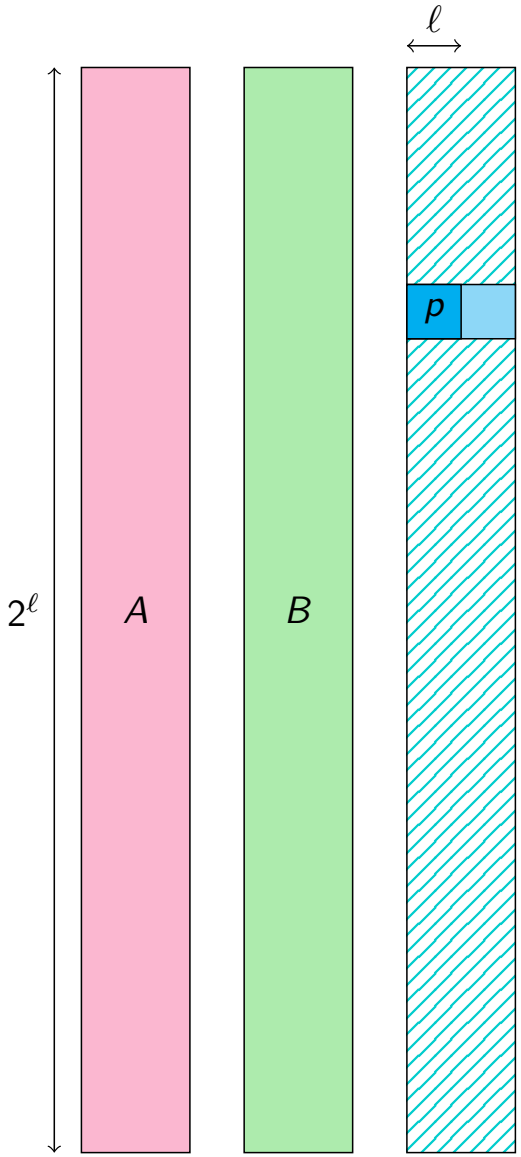
$\ell \simeq \log(N)$
Time and space: $\mathcal{O}(N)$

Wagner [Wagner02]



- $N = 2^{n/2}$, $\ell = n/2$
- p: arbitrary chosen
- Expected $|C|$: 1
- Time/Space: $\mathcal{O}\left(2^{n/2}\right)$.
- No improvement compared to Quad

Nicolić and Sasaki [NS14]



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- Begin with $|C| = 2^\ell$
- p: Most frequent prefix in whole C

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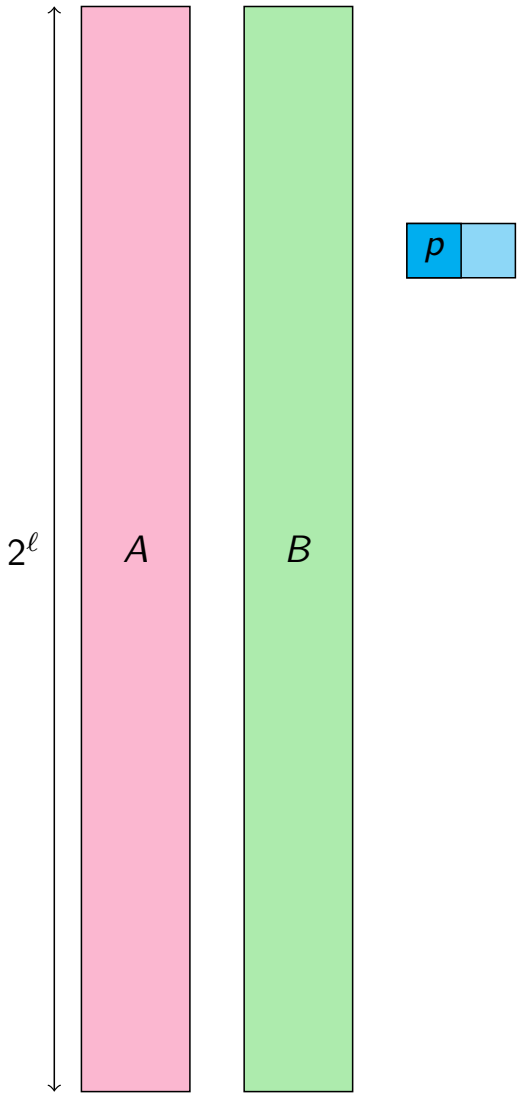
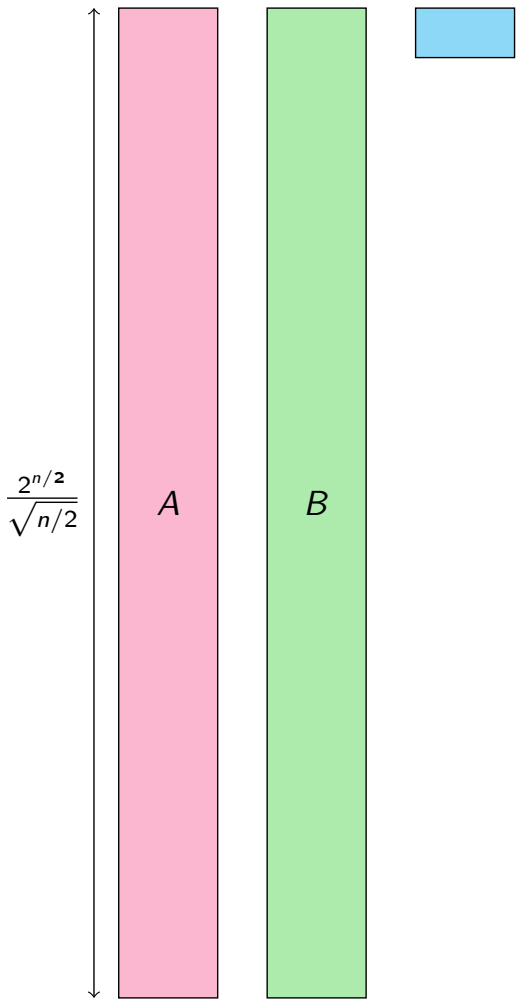


- $N \simeq 2^{n/2}/\sqrt{(n/2)/\ln(n/2)}$, $\ell = \log(N)$
- Begin with $|C| = 2^\ell$
- p: Most frequent prefix in whole C
- Discard other elements of C
- Expected $|C|$: $\ell/\ln(\ell)$
- Time/Space: $\mathcal{O}\left(2^{n/2}/\sqrt{n/\ln(n)}\right)$
- $\simeq \sqrt{\frac{n}{\ln(n)}}$ speedup compared to Wagner

Joux [Joux09]



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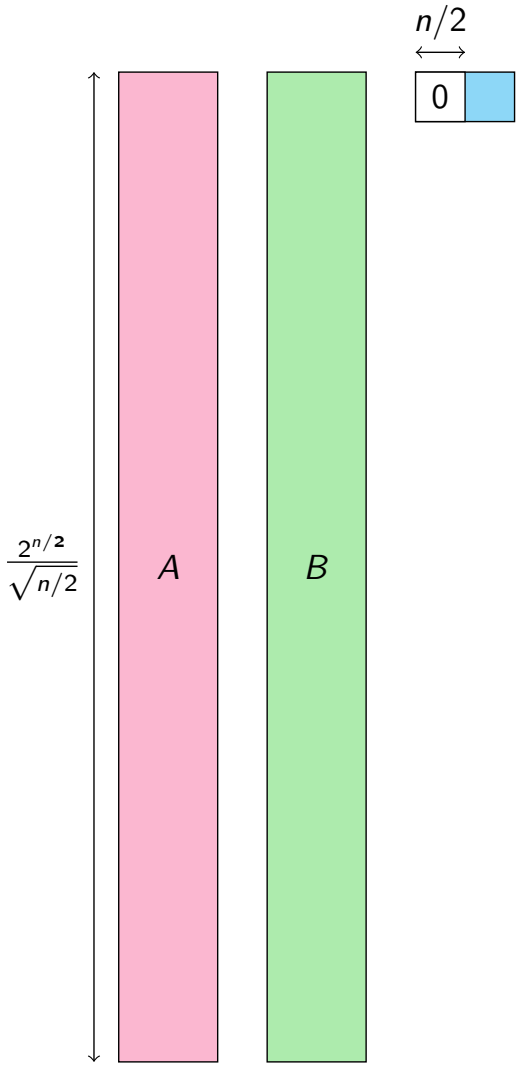

- $N = 2^{n/2}/\sqrt{n/2}$, $\ell = n/2$
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- Perform a basis change that forces p to 0
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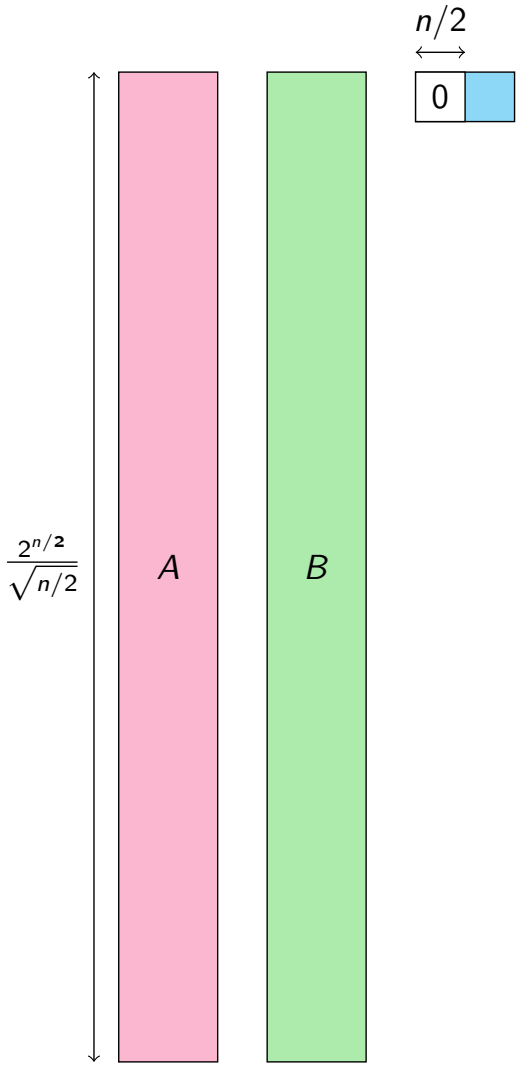

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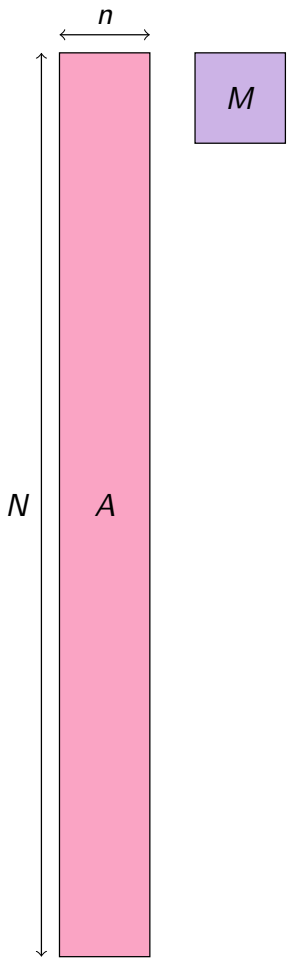
What about the cost of basis change?

Computing the Basis Change



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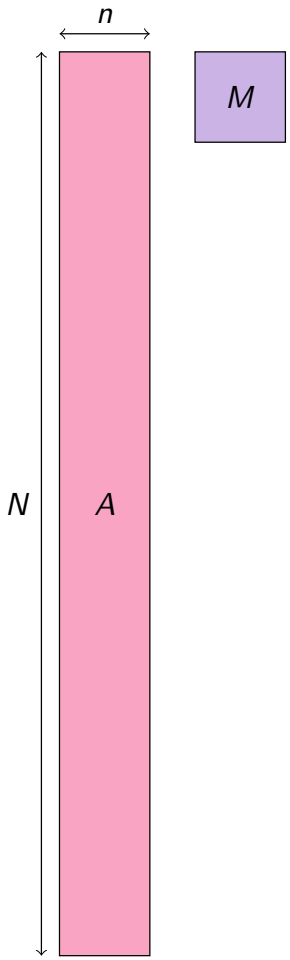

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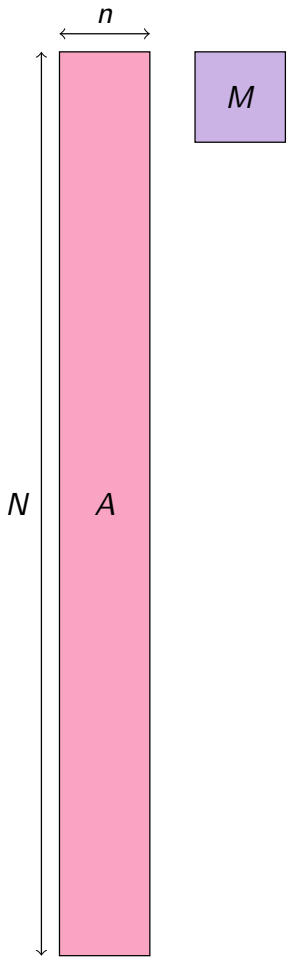

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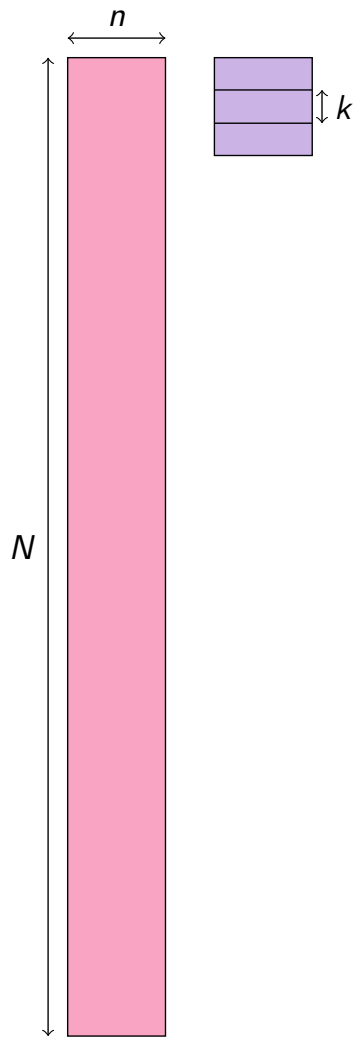
We use a 4-Russian trick:

The Trick



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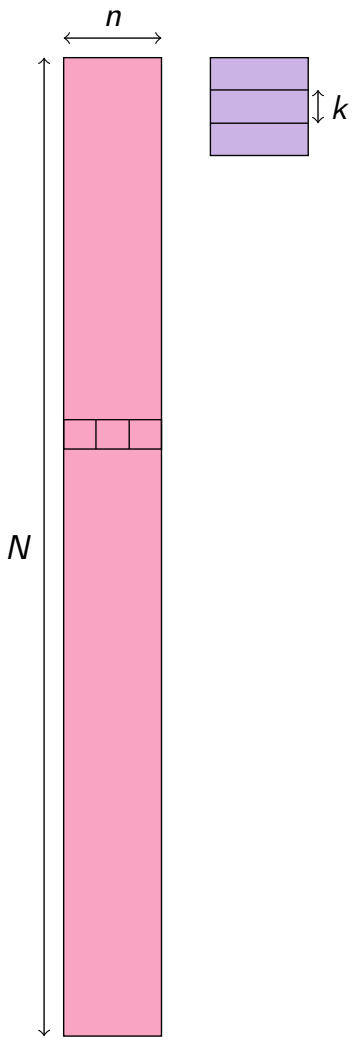

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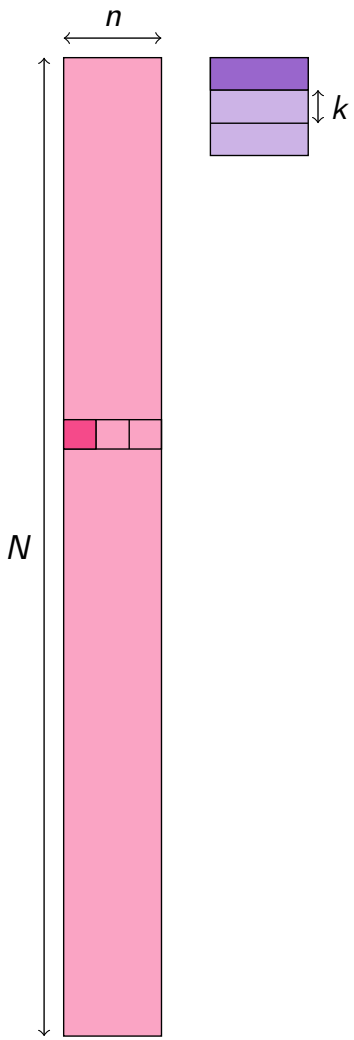

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  - Get the corresponding linear combinaisons $T_1[a_1], \ldots, T_{n/k}[a_{n/k}]$
  - Xor them

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$\Rightarrow$ Keep the lists small!

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$2^{n/2}$ Queries

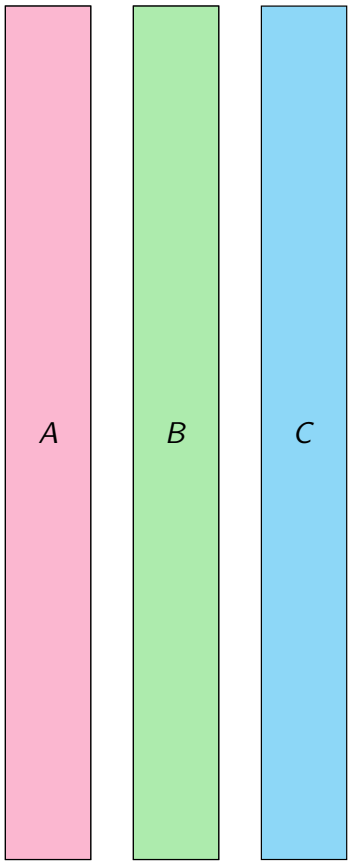
- $\ell = n/4$, $n' = 3n/4$
- **Stored data:** $\mathcal{O}\left(2^{n/4}\right)$ words
- **Time Complexity:** $\mathcal{O}\left(2^{n/2}\right)$ with Quadratic Algorithm

1. Background

2. Our New Algorithm

3. Adaptation of BDP Algorithm for the 3SUM problem

Our Work: A Generalization of Joux Algorithm



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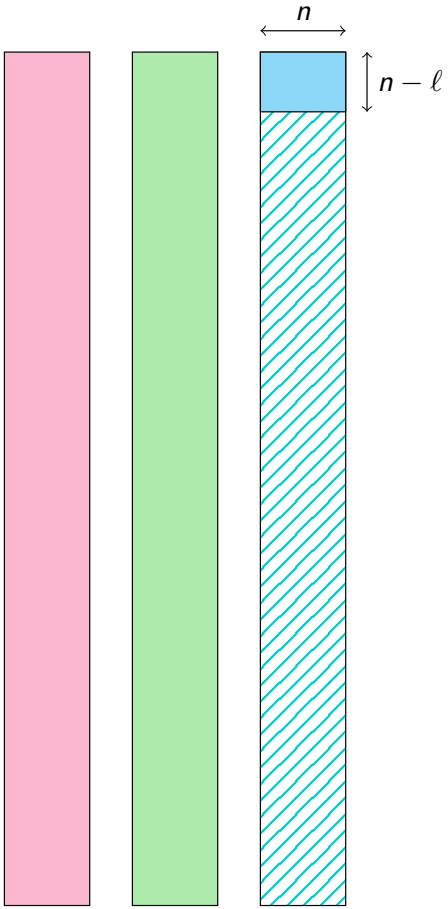

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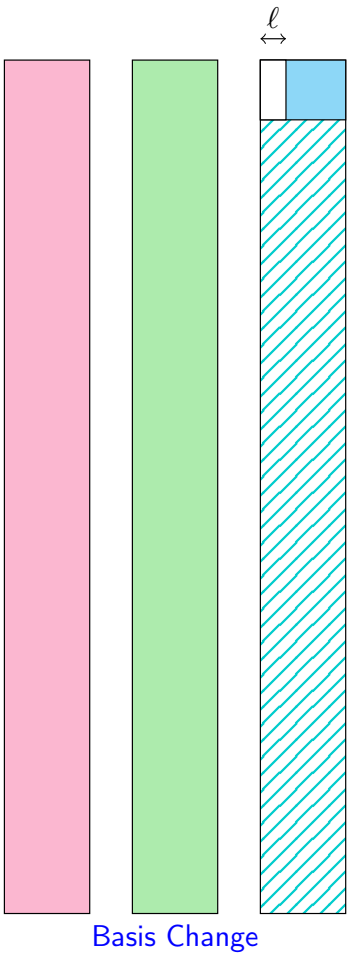

Generalization to any size of input lists

- Pick $n - \ell$ arbitrary entries in C (the first ones)
- Apply Joux's Algorithm

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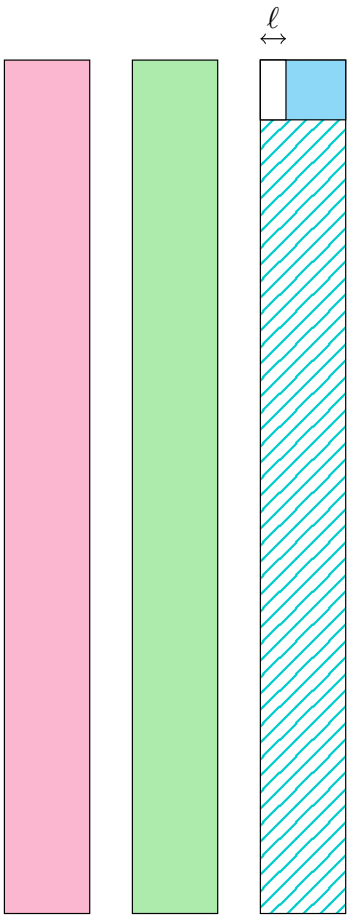

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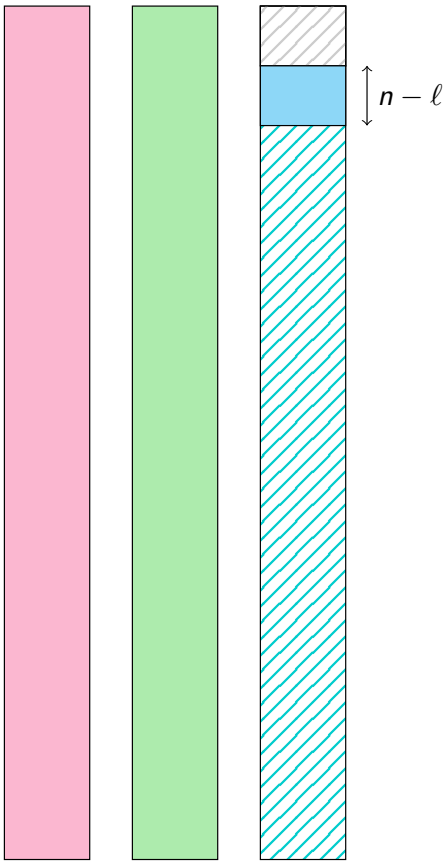

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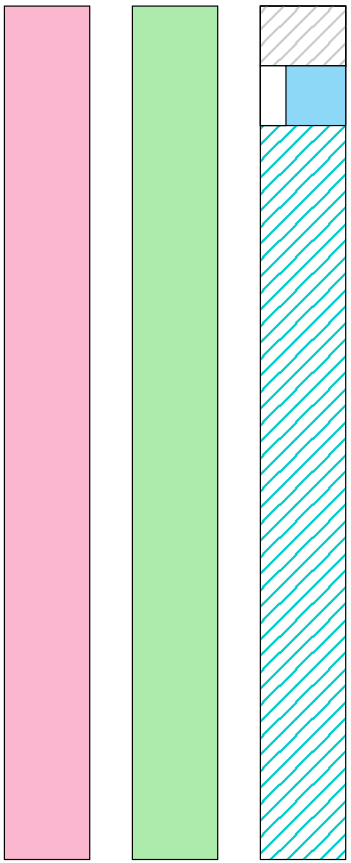

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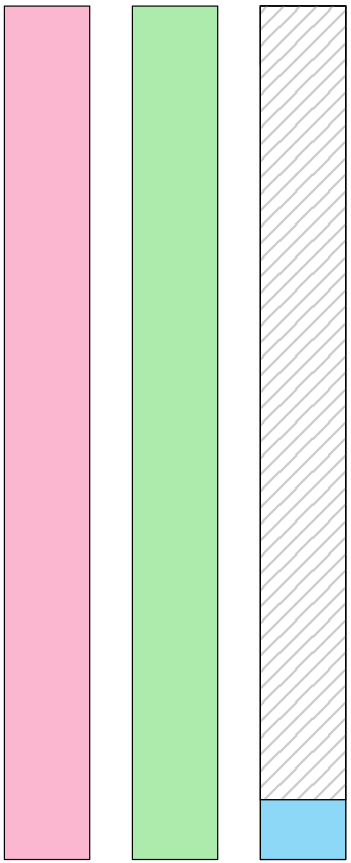

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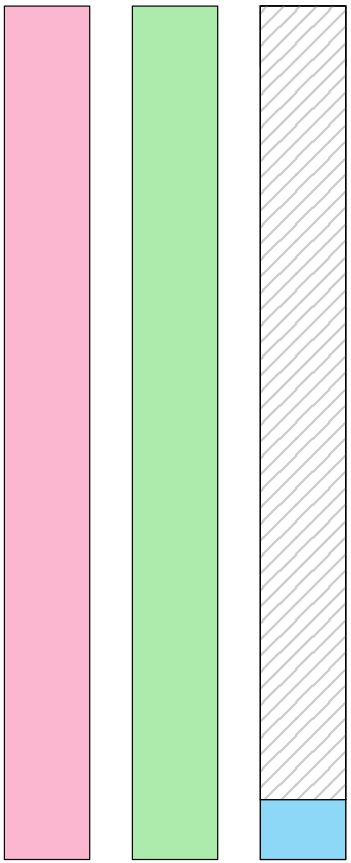

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$$\ell = \log(\min(|A|, |B|)), \text{ Time: } \mathcal{O}\left((|A| + |B|) \cdot \frac{|C|}{n}\right)$$

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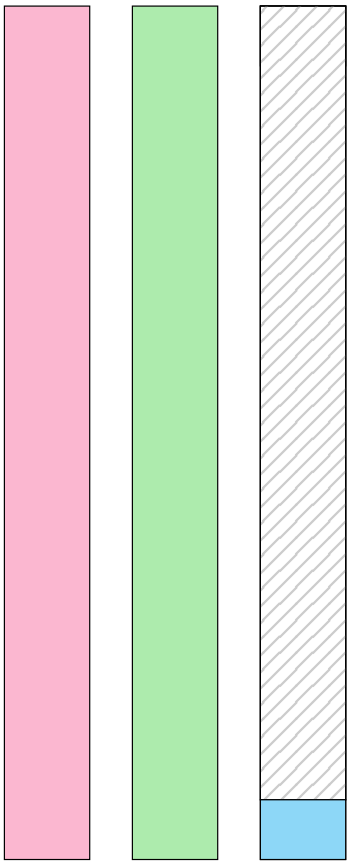

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$$|A| = |B| = |C| = 2^{n/3}; \ell = n/3, \text{ Time: } \mathcal{O}\left(\frac{2^{2n/3}}{n}\right)$$

Using the Clamping Trick

$2^{n/2-\epsilon}$ **Queries**

- Clamping performed over $(n/2 - 3 \cdot \epsilon)/2$ bits
- Size of lists: $\simeq 2^{n/4+\epsilon/2}$
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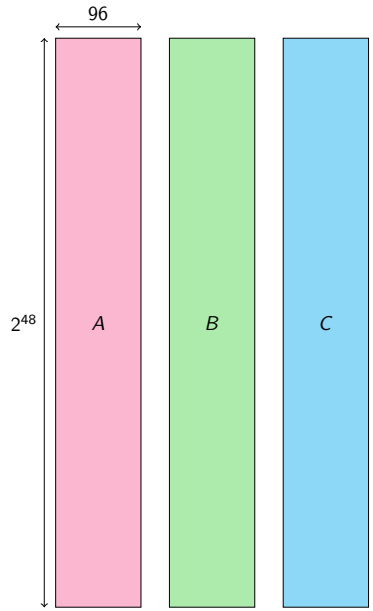
Nikolić and Sasaki's setting

- $\epsilon = 1/2 \cdot \log((n/2)/\ln(n/2))$
- Time: $\mathcal{O}\left(2^{n/2}/\sqrt{n \cdot \ln(n)}\right)$, space $\mathcal{O}\left(2^{n/4+\epsilon/2}\right)$
- $\ln(n)$ speedup compared to [NS14]

Concrete Example

A 96-bit 3XOR



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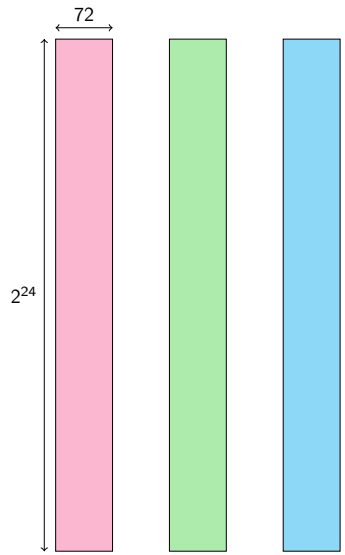

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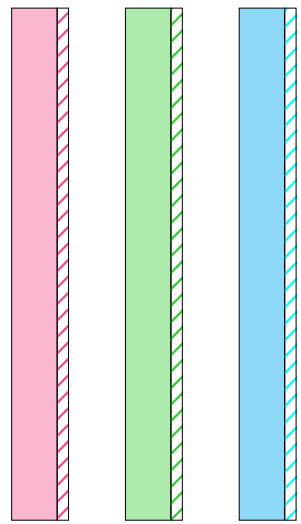

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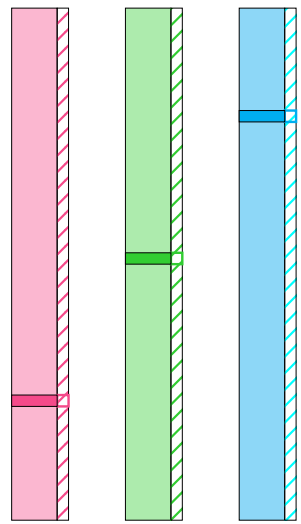

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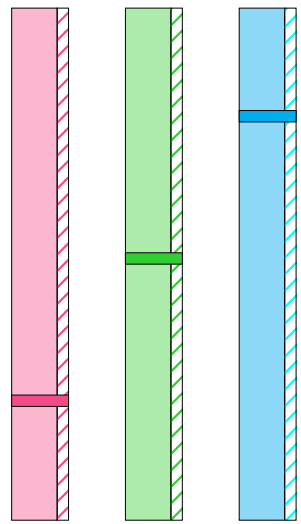

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Experimentations

- 3XOR of 96 bits of SHA-256
- Tests performed on a Haswell Core i5 CPU

Timing

	Quadratic	Our Algorithm
CPU hours	340	105
Data	576 MB	576 MB

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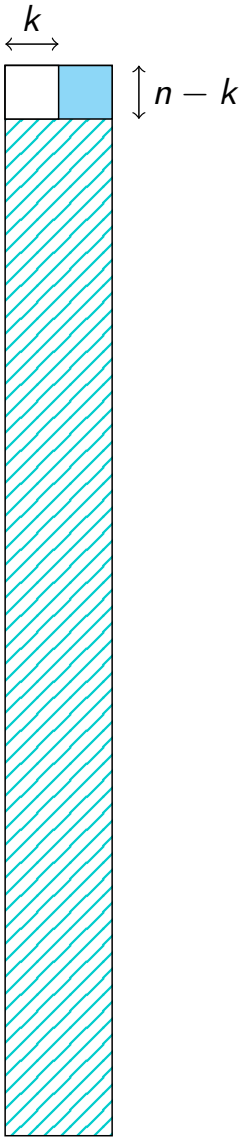
Creation of the lists: $\times 100$ slower than processing them!

In a Nutshell

This Algorithm...

- can be applied to any size of input list
- has a $\times n$ speed-up compared to the Quadratic Algorithm
- is about 3 times faster, in practice ($n = 96$)
- is faster than [NS14] with the same amount of data, in theory
- is the same than [Joux09] with the same amount of data

Possible Improvement



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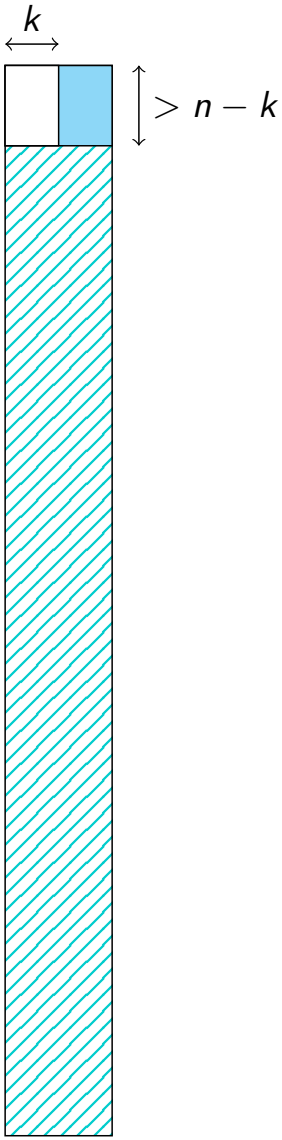

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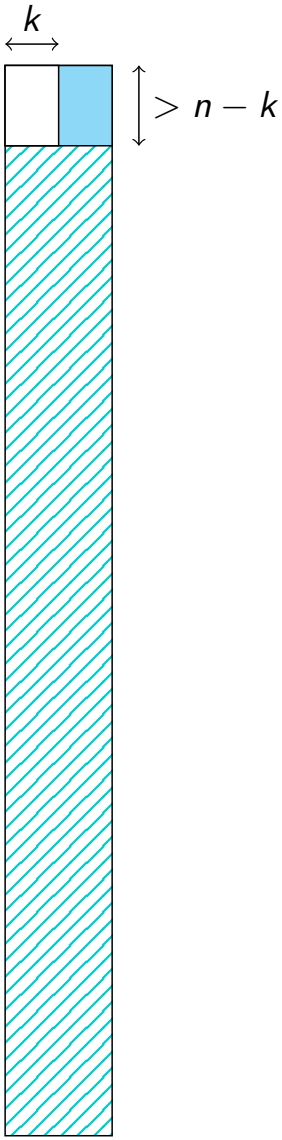

Problem

Given a list C of size N of n-bit uniformly random elements, find a k-dimensional vectorial subspace $\mathcal{V}$ of $\mathbb{F}_2^n$, s.t. $\mathcal{V} \cap C$ is maximized.

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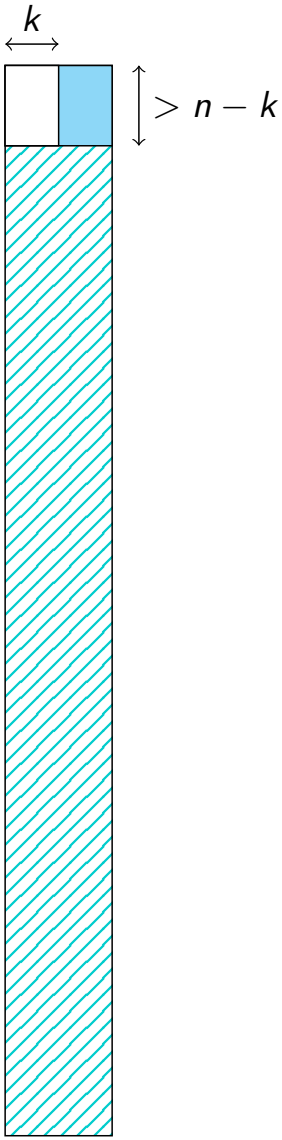

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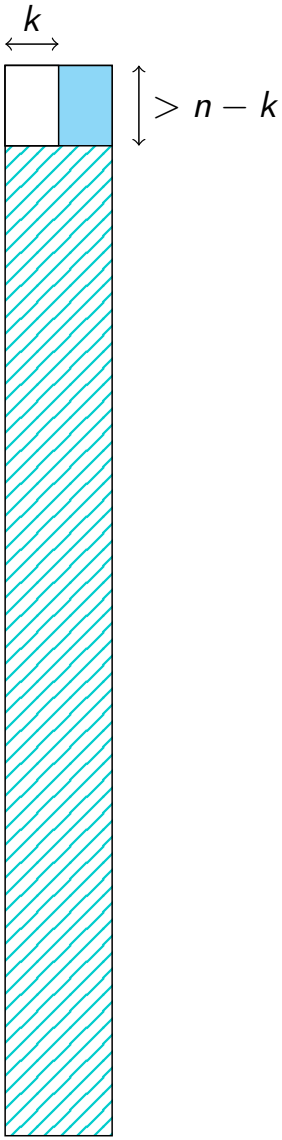

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$\Rightarrow$ Our goal: Find many $\mathcal{V}_i$ such that $\mathcal{V}_i \cap C$ is large.

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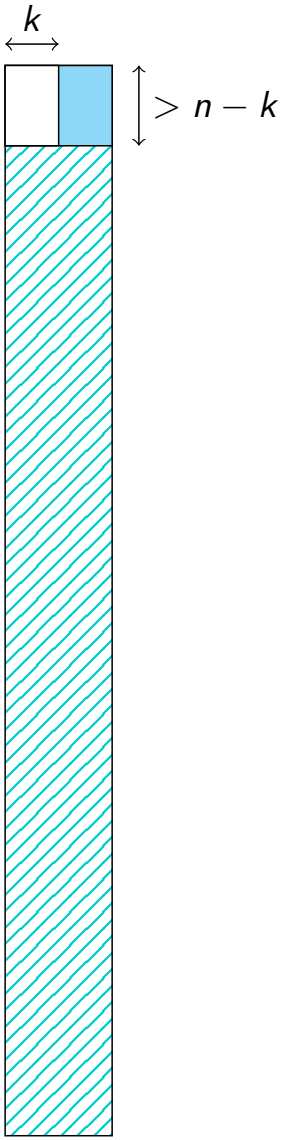

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For now: Only constant time improvement

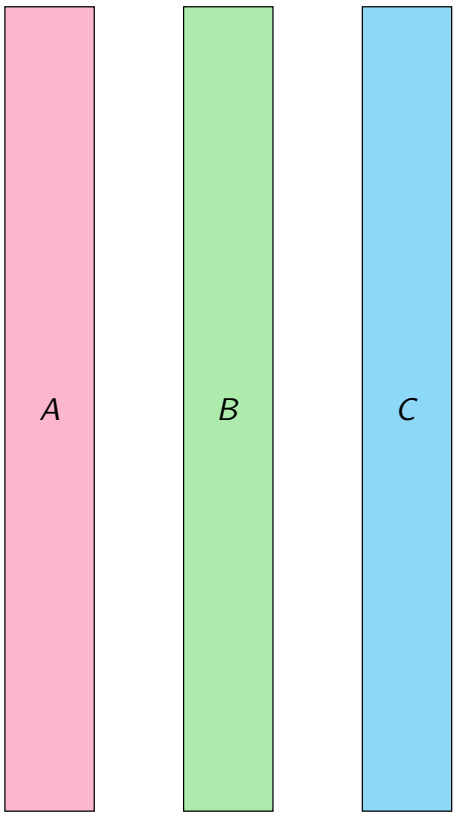
1. Background

2. Our New Algorithm

3. Adaptation of BDP Algorithm for the 3SUM problem

A 3XOR Adaptation of [BDP05]

- Originally designed for the 3SUM Problem over $(\mathbb{Z}, +)$
- We transposed it for the 3XOR Problem



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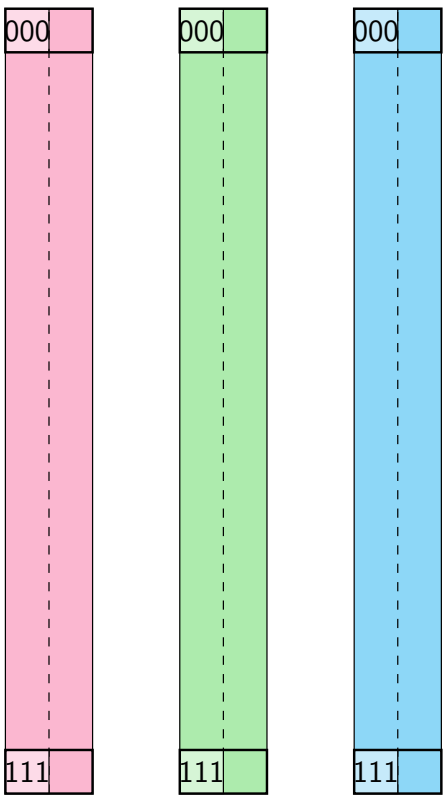

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- A^u: Bucket of elements of A starting by u

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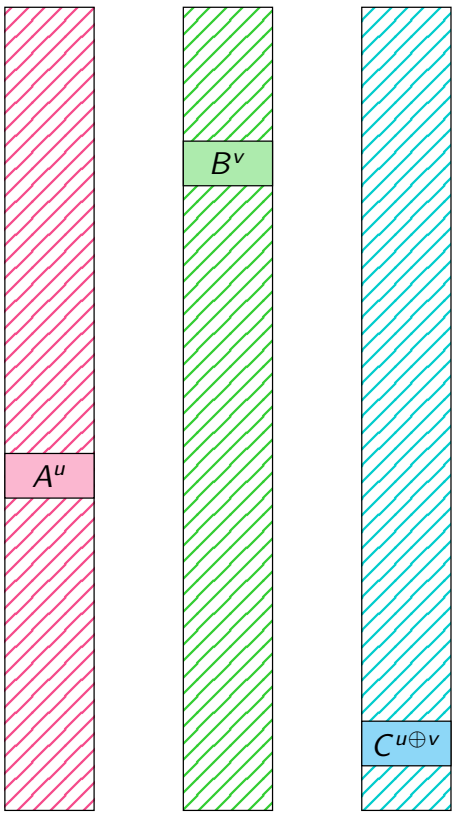
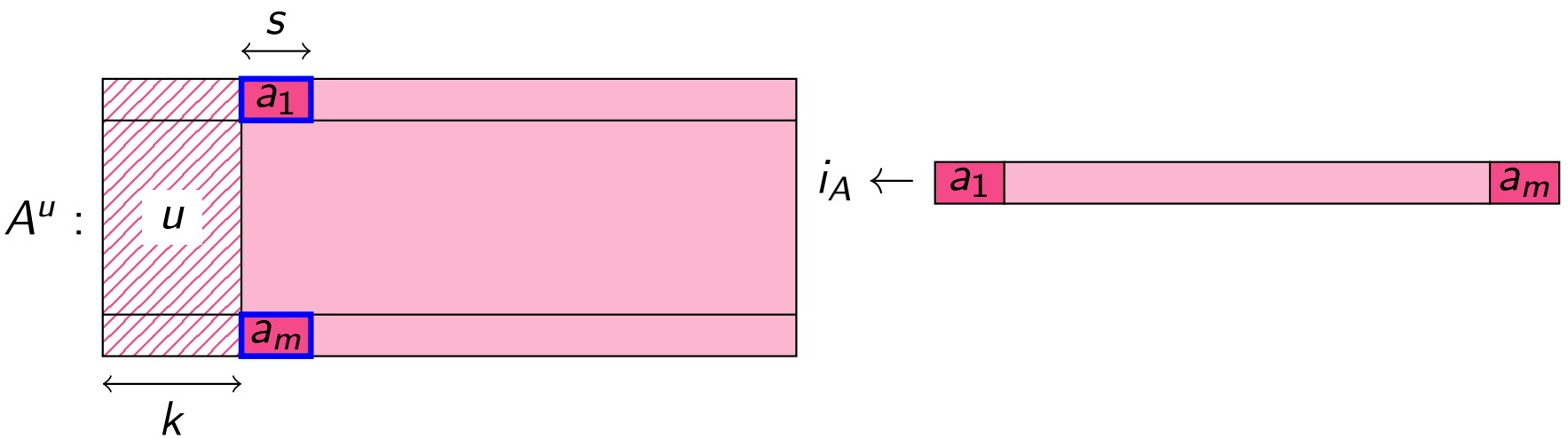


- Dispatch entries according to first k bits
- A^u: Bucket of elements of A starting by u
- For each triplet $(A^u, B^v, C^{u\oplus v})$ perform constant time preliminary test
- If the test fail: no solution for sure
- If the test succeed: there may be a solution
  - Solve the small instance

Preliminary Test

Instance $(A^u, B^v, C^{u \oplus v})$



$$T[i] = 1 \iff \exists j, k, \ell \text{ s.t. } a_j \oplus b_k \oplus c_\ell = 0$$

$$T[i] = 0 \Rightarrow \text{No solution in } (A^u, B^v, C^{u \oplus v})$$

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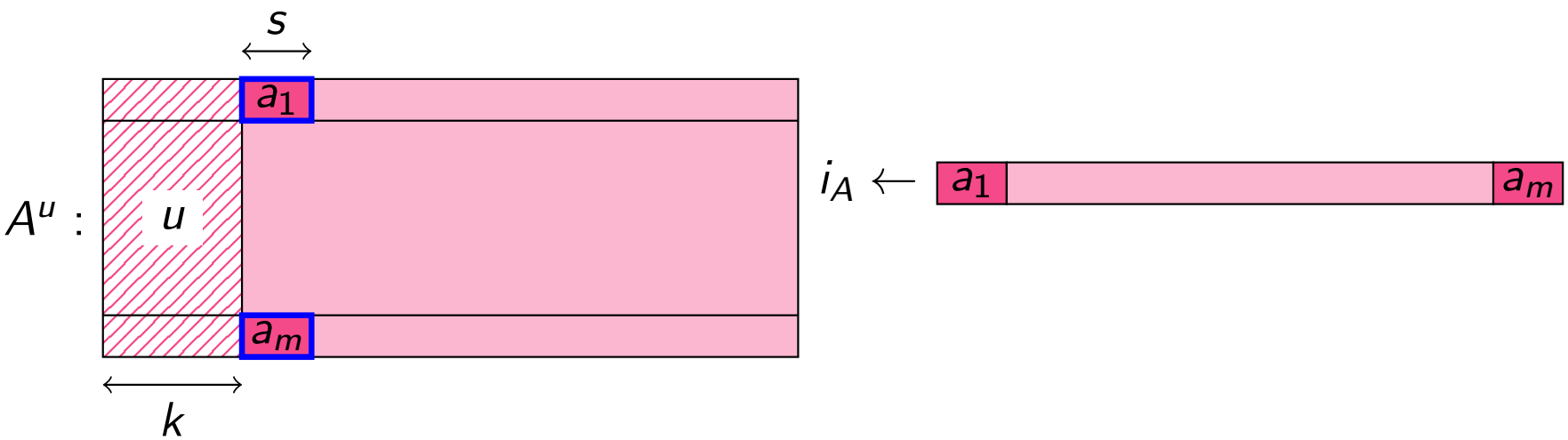

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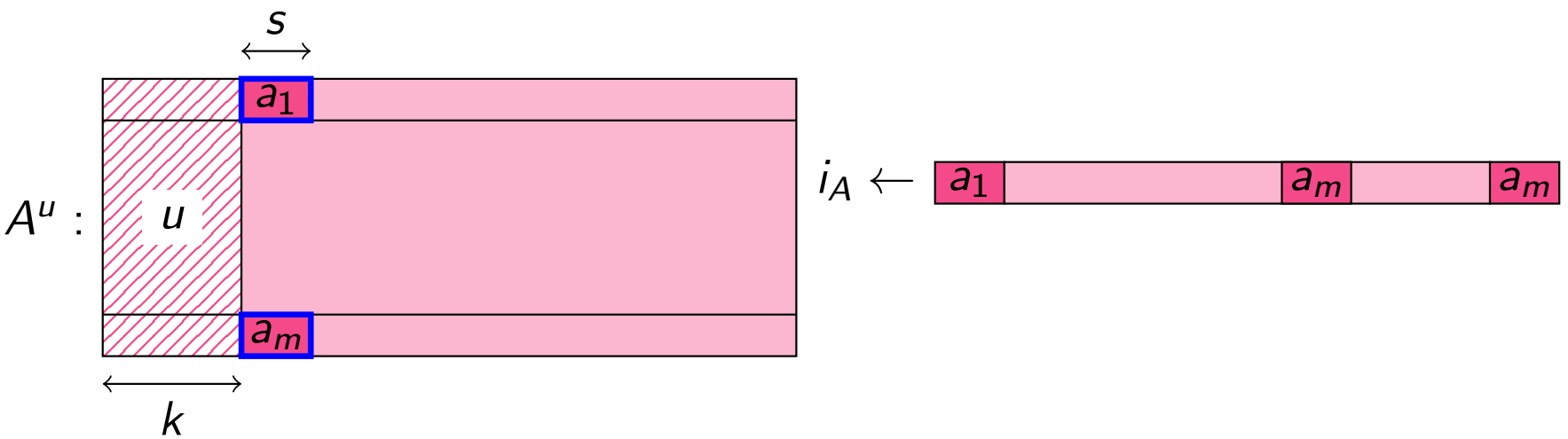

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The Gory Part Starts Now...

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The following slides

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- introduces a lot of notations
- talk about probability

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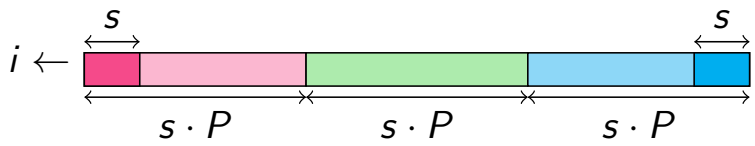
But...

Don't worry: This is nearly over!

BDP in Theory

For each $(A^u, B^v, C^{u \oplus v})$:



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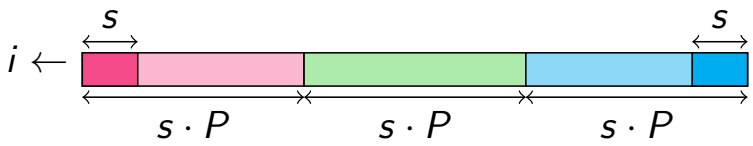

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- m: expected size of a bucket
- w: size of a machine word
- $P = c1 \cdot m$: upper bound on size of a bucket w.h.p. (Chernoff)
- $s = c2 \cdot \log(w)$: only one triplet passes the test for large n
- $3 \cdot s \cdot P \leq n/3$: No space overhead.

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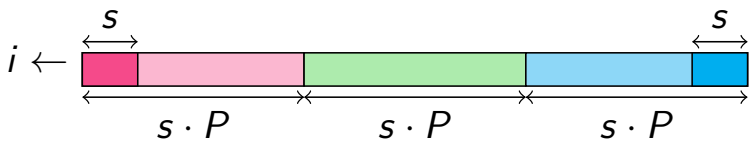

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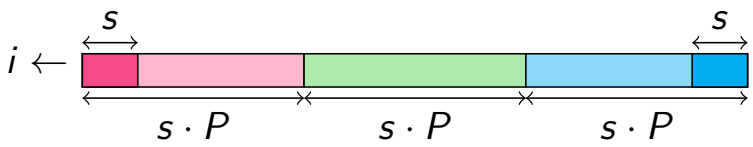
Complexity

When n grows up to infinity:

$$\text{Time: } \mathcal{O}\left(\frac{2^{2n/3}\log^2(n)}{n^2}\right), \text{Space: } \mathcal{O}\left(2^{n/3}\right)$$

BDP In Practice

For each $(A^u, B^v, C^{u \oplus v})$:



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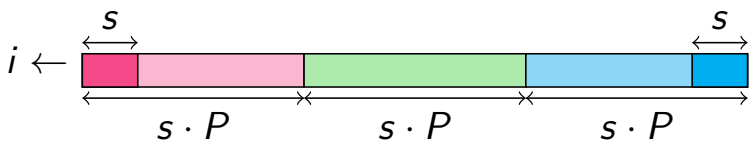

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Concrete example

n = 96:

Expected size of a bucket: $m = 0.14$

$\implies$ Completely impractical

Conclusion

This work

- Discusses issues arising from the 3XOR problem
- Propose a new practical algorithm for the 3XOR problem, that is
  - $n\times$ faster than the quadratic algorithm in theory
  - $3\times$ faster than the quadratic algorithm with our benchmarks
  - Code available:
    - `https://github.com/cbouilla/3XOR`
- Propose an adaptation of [BDP05] algorithm that is
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What's Next?

- Compute a 128-bit 3XOR on SHA-256
  - Expect to have the lists in about 1.5-2 years (using one Antminer S7)
- Look at different settings (e.g. Sparse entries, 3SUM over $(\mathbb{Z}, +)$)

Thank you for your time!