Conflict measure for the discounting operation on belief functions

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***Abstract*—In the belief function theory, the concept of conflict appearing while confronting several experts' opinions can serve for many purposes, and in particular it can be used as an indicator of the relative reliability of the experts. The traditional definition of conflict as the basic belief assigned to the empty set during the combination has several issues and in particular it may not adequately represent the disagreement between the experts in presence.**

Hence, we propose some alternative measures of conflict as the distance between belief functions. These measures of conflict are further used for an *a posteriori* estimation of the relative reliability between the sources of information. This estimation of the reliability does not need any training or prior knowledge and can then be used to discount the unreliable sources before the combination step. These measures are evaluated and debated on random basic belief assignments and on real radar data.

Keywords: Belief functions theory, conflict, distance, discounting.

I. Introduction

Many fusion theories have been studied for the combination of the experts opinions such as voting rules [33], [15], possibility theory [34], [7], and belief functions theory [6], [25]. We can divide all these fusion approaches into four steps: *modelization*, *parameters estimation* depending on the model (not always necessary), *combination*, and *decision*. The most difficult step is presumably the first one. However, the conflict between the expert's responses can only be defined considering the ensemble of the responses. This is the reason why it is generally integrated in the combination step.

The voting rules are not adapted to the modelization of conflict between experts. If both possibility and probability-based theories can model imprecise and uncertain data at the same time, in a lot of applications experts can express their certainty on their perception of the reality. As a result, probabilities-based theory such as the belief functions theory is more adapted.

Belief function theory (also commonly referred to as evidence theory or Dempster-Shafer theory) is one of the most popular one among the quantitative approaches because it can be seen as a generalization of others. Its strength lies in (1) its richer representation of uncertainty and imprecision compared to probability theory and (2) its higher ability to combine pieces of information. In particular, a crucial task in information fusion is the management of conflict between different (partially or totally) disagreeing sources. Dempster's rule is the oldest combination rule of belief function theory [6] and has been the subject of many discussion and critics, arguing (for or against) a possible counter-intuitive behavior. As a consequence, a plethora of alternative combination rules to Dempster's one were born, in particular proposing alternative repartitions of conflict [32], [8], [28], [11], [29], [12], [26], [10], [20], [4]. Last years some unification rules have been proposed [30], [16], [1], [21].

The weight of conflict between some belief functions is indeed an important quantity as it aims at representing the disagreement between the corresponding sources of information. In belief functions theory, the global conflict is traditionally defined by the weight assigned to the empty set after a conjunctive rule, noted k. However, this quantity fails to adequately represent the disagreement between experts in particular when noticing that the conflict between identical belief functions is not null due to the non-idempotence of the majority of the rules (except the rules proposed in [4], [5]). Intuitively, some experts expressing their opinion through the same belief function should be in total agreement. Indeed, as it has been noticed in [24], k includes an amount of *auto-conflict*. Hence the majority of the combination rules does not the difference between the conflict (global or local conflict) and the auto-conflict due to the non-idempotence of the rules.

In a lot of applications, we cannot learn the reliability of each expert, and this reliability cannot be considered before the combination in a discounting procedure. The disagreement between two experts is an indicator of the unreliability of at least one of them: If they totally disagree then, at least one of them is unreliable regarding its opinion, while if they totally agree it can be assumed without any contradictory information that both of them are reliable. Based on this interpretation of the conflict between sources several combination rules have been proposed to automatically and adaptively account for the reliability of the sources [8], [10]. Adaptive combination rules are alternatives to discounting operations when the reliability of the sources cannot be estimated beforehand.

In this paper, we propose an estimation of the relative reliability of a set of sources of information based on the conflict between each other. We make the assumption that the more one expert is in conflict with the others, the more he is unreliable, an assumption which implies another one *i.e.*, that a majority of experts are reliable. This latter assumption is currently made in a fusion process. In Section II, after a recall of the theoretical background of the belief function theory, we discuss the definition of the *auto-conflict*. In Section III we argue that a distance between belief functions such as the ones proposed in [13], [17] is better suited to quantifies the disagreement between two experts. Hence, we propose conflict measures based on this distance, illustrated with randomly generated basic belief assignments. These conflict measures are further used to estimate the reliability of the sources as detailed in Section IV. The estimated reliability assigned to each of the experts is finally used to discount the corresponding belief functions expressing their opinion and illustrated on randomly generated basic belief assignments. We discuss of the interest of the discounting procedure for the combination in terms of complexity. In Section V, the proposed method is used to combine three classifiers for radar targets recognition with real radar data obtained in anechoic chamber. The three classifiers are reliable, hence we generate reliable and not reliable experts to illustrate our approach.

II. Belief Function Theory

A. Theoretical background

Let Θ be a frame of discernment. A *basic belief assignment* (bba) m is the mapping from elements of the power set 2^Θ onto $[0,1]$ such that:

$$m(\emptyset)=0, \text{ and } \sum_{X\in 2^\Theta} m(X)=1. \tag{1}$$

A focal element X is an element of 2^Θ such that $m(X)\neq 0$. Constraining $m(\emptyset)=0$ corresponds to a closed-world assumption [25], while allowing $m(\emptyset)\geq 0$ corresponds to an open world assumption [27]. In order to change from an open world to a closed world assumption, one can simply add an element to the frame of discernment.

From a given bba m, the corresponding credibility and plausibility functions are respectively defined as:

$$\mathrm{bel}(X)=\sum_{A\subseteq X} m(A) \tag{2}$$

and

$$\mathrm{pl}(X)=\sum_{A\cap X\neq\emptyset} m(A). \tag{3}$$

The pignistic probability transformation [27] is generally considered as a good basis for a decision rule. It is defined for all $X\in 2^\Theta$, with $X\neq\emptyset$ by:

$$\mathrm{BetP}(X)=\sum_{Y\in 2^\Theta, Y\neq\emptyset}\frac{|X\cap Y|}{|Y|}\frac{m(Y)}{1-m(\emptyset)}. \tag{4}$$

The first as best known combination rule of the belief function theory has been proposed by Dempster [6] and is defined for two bbas m_1 and m_2, for all $X\in 2^\Theta$, with $X\neq\emptyset$ by:

$$m_{\mathrm{DS}}(X)=\frac{1}{1-k}\sum_{A\cap B=X} m_1(A)m_2(B), \tag{5}$$

where $k = \sum_{A\cap B=\emptyset} m_1(A)m_2(B)$ is generally called the global conflict of the combination or the inconsistence of the combination.

The problem enlightened by the now famous Zadeh's example is the repartition of the global conflict. Indeed, consider $\Theta=\{A,B,C\}$ and two experts opinions given by $m_1(A)=0.9$, $m_1(C)=0.1$, and $m_2(B)=0.9$, $m_1(C)=0.1$, the bba resulting in the combination using Dempster's rule is $m(C)=1$.

In order to solve partially this paradox, Smets [28] proposes to consider an open world, therefore the conjunctive rule is non-normalized and we have for two basic belief assignments m_1 and m_2 and for all $X\in 2^\Theta$ by:

$$m_{\mathrm{Conj}}(X)=\sum_{A\cap B=X} m_1(A)m_2(B) := (m_1\oplus m_2)(X). \tag{6}$$

$k=m_{\mathrm{Conj}}(\emptyset)$ can be interpreted as a non-expected solution. However this is still a problem for the combination of conflicting belief functions [31].

Yager [32] interpreted k as ignorance Θ and proposed the rule given for two basic belief assignments m_1 and m_2 and for all $X\in 2^\Theta$ by:

$$\left\{\begin{array}{rcl} m_{\mathrm{Y}}(X) &=& m_{\mathrm{Conj}}(X), \forall X\in 2^\Theta\setminus\{\emptyset,\Theta\}\\ m_{\mathrm{Y}}(\Theta) &=& m_{\mathrm{Conj}}(\Theta)+m_{\mathrm{Conj}}(\emptyset)\\ m_{\mathrm{Y}}(\emptyset) &=& 0. \end{array}\right. \tag{7}$$

In [23], Murphy proposes a combination rule as the average of the basic belief assignments:

$$m_{\mathrm{Mean}}(X)=\frac{1}{M}\sum_{i=1}^{M} m_i(X). \tag{8}$$

B. The auto-conflict

As observed in [17], the weight of conflict given by $k=m_{\mathrm{Conj}}(\emptyset)$ is not a conflict measure between the basic belief assignments. Indeed, in case of non-idempotent rules, the combination of identical basic belief assignments leads generally to a positive value of k. To highlight this behavior, we defined in [24] the *auto-conflict* which quantifies the intrinsic conflict of a bba. The auto-conflict of order n for one expert is given by:

$$a_n=(\overset{n}{\underset{i=1}{\oplus}} m)(\emptyset), \tag{9}$$

where $\oplus$ is the conjunctive operator of Equation (6). The following property holds:

$$a_n\leq a_{n+1}, \tag{10}$$

meaning that due to the non-indempotence of $\oplus$, the more m is combined with itself the nearer to 1 k is, and so in a general case, the more the number of experts is high the nearer to 1 k is.

In order to study the distribution of the auto-conflict we randomly generated non-dogmatic belief functions (*i.e.* such that $m(\Theta)\neq 0$), considering all the singletons of Θ and the ignorance Θ itself as only focal elements. Figure 1 shows the average of the auto-conflict over 1000 masses according to the order n and for different cardinalties of Θ. It appears that the auto-conflict comes quickly near to 1 according to $|\Theta|$ and the order n. Figure 2 focuses on the distributions of the auto-conflict for $|\Theta|$ =3, 4, 5 and 6 and for an integer $n \in [2, 7]$. For $|\Theta| \geq 4$ and $n \geq 4$ the distribution can be approximated by a function of the form of $\dfrac{1}{1-\exp(x)}$. This shows once again that the auto-conflict tends quickly to 1.

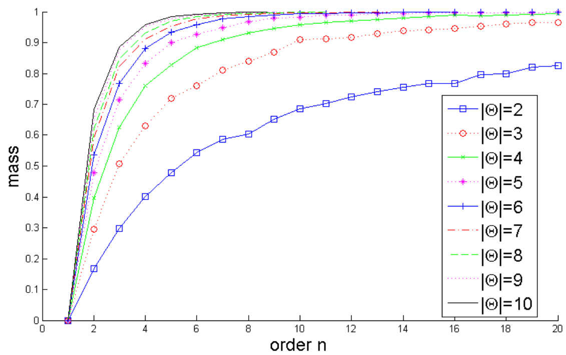

Figure 1. Average of the auto-conflict for randomly generated bbas.

Figure 3 considers the average of the conflict k for successive random-generated masses according to both $|\Theta|$ and the number of experts n. We can note that k tends more quickly toward 1 than the auto-conflict. The distribution form of k is also very similar to the distribution of the auto-conflict for a given order n. These results illustrate that k does not

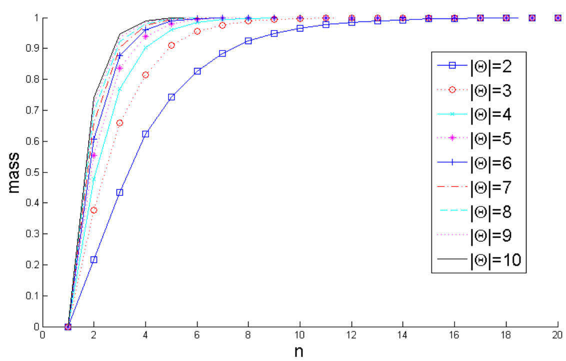

Figure 3. Average of the conflict for randomly generated bbas.

adequately defines a conflict measure between a set of experts. Although we must take into account the internal inconsistency k in the combination, we also want to take into account the conflict among the experts.

III. Conflict measure

A. Distance between experts for quantifying the conflict

Rather than the measure k, we propose here to define the conflict between experts opinions through a *distance* between their respective bbas. Hence, if the opinions of two experts are far from each other, we consider that they are in conflict.

We use in this paper the distance defined in [13], distance used in several works [2], [3], [5]. This distance is defined for two basic belief assignments m_1 and m_2 by:

$$d(m_1, m_2) = \sqrt{\frac{1}{2}(\mathbf{m}_1 - \mathbf{m}_2)^t \underline{\underline{D}}(\mathbf{m}_1 - \mathbf{m}_2)}, \tag{11}$$

where $\underline{\underline{D}}$ is an $2^{|\Theta|} \times 2^{|\Theta|}$ matrix whose elements are:

$$D(A,B) = \begin{cases} 1, \text{ if } A = B = \emptyset, \\ \dfrac{|A \cap B|}{|A \cup B|}, \ \forall A, B \in 2^\Theta. \end{cases} \tag{12}$$

Our assumption is that the more two bbas are far from each other and the more they are in conflict. Hence, the conflict measure between 2 experts can be defined by:

$$Conf(1,2) = d(m_1, m_2). \tag{13}$$

To assign a weight to each expert, we must quantify how much a given expert in a set of experts $\mathcal{E} = \{1, \ldots, M\}$ is in conflict with the rest of the set. Thus, we can define the conflict measure between one expert i and the other $M-1$ experts by:

$$Conf(i, \mathcal{E}) = \frac{1}{M-1} \sum_{j=1, i \neq j}^{M} Conf(i,j). \tag{14}$$

Another possible definition is:

$$Conf(i, M) = d(m_i, m_M), \tag{15}$$

where m_M is the bba of the artificial expert representing the combined opinions of all the experts in $\mathcal{E}$ except i. The combination referred here can be the conjunctive combination (6), the normalized conjunctive combination (5), the Yager's rule (7), the average of the bbas (8), etc. Which combination rule to choose for computing m_M is not obvious.

Here in the extension of the conflict measure from two bbas to M bbas, we make the implicit assumption that more than half of the number of experts are reliable. Indeed, one expert, reporting a bba m, is in conflict with the others, if the bba m is far away from the bbas reported by the other experts.

B. Simulations

Let us consider 130 experts expressing their opinions by means of belief functions defined on 2^Θ with $\Theta = \{S_1, S_2\}$. We randomly generated bbas for 100 experts assigning a mass to both S_1 and Θ and 30 experts assigning a mass to both S_2 and Θ. Figure 4 presents the conflict obtained for each expert according to the mass on S_1 and S_2 given by each expert for different ways to calculate the conflict (*i.e.* using different combination rules and the mean of conflicts). We considered the conflict measures defined by Equation (14) and by Equation (15) with the conjunctive rule (6), the normalized conjunctive rule (5), the Yager's rule (7) and the average of the bbas (Equation (8)). Because of the high number of experts,

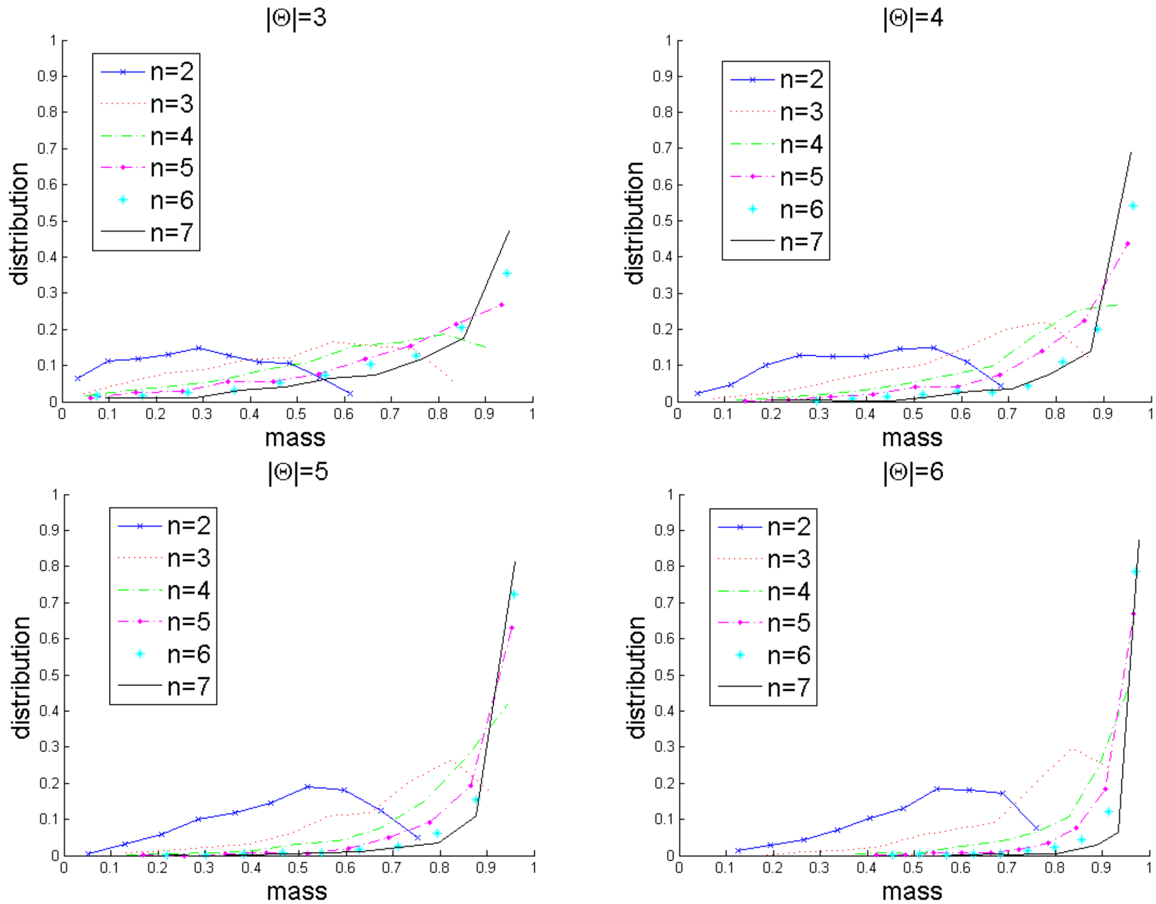

Figure 2. Distributions of the auto-conflict according to $|\Theta|$.

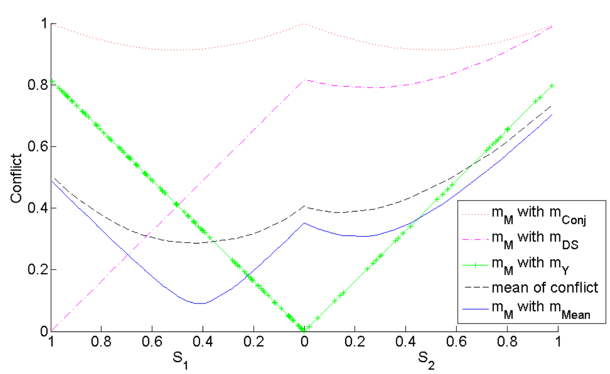

Figure 4. Conflict measure according to the mass on S_1 and S_2 for different combination rules.

the value of k is close to 1. Hence, the conjunctive and the normalized conjunctive rules do not work well. Yager's rule transfers k to Θ and then the conflict of one expert according to Equation (15) becomes linear with respect to the mass of Θ (and so with respect to the singleton S_1, next S_2 because we have only two focal elements). In this case with many experts only the conflict given by Equations (14) and (15) with the mean of the bbas lead to good results. Here, some experts are not sure because the random mass on the singleton can be smaller than 0.5 (and so the mass on Θ is bigger than 0.5).

Let us now consider a sightly different example: Over the 130 randomly generated experts, we only keep the experts whose masses assigned to singletons are higher than 0.5. Remains 44 experts expressing their opinions in favor of S_1 and 18 on S_2. Figure 5 presents the obtained conflict using the same method than for Figure 4. Here the conflict given by Equations (14) and (15) together with the use of the mean of the bbas leads to a separation between the two groups of experts. Indeed, the threshold for the first group is around 0.5 while it is around 0.4 for the second one.

We now consider only 5 experts whose respective bbas are given in Table I: 3 experts are very favorable to S_1 and 2 to S_2. We note that even with few experts, the conflicts given by Equation (15) with the conjunctive, the normalized conjunctive and Yager's rules are conclusive. Moreover, with Equations (14) and (15) with the mean of the bbas, the conflicts for the three experts favorable to S_1 are weaker than the conflict for the two experts favorable to S_2.

Let us now consider the example of Table II with 6 experts: Two experts express a favorable opinion toward S_1

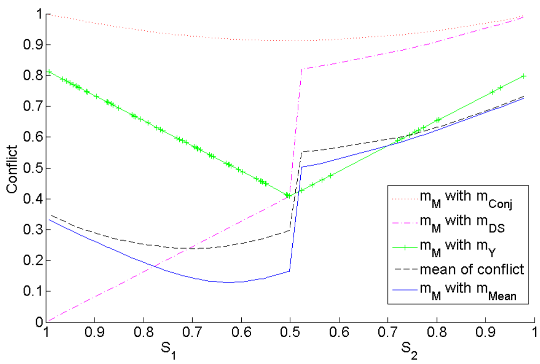

Figure 5. Conflict measure according to the mass on S_1 and S_2 for different ways to calculate the conflict for sure experts.

	Experts				
	1	2	3	4	5
S_1	0.8147	0.9058	0.9134	0	0
S_2	0	0	0	0.9706	0.9572
$S_1 \cup S_2$	0.1853	0.0942	0.0866	0.0294	0.0428
m_M with m_{Conj}	0.9426	0.9615	0.9629	0.9678	0.9700
m_M with m_{DS}	0.7831	0.8827	0.8916	0.9521	0.9307
m_M with m_{Y}	0.6626	0.7354	0.7413	0.7833	0.7747
mean of conflict	0.3888	0.3903	0.3930	0.5582	0.5539
m_M with m_{Mean}	0.4282	0.4779	0.4826	0.6965	0.6874

Table I
BBAS AND RESULTING CONFLICT FOR ONLY 5 EXPERTS.

with approximately the same mass 0.7, two experts express a favorable opinion toward S_2 with approximately the same mass 0.65, one expert express a favorable opinion toward S_2 with a high mass 0.93 and the last expert has a high ignorance. In this example, one more time, the conjunctive rule does not work well. Dempster's rule provides low conflict for both experts (3 and 4) with the same mass for S_2 and a high conflict for both experts 1 and 2. The Yager's rule provides a very low conflict for the expert number 6 and the higher conflict for the expert 5. The conflict measures given by the Equation (14) are very near for the 6 experts, with the lower conflict for the experts 3 and 4. The last conflicts given by the Equation (15) with the mean of the bbas, are low for the experts 3, 4 and 6.

	Experts					
	1	2	3	4	5	6
S_1	0.7060	0.6948	0	0	0	0.1082
S_2	0	0	0.6557	0.6787	0.9340	0.1386
$S_1 \cup S_2$	0.2940	0.3052	0.3443	0.3213	0.0660	0.7532
m_{Conj}	0.7930	0.7940	0.8351	0.8368	0.8590	0.8577
m_{DS}	0.8538	0.8491	0.2129	0.1997	0.5127	0.6342
m_{Y}	0.5599	0.5503	0.4680	0.4866	0.6890	0.1009
eq. (14)	0.4482	0.4441	0.3354	0.3390	0.4551	0.4011
m_{Mean}	0.5151	0.5077	0.3036	0.3181	0.5186	0.3224

Table II
BBAS AND RESULTING CONFLICT FOR ONLY 6 EXPERTS.

The expert 5 is sure of its response S_2, but the other experts 3 and 4 are not sure. Hence, the conflict of the expert 5 can be high, even he seems to say true. The expert 6 with a lot of ignorance can be in small conflict with the other experts because his bba is different. The distance given by the Equation (11) takes into account the specificity of the responses with the calculus of the matrix $\underline{\underline{D}}$ given by the Equation (12). And so, the conflict measure takes also into account the specificity. Here, we could change the definition of the matrix $\underline{\underline{D}}$ or the distance to weight more the higher specificities.

IV. Reliability based on conflict measure

The conflict appearing while confronting several experts' opinions can be used as an indicator of the relative reliability of the experts. We have seen that there exist many rules in order to take into account the conflict during the combination step. These rules do make not the difference between the conflict (global or local conflict) and the auto-conflict due to the non-idempotence of the majority of the rules. We propose here the use of a conflict measure in order to define a reliability measure, that we consider before the combination, in a discounting procedure.

When we can quantify the reliability of each expert, we can weaken the basic belief assignment before the combination by the discounting procedure:

$$\left\{\begin{array}{l} m_i^\alpha(X) = \alpha_i m_i(X),\, \forall X \in 2^\Theta \setminus \{\Theta\} \\ m_i^\alpha(\Theta) = 1 - \alpha_i(1 - m_i(\Theta)). \end{array}\right. \tag{16}$$

$\alpha_i \in [0,1]$ is the discounting factor of the expert i that is, in this case, the reliability of the expert i, eventually as a function of $X \in 2^\Theta$.

Other discounting procedures are possible such as the contextual discounting [22], or a discounting procedure based on the credibility or the plausibility functions [35].

A. Reliability estimation

According to the applications, we can search to learn the discounting factors α_i for example from the confusion matrix [19]. In a lot of applications, we cannot learn the reliability of each expert. A general approach in order to evaluate without learning the discounting factor is given in [9]. For a given bba the discounting factor is obtained by the minimization on α of a distance given by:

$$Dist^{\alpha_i} = \sum_{A \in \Theta} \left(\mathrm{BetP}_i(A) - \delta_{A,i}\right)^2, \tag{17}$$

where BetP_i is the pignistic probability (Equation (4)) of the bba given by the expert i and $\delta_{A,i} = 1$ if the expert i supports A and 0 otherwise.

This approach is interesting with the goal of a pignistic decision. However, if the expert i does not support a singleton of Θ, the minimization on α_i does not work well.

In order to combine the bbas of all experts together, we propose here to estimate the reliability of each expert i from the conflict measure $Conf$ between the expert i and the others by:

$$\alpha_i = f(Conf(i, M)), \tag{18}$$

where f is a decreasing function. We can choose:

$$\alpha_i = \left(1 - Conf(i,M)^{\lambda}\right)^{1/\lambda}, \tag{19}$$

where $\lambda > 0$. We illustrate this function for $\lambda = 2$ and $\lambda = 1/2$ on figure 6. This function allows to give more reliability to the experts with few conflict with the other.

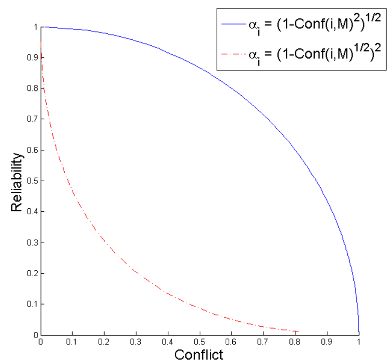

Figure 6. Reliability of one expert according to the conflict of the expert with the other experts.

Other definition are possible. The credibility degree defined in [2], also based on the distance given in the Equation (11), could also be interpreted as the reliability of the expert. However the credibility degree is integrated directly in the combination with a weighted average. Our reliability measure allows the use of all the existing combination rules.

If we take again the previous example given by the table II, the obtained values of α_i are given in the table III.

	Experts					
	1	2	3	4	5	6
m_{Conj}	0.6092	0.6079	0.5501	0.5475	0.5120	0.5142
m_{DS}	0.5206	0.5282	0.9771	0.9799	0.8586	0.7732
m_{Y}	0.8286	0.8350	0.8837	0.8736	0.7248	0.9949
eq. (14)	0.8939	0.8960	0.9421	0.9408	0.8904	0.9160
m_{Mean}	0.8571	0.8615	0.9528	0.9481	0.8550	0.9466

Table III
RELIABILITY MEASURE BASED ON CONFLICT MEASURES DEFINED BY THE EQUATION (19) WITH $\lambda = 2$ FOR ONLY 6 EXPERTS.

In the the special case of only 2 experts, the conflict measure is directly given by the Equation (13) and is the same for both experts. Hence, in the one hand, if the conflict measure is high (*i.e.* the distance between the two experts is high), the reliability measures will be weak. So we increase the mass on the ignorance for both basic belief assignments. In the other hand, if the he conflict measure is weak (that means that both experts say approximately the same thing) the reliability measures will be weak. Hence, we consider both bbas in the combination rule.

B. Complexity interest for the combination

Unlike conflict redistributing combination rules, the discounting operation is applied as a separated step. So, if we use an associative combination rule, we can proceed by taking M experts one by one, and make $M-1$ calls to a combination procedure between two experts instead of one call to a combination procedure between M experts, usually a lot more time consuming.

Based on the conjunctive rule (6), one can build a conflict redistribution rule, which is non-associative, but can take any number of experts in parameter. Such a rule is illustrated by the PCR6, in [20].

$$m_{\text{PCR6}}(X) = m_{\text{Conj}}(X) + \sum_{i=1}^{M} m_i(X)^2 \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X = \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (2^{\Theta})^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right), \tag{20}$$

where $Y_j \in 2^{\Theta}$ is the response of the expert j, $m_j(Y_j)$ is the associated belief function and σ_i counts from 1 to M avoiding i:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j+1 & \text{if } j \geq i. \end{cases} \tag{21}$$

Let n be the cardinal of Θ, and p a "standard" number of focal elements for an expert.

To combine the bbas from two experts, most rules will use $\mathcal{O}(p^2)$ elementary operations. If the experts only use singletons and Θ as focal elements, the resulting bba has less than $n+2$ focal elements, including $\emptyset$ and the ignorance. When considering larger input focal elements or other combination rules, like Dubois & Prade's [8] or the disjunctive rule, one can get up to p^2 focal elements. The mean operator is cheaper, with only $\mathcal{O}(p)$ operations, and $\mathcal{O}(p)$ focal elements.

Calculating $d(m_1, m_2)$ by the formula (11) may be costly. The matrix $\underline{\underline{D}}$ has 2^{2n} entries; half of them are zero, and a half of the remaining ones are determined by symmetry properties. The memory needed to simply store the whole matrix is 2^{2n-2}. However, the vector $\mathbf{m}_1 - \mathbf{m}_2$ has only at most $2p$ non-zero entries over the 2^n it contains. So $d(m_1, m_2)$ can be calculated in $\mathcal{O}(p^2)$ operations.

The bba m_M will typically have more focal elements than the input ones. We will consider this parameter as $\mathcal{O}(n)$, reflecting the style of the experts of the preceding examples. So calculating $d(m_i, m_M)$ costs $\mathcal{O}(np)$ operations.

Therefore, the discounting procedure needs $\mathcal{O}(Mnp)$ for calculating m_M, $\mathcal{O}(n^2 + Mp))$ operations for calculating the α_i by the formula (19), and $\mathcal{O}(Mp)$ operations to apply the procedure (16). We just have to combine the discounted bbas ($\mathcal{O}(Mnp)$ operations) to obtain the result. The overall complexity of the discounting combination is $\mathcal{O}(Mnp)$.

Now, if we compute the α_i with the Equation (17), considering that the calculus of the minimum needs K operations, the distance $Dist^{\alpha_i}$ costs $\mathcal{O}(np)$ operations and so we obtain α_i with $\mathcal{O}(np + Kp)$.

As the auto-conflict of order k for one expert can be calculated in $\mathcal{O}(kp)$ operations, taking it into account during the procedure does not make it more costly.

Conflict redistributing rules are not associative: all the experts must be combined in an unique step. It takes $\mathcal{O}(Mp^M)$ operations.

Both procedures tend to the same result: discounting enforces ignorance for minority experts, giving more weight to a focal element A of an other expert, when assigning the mass $m_1(\Theta)m_2(A)$ to $A = A \cap \Theta$. Conflict redistribution enforces the majority when a local conflict arises around the focal element X of the expert k:

$$\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X = \emptyset. \tag{22}$$

So the discounting procedure should be preferred in the cases where many experts (typically, more than 5) are implied. Conflict redistribution should be preferred when a sharp treatment of local conflict is needed, to avoid information loss. It allows to extract the real part of truth when experts are only partially wrong.

V. Illustrations

As an illustration, we consider 5 scale reduced (1:48) targets (Mirage, F14, Rafale, Tornado, Harrier) to classify. The real data were obtained in the anechoic chamber of ENSIETA (Brest, France) using the experimental setup [18].

Each target is illuminated in the acquisition phase with a frequency stepped signal. The data snapshot contains 32 frequency steps, uniformly distributed over the band $B = [11650, 17850]MHz$, which results in a frequency increment of $\Delta f = 200MHz$. Consequently, the slant range resolution and ambiguity window are given by:

$$\Delta R_S = c/(2B) \simeq 2.4m,\ W_S = c/(2\Delta f) = 0.75m \tag{23}$$

The complex signature obtained from a backscattered snapshot is coherently integrated via FFT in order to achieve the slant range profile corresponding to a given aspect of a given target. For each of the 10 targets 150 range profiles are thus generated corresponding to 150 angular positions, from -5° to 69.5°, with an angular increment of 0.50.

We classify these data with three supervised classifiers (a classical k-nearest neighbor, a fuzzy k-nearest neighbor [14], and a multi perceptron [18]). The training set is formed by randomly selecting 2/3 of the range profiles, the others being considered as the test set. Then we fuse the three responses of the classifiers. We can interpret the outputs of the three classifiers as the mass of the target-singleton. We just apply a discounting with $\alpha = 0.95$ in order to combine these basic belief assignments.

Hence with 250 range profiles for testing, we obtain the following good classification rates: 96.4% for the classical k-nearest neighbor, 92.4% for the fuzzy k-nearest neighbor and 82.0% for the multi perceptron. We can interpret these rates as reliabilities of the three classifiers.

Table IV gives the reliability measure based on the conflict measures defined by the Equation 19 with $\lambda = 1.5$ for the three classifiers. We can observe that the reliability of the multi perceptron is the lowest, except for the reliability given by the m_{Conj}. This is the classifier given the lowest rate. The good classification rates of the k-nearest neighbor and the fuzzy k-nearest neighbor are very near. The reliabilities for these both classifiers are inversed, compared to the good classification rates, but are also very near.

	Classifiers		
	k-nn	fuzzy k-nn	multi perceptron
m_{Conj}	0.8459	0.8682	0.8572
m_{DS}	0.9655	0.9585	0.8634
m_{Y}	0.8854	0.9047	0.8693
eq. (14)	0.9686	0.9715	0.9406
m_{Mean}	0.9462	0.9572	0.8900

Table IV
Reliability measure based on conflict measures defined by the Equation 19 with $\lambda = 1.5$ for the three classifiers.

In fact, the three classifiers are quite reliable. To study the proposed reliability measure, we generate random bbas.

In the first case (Table V), we generate bbas with only two focal elements with $\Theta = \{C_1, C_2, C_3\}$; one focal element is C_1 for the three fist bbas and C_3 for the fourth one, and the second focal element is Θ. The table shows that all the reliability measures give the expert 4 not reliable in this case.

	Experts			
	1	2	3	4
m_{Conj}	0.6986	0.7156	0.7204	0.4438
m_{DS}	0.7637	0.8061	0.7932	0.4438
m_{Y}	0.8468	0.8683	0.8599	0.4438
eq. (14)	0.8904	0.8851	0.8891	0.7886
m_{Mean}	0.8782	0.8647	0.8766	0.6809

Table V
Reliability measure based on conflict measures defined by the Equation 19 with $\lambda = 1.5$ for the four experts (three reliable and one not).

In the second case (Table VI), we generate bbas with only four focal elements with $\Theta = \{C_1, C_2, C_3\}$; one focal element is C_1 for the three fist bbas and C_3 for the fourth one, and another focal element is Θ, the two other are in 2^Θ. In this case, the expert 4 is still the less reliable, but the difference with the reliability of the other experts is weaker. Indeed, the generated bbas for the first three experts can provide a bigger mass on C_3 than on C_1, and so the bbas can be very similar in some cases than the bbas of the expert 4.

VI. Conclusions

In this paper, we proposed some conflict measures of a group of experts based on the distance of basic belief assignments. In particular, the conflict is evaluated for one expert i against the rest of the group according to two distinct

	Experts			
	1	2	3	4
m_{Conj}	0.6847	0.7112	0.6750	0.6506
m_{DS}	0.7847	0.7975	0.7860	0.7136
m_{Y}	0.8461	0.8441	0.8439	0.8333
eq. (14)	0.8873	0.8895	0.8870	0.8752
m_{Mean}	0.8703	0.8759	0.8665	0.8430

Table VI
RELIABILITY MEASURE BASED ON CONFLICT MEASURES DEFINED BY THE EQUATION 19 WITH $\lambda = 1.5$ FOR THE FOUR EXPERTS (THREE RELIABLE AND ONE NOT).

approaches: (1) the average of all the distances between i's bba and each bba of the other experts of the group (except i), and (2) the distance of i's bba to the bba obtained by the combination of the bbas of the other experts (except i). These measures of conflict are further used for an *a posteriori* estimation of the relative reliability between the sources of information: The more i is in conflict with the rest of the group of experts, the less i is reliable.

Our proposed measures of conflict and the associated reliability are evaluated and debated on random basic belief assignments but also on a real radar target recognition application. It appears that the reliability estimation provides a good alternative measure to be used in the discounting procedure on belief functions when the reliability is unknown and cannot be estimated *a priori*. Moreover, beside their link to the reliability estimation, the proposed conflict measures could be employed for example to alert the decision maker in a decision support system.

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