Constructing of a consonant belief function induced by a multimodal probability density function

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Abstract – *In this paper, we generalize the approach of Ph. Smets on the continuous belief functions. Instead of having only connected sets as focal set, we put basic belief assignement on elements of the Borel sigma-algebra of* $\overline{\mathbb{R}}^n$ *(the extended real space set of dimension* n*). We decide to analyse belief functions with an index function which describes all the focal sets. We focus on the consonant belief functions and we show some of their properties. They are useful to define belief function associated to multimodal probability density function. We apply the obtained results to compute a consonant belief function linked to a Gaussian mixture.*

Keywords: Continuous belief function, multimodal probability density function, consonant belief function.

1 Introduction

The theory of belief functions is a powerful formalism to describe the imperfections of the data given by an information's source. They are widely used in classification to merge the decisions comming from several classifiers. This classifiers can have in input sensors' data. They use the estimation of continuous parameters to take a decision. It could be a good idea to estimate these parameters with the theory of belief functions. Recently, some works deal with the question of representation of belief functions on real numbers. Several approaches exist [12, 5, 6, 10]. In this paper, we will focus on the way that belief functions have been modeled in [12, 6, 10]. It allows us to link belief functions with probability. However, in these works, basic belief assignement are allocated only on connected sets of $\overline{\mathbb{R}}^n$ (the extended real space set of dimension n). This choice has been done to work with user-friendly objects. Unfortunately, when we manipulate belief functions linked to multimodal probability, it seems better to assign beliefs on unions of disjointed sets. Thus we need a formalism to describe a more complex frame of discernment.

In the following, after a brief reminder of the result about the belief functions, we will present the Smets'approach [10] of continuous belief functions (*cf.* section 3). Then, we will propose a new way to represent belief function in order that focal elements belong to $\mathcal{B}\left(\overline{\mathbb{R}}^n\right)$, the Borel sigma-algebra of $\overline{\mathbb{R}}^n$ (*cf.* section 4). To work with user-friendly object, we will focus, as it has been done in [10, 1, 6], with consonant belief functions. We will use them to model the information transmitted by multimodal probability density function. To finish, using the result in [1], we will compare this new approach with the one suggested in [10, 1] by computing the consonant belief function linked to a Gaussian mixture (*cf.* section 5).

2 Discrete belief functions on $\overline{\mathbb{R}}^n$

In a classical way [2, 7, 8], a frame of discernment Ω is a finite set of disjointed elements. The set built with all the subset of Ω is noted 2^Ω. The belief functions m^Ω allow us to model information on all the elements of 2^Ω. A focal element is an element A of 2^Ω whose the basic belief assignment $m^\Omega(A)$ is not equal to zero. A basic belief assignment function verifies the condition $\sum_{A\subseteq\Omega} m^\Omega(A) = 1$. It is linked to the following belief functions define for each $X \subseteq \Omega$ by:

- belief function

$$bel^\Omega(X) = \sum_{A\subseteq X, A\neq\emptyset} m^\Omega(A) \tag{1}$$

- plausibility function

$$pl^\Omega(X) = \sum_{A\subseteq\Omega, A\cap X\neq\emptyset} m^\Omega(A) \tag{2}$$

- communality function

$$q^\Omega(X) = \sum_{X\subseteq A} m^\Omega(A) \tag{3}$$

- pignistic probability [9]

$$BetP^\Omega(X) = \sum_{A\subseteq\Omega, A\cap X\neq\emptyset} \frac{|A\cap X|}{|A|}\frac{m^\Omega(A)}{1-m^\Omega(\emptyset)} \tag{4}$$

By combining m_1^Ω and m_2^Ω according to the conjonctive rule, we have $m_{1 \circledcirc 2}^\Omega$ such as for all $A \subseteq \Omega$:

$$m_{1 \circledcirc 2}^\Omega(A) = \sum_{X \cap Y = A} m_1^\Omega(X) \, m_2^\Omega(Y) \quad (5)$$

An other way to write this rule is to consider the communality:

$$q_{1 \circledcirc 2}^\Omega(A) = q_1^\Omega(A) \cdot q_2^\Omega(A) \quad (6)$$

Thanks to this definition of belief functions, we can model a discrete belief on $\overline{\mathbb{R}}^n$ with a basic belief assignment m^Θ. The frame of discernment Θ is made of a countable set of disjointed elements of $\mathcal{B}\left(\overline{\mathbb{R}}^n\right)$. The set of focal elements of m^Θ (*i.e.* the subset A of 2^Θ such as $m^\Theta(A) \neq 0$), which is written $\mathcal{F}\left(m^\Theta\right)$, is a countable set of elements of 2^Θ. These elements can be written F_i, with $i \in \mathbb{N}$. m^Θ verify the condition $\sum_i m^\Theta(F_i) = 1$. We can define for all A in $\mathcal{B}\left(\overline{\mathbb{R}}^n\right)$ the following belief functions:

$$bel^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \sum_{F_i \subseteq A} m^\Theta(F_i) \quad (7)$$

$$pl^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \sum_{F_i \cap A \neq \emptyset} m^\Theta(F_i) \quad (8)$$

$$q^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \sum_{A \subseteq F_i} m^\Theta(F_i) \quad (9)$$

$$BetP^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \sum_{A \cap Fi} \frac{|A \cap F_i|}{|A|} \frac{m^\Theta(A)}{1 - m^\Theta(\emptyset)} \quad (10)$$

These belief functions are the same that those describe previously. All the usual properties on belief functions can be applied on them. However, these functions are discrete. Hence they are not well fit to estimate a continuous parameter. Modeling belief functions with a continuous function is a way to resolve the problem.

3 Continuous belief functions with connected focal elements in $\overline{\mathbb{R}}^n$

In order to model information given by a continuous belief function, we can use a basic belief density function $m^{\overline{\mathbb{R}}^n}$ (*cf.* [12, 10]), as in the theory of of probabilities we use a probability density function (pdf). The basic belief density functions (bbd) allocate a density to sets of $\overline{\mathbb{R}}^n$. Unfortunately, a belief function does not verify the additivity property (*i.e.* $bel(A \cup B) \neq bel(A) + bel(B) - bel(A \cap B)$). To use continuous belief functions, we need to work with belief functions whose the focal elements are easily described.

3.1 Connected set of $\overline{\mathbb{R}}^n$

Ph. Smets [10] suggests to model continuous belief functions on $\overline{\mathbb{R}}$ by applying mass only on intervals of $\overline{\mathbb{R}}$. He links a bbd $m^{\overline{\mathbb{R}}}$ on $\overline{\mathbb{R}}$, to a pdf $f^\mathcal{T}$ on $\mathcal{T} = \{(x, y) \in \mathbb{R}^2 | x \leq y\}$ (we set $[x, y] = \emptyset$ si $y < x$). By analogy with the discrete belief functions, we obtain:

$$bel^{\overline{\mathbb{R}}}([a, b]) = \int_{x=a}^{x=b} \int_{y=x}^{y=b} f^\mathcal{T}(x, y) \, dy \, dx \quad (11)$$

$$pl^{\overline{\mathbb{R}}}([a, b]) = \int_{x=-\infty}^{x=b} \int_{y=\max(a,x)}^{y=+\infty} f^\mathcal{T}(x, y) \, dy \, dx \quad (12)$$

$$q^{\overline{\mathbb{R}}}([a, b]) = \int_{x=-\infty}^{x=a} \int_{y=b}^{y=+\infty} f^\mathcal{T}(x, y) \, dy \, dx \quad (13)$$

$m_{1 \circledcirc 2}^{\overline{\mathbb{R}}}$ is the bbd resulting from conjonctive combination of $m_1^{\overline{\mathbb{R}}}$ and $m_2^{\overline{\mathbb{R}}}$. The product $m_1^{\overline{\mathbb{R}}}(A) \cdot m_2^{\overline{\mathbb{R}}}(B)$ is allocated to $m_{1 \circledcirc 2}^{\overline{\mathbb{R}}}(A \cap B)$. For each closed set A of $\overline{\mathbb{R}}$:

$$q_{1 \circledcirc 2}^{\overline{\mathbb{R}}}(A) = q_1^{\overline{\mathbb{R}}}(A) \cdot q_2^{\overline{\mathbb{R}}}(A) \quad (14)$$

This is only an introduction to the results obtain by Ph. Smets [10]. We can extend this model to $\overline{\mathbb{R}}^n$.

3.2 Consonant bbd

The study of consonant bbd is the object of several papers [6, 10, 1]. The focal elements of this kind of belief function are nested. For each A et B, focal elements of $m^{\overline{\mathbb{R}}^n}$, we have $A \subset B \Longleftrightarrow q^{\overline{\mathbb{R}}^n}(B) < q^{\overline{\mathbb{R}}^n}(A)$. Therefore it is quite normal to assign an index y to an element $F(y) \in \mathcal{F}\left(m^{\overline{\mathbb{R}}^n}\right)$ such as $y < y'$ imply $F(y) \subseteq F(y')$.

3.3 Least Committed bbd induced by unimodal pdf

The estimation of a parameter is done thanks to data given by several sensors. The information comming from sensors is often modelized by pdf. To merge this information with the framework of the theory of belief functions, we have to linked a bbd to a pdf. Ph. Smets introduces in [9] the concept of pignistic transformation. A bbd $m^{\overline{\mathbb{R}}^n}$ corresponds to a pdf $Betf$ and a pignistic probability $BetP$. For each interval $[a, b]$ in $\overline{\mathbb{R}}$, we have according to [10]:

$$BetP([a, b]) = \int_{x=-\infty}^{x=\infty} \int_{y=x}^{y=\infty} \frac{y \wedge b - x \vee a}{y - x} f^\mathcal{T}([x, y]) \, dy \, dx \quad (15)$$

The set of bbd whose pignistic probability is equal to $BetP$ is written $\mathscr{B}Iso(BetP)$. The issue is to choose a function in this set. Several works deal with the problem of the ordering of continuous belief functions [10, 3]. Ph. Smets [10] has choosen the least commitment criteria. The aim is to choose the belief function committing the least the source. One optimization criteria can be to maximize the communality function. We can use the partial ordering:

$$\left(\forall A \subseteq \overline{\mathbb{R}}^n, q_1^{\overline{\mathbb{R}}^n}(A) \leq q_2^{\overline{\mathbb{R}}^n}(A)\right) \Longrightarrow \left(m_1^{\overline{\mathbb{R}}^n} \sqsubseteq_q m_2^{\overline{\mathbb{R}}^n}\right) \quad (16)$$

Ph. Smets [10] has proven that the Least Committed basic belief density (LC bbd) linked to a pdf on $\overline{\mathbb{R}}$ whose the graph is “bell-shaped” is consonant. For each interval $[a,b]$ of $\overline{\mathbb{R}}$ we have:

$$m^{\overline{\mathbb{R}}}([a,b]) = (\gamma(b) - b)\frac{dBetf(b)}{db}\delta(a - \gamma(b)) \quad (17)$$

with ν the mode of $Betf$ (the axis $x = \nu$ is the symetrical axis of the curve), b in $[\nu, \infty]$, and $\gamma(b)$ in $[-\infty, \nu]$ such as $Betf(b) = Betf(\gamma(b))$. The focal elements of this belief function are the α-cuts[1] of $Betf$. In [1], F. Caron *et al.* have given the expression of the bbd linked to the Gaussian of $\mathbb{R}^n$. They have proven that their focal elements are the confidence sets of the linked Gaussian. These approaches to model continuous belief functions on real numbers are all fonded on the idea to describe focal elements thanks to a continuous function. However, they only take into account the frames of discernment built with connected set of $\overline{\mathbb{R}}^n$. It raises some problems. One of them is that the α-cuts of a multimodal function are not connected sets (*cf.* example 2 in section 5). If our frame of discernment is $\mathcal{B}(\mathbb{R}^n)$, we cannot compute the consonant belief function linked to a multimodal pdf.

4 Continuous belief functions with focal elements in $\mathcal{B}\left(\overline{\mathbb{R}}^n\right)$

Using an index to describe the focal elements is handy when we want to use efficiently the belief functions. In this section, we will present an approach of belief functions founded on the definition of a function describing the set of focal elements.

4.1 Credal measure

We try to model a belief on $\overline{\mathbb{R}}^n$, $\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}$. The set of focal elements linked to this belief is written $\mathcal{F}\left(\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}\right)$. If we can define a onto mapping f^I named index function such as:

$$\begin{array}{llll} f^I: & I \in \mathcal{B}\left(\overline{\mathbb{R}}^l\right) & \longrightarrow \mathcal{F}\left(\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}\right) \\ & y & \longmapsto f^I(y) \end{array} \quad (18)$$

We can consider $\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ as a positive measure on a measurable space $(I, \mathcal{B}(I))$ which verifies the condition $\int_I d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \leq 1$. If for each $A \in \mathcal{B}\left(\overline{\mathbb{R}}^n\right)$, the following sets belong to $\mathcal{B}(I)$:

$$F_{\subseteq A} = \{y \in I | f^I(y) \subseteq A\} \quad (19)$$

$$F_{\cap A} = \{y \in I | \left(f^I(y) \cap A\right) \neq \emptyset\} \quad (20)$$

$$F_{\supseteq A} = \{y \in I | A \subseteq f^I(y)\} \quad (21)$$

We name the measurable space $\left(I, \mathcal{B}(I), \mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}\right)$ *credal space* and the positive measure linked *credal measure*. We define for all $A \in \mathcal{B}\left(\overline{\mathbb{R}}^n\right)$ the following belief functions:

$$bel^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \int_{F_{\subseteq A}} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \quad (22)$$

$$pl^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \int_{F_{\cap A}} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \quad (23)$$

$$q^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \int_{F_{\supseteq A}} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \quad (24)$$

We note that l, the dimension of I (the index space), do not depend on n, the dimension of the space where we exprime a belief. To quote an example, in [10], Ph. Smets suggests to use subsets of $\overline{\mathbb{R}}^2$ to describe focal elements of a belief on $\overline{\mathbb{R}}$ while F. Caron *et al.* in [1] use an index space of one dimension to describe the focal elements of a Gaussian belief function on $\overline{\mathbb{R}}^n$.

When several sources of information are available, we can use the conjonctive rule of combination to merge them. We prove the theorem:

Theorem 1. *Let $\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ and $\mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ be two credal measures. We obtain after a conjonctive combination the credal measure $\mu_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ which verifies the equality:*

$$q_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = q_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) \cdot q_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) \quad (25)$$

Proof. Let A be in $\mathcal{B}\left(\overline{\mathbb{R}}^n\right)$. We have:

$$q_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) \cdot q_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \int_{F_{\supseteq A}^1} d\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y_1) \cdot \int_{F_{\supseteq A}^2} d\mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y_2) \quad (26)$$

According to the theorem of Fubini, we have:

$$\begin{array}{l} q_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) \cdot q_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \\ \int_{F_{\supseteq A}^1} \int_{F_{2 \supseteq A}} d\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y_1)\, d\mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y_2) = \\ \int_{F_{\supseteq A}^1} \int_{F_{2 \supseteq A}} d\left(\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)} \otimes \mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}\right)(y_1, y_2) \end{array} \quad (27)$$

Let $f^{I_{1 \circledcirc 2}}$ be a mapping such as:

$$\begin{array}{lll} f^{I_{1 \circledcirc 2}}: & I_{1 \circledcirc 2} = I_1 \times I_2 & \longrightarrow \mathcal{F}\left(\mu_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)}\right) \\ & y = (y_1, y_2) & \longmapsto f^{I_1}(y_1) \cap f^{I_2}(y_2) \end{array} \quad (28)$$

We have:

$$F_{\subseteq A}^{1 \circledcirc 2} = \left(F_{\subseteq A}^1 \times I_2\right) \cup \left(I_1 \times F_{\subseteq A}^2\right) \quad (29)$$

$$F_{\cap A}^{1 \circledcirc 2} = F_{\cap A}^1 \times F_{\cap A}^2 \quad (30)$$

$$F_{\supseteq A}^{1 \circledcirc 2} = F_{\supseteq A}^1 \times F_{\supseteq A}^2 \quad (31)$$

These sets belong to a σ-algebra, so $f^{I_{1 \circledcirc 2}}$ can be seen as an index function. Therefore we can build a credal measure $\mu_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ such as:

$$\mu_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)} = \mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)} \otimes \mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)} \quad (32)$$

[1] the α-cuts of a function f from $\overline{\mathbb{R}}^n$ to $\mathbb{R}^+$ are the sets $\{y \in \overline{\mathbb{R}}^n | f(y) \geq \alpha\}$.

Hence:
$$q_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A)=\int_{F_{\supseteq A}^{1 \circledcirc 2}} d\mu_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \tag{33}$$

We obtain:

$$q_{1 \circledcirc 2}^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = q_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) \cdot q_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) \tag{34}$$

□

Let f^{I_1} et f^{I_2} be two index functions linked to the credal measures $\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ et $\mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}$. Let φ be an onto mapping from I_1 to I_2 such as $\varphi(y_1) = y_2$ implies $f^{I_1}(y_1) = f^{I_2}(y_2)$. Let $H_1 \subset I_1$ and $H_2 \subset I_2$ be two elements of Borel sigma-algebra. If $\varphi(H_1) = H_2$ and $\varphi^{-1}(H_2) = H_1$ imply $\int_{H_1} d\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y_1) = \int_{H_2} d\mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y_2)$, then the two beliefs linked to the credal measures are equal.

Theorem 2. *Let f^{I_1} et f^{I_2} be two index functions linked two credal measures $\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ et $\mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}$. Let φ be a bijection such as $\varphi(y_1) = y_2$ implies $f^{I_1}(y_1) = f^{I_2}(y_2)$. These credal measures are equal if:*

$$d\mu_1^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y_1) = |det(\varphi'(y_1))|\, d\mu_2^{\mathcal{B}(\overline{\mathbb{R}}^n)}(\varphi(y_1)) \tag{35}$$

4.2 Consonant credal measure

The consonant credal measures are a particular case of credal measure. Indeed their index functions f_{cs}^I are bijections such as:

$$\begin{array}{llll} f_{cs}^I: & I \subset \overline{\mathbb{R}}^+ & \longrightarrow & \mathcal{F}\left(\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}\right) \\ & y & \longmapsto & f_{cs}^I(y) \\ \text{and} & y_2 < y_1 & \Longleftrightarrow & f_{cs}^I(y_1) \subset f_{cs}^I(y_2) \end{array} \tag{36}$$

Thanks to the index function f_{cs}^I, we can express the value of belief function. By example, if the index space is an interval, *i.e.* $I = [0, y_{\max}]$, we have:

$$bel^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \int_{y_{\max}}^{y_1} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \tag{37}$$
with y_1 the smallest element of $F_{\subseteq A}$

$$pl^{\mathcal{B}(\overline{\mathbb{R}}^n)}(A) = \int_{y_1}^{0} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \tag{38}$$
with y_1 the biggest element in $F_{\cap A}$

$$q^{\mathcal{B}(\overline{\mathbb{R}}^n)}(B) = \int_{y_1}^{0} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \tag{39}$$
with y_1 the biggest element in $F_{\supseteq A}$

We notice that the conjonctive combination of two consonant credal measures is not consonant. It is a problem if we want to merge a big quantity of information or realize a dynamic merging. A solution is to substitute the credal measure by the isopignistic consonant credal measure.

4.3 Consonante credal measure induced by a multimodal probability density

We model the different sources of information with multimodal pdf. To merge these sources using the theory of belief functions, we have to link these probabilities to belief functions. The pignistic transformation in the case of credal measure is written for each $A \in \mathcal{B}\left(\overline{\mathbb{R}}^n\right)$:

$$BetP(A) = \int_{F_{\cap A}} \frac{\lambda\left(A \cap f^I(y)\right)}{\lambda\left(f^I(y)\right)} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \tag{40}$$

We notice in this case that $\lambda(B)$ is Lebesgue's measure of the hypervolume B, element of $\mathcal{B}\left(\overline{\mathbb{R}}^n\right)$ (we decide that $0/0 = 1$). Let $Betf$ be a continuous pdf on $\overline{\mathbb{R}}^n$. We have $Betf\left(\overline{\mathbb{R}}^n\right) = [0, \alpha_{\max}] = I$. We obtain the index function f_{cs}^I such as $f_{cs}^I(\alpha)$ (with α in I) is the α-cut of $Betf$ and define a consonant credal measure $\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}$ linked to $Betf$.

Theorem 3. *Let $Betf$ be a continuous pdf. We can associate it to $\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}$, a consonant credal measure whose the focal element are the α-cuts of $Betf$ such as:*

$$d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(\alpha) = \lambda\left(f_{cs}^I(\alpha)\right) d\lambda(\alpha) \tag{41}$$

Proof. We will use two different expressions of $BetP\left(f_{cs}^I(\alpha)\right)$. Thanks to the pignistic transformation, we have:

$$BetP\left(f_{cs}^I(\alpha)\right) = \int_{\alpha_{max}}^{0} \frac{\lambda\left(f_{cs}^I(\alpha) \cap f_{cs}^I(y)\right)}{\lambda\left(f_{cs}^I(y)\right)} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \tag{42}$$

Let ν be the measure such as:

$$\lambda\left(f_{cs}^I(\alpha)\right) = \int_{\alpha_{max}}^{\alpha} d\nu(y) \tag{43}$$

Then:
$$BetP\left(f_{cs}^I(\alpha)\right) = \int_{\alpha_{max}}^{\alpha} y\, d\nu(y) \tag{44}$$

By differentiating these two expressions, we have:

$$d\nu(\alpha) \int_{\alpha_{max}}^{\alpha} \frac{1}{\lambda\left(f_{cs}^I(y)\right)} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) = \alpha\, d\nu(\alpha) \tag{45}$$

Hence:

$$\alpha = \int_{\alpha_{max}}^{\alpha} \frac{1}{\lambda\left(f_{cs}^I(y)\right)} d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(y) \tag{46}$$

By differentiating, we have:

$$d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(\alpha) = \lambda\left(f_{cs}^I(\alpha)\right) d\lambda(\alpha) \tag{47}$$

□

We can build a consonant credal measure for each continuous pdf. We will apply this result by building the consonant credal measure induced by a Gaussian mixture.

5 Applications

To illustrate our results, we will use the previous theorems to build the consonant credal measure induced by a continuous pdf. To begin, we will demonstrate a classic result with continuous belief functions, the value of the consonante belief function induced by a Gaussian.

Example 1 (Application to a Gaussian pdf). Let $Betf$ be the pdf of a standard Gaussian distribution. We define $Betf^{-1}$ as the inverse function of $Betf$ restrained to $\mathbb{R}^+$. It is a bijection. According to the theorem 3, it induces a credal measure such as:

$$d\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}(\alpha) \begin{array}{l} = \lambda\left(f_{cs}^I(\alpha)\right) d\lambda(\alpha) \\ = 2Betf^{-1}(\alpha)\, d\lambda(\alpha) \end{array} \tag{48}$$

As $\alpha = Betf(x)$, we have:

$$d\lambda(\alpha) = Betf'(x)\, d\lambda(x) = xBetf(x)\, d\lambda(x) \tag{49}$$

Hence, according to the theorem 2, the credal measure:

$$d\tilde{\mu}^{\mathcal{B}(\overline{\mathbb{R}}^n)}(x) = 2x^2 Betf(x)\, d\lambda(x) \tag{50}$$

describe the same belief as $\mu^{\mathcal{B}(\overline{\mathbb{R}}^n)}$. That is the result given by Ph. Smets in [10].

We can use theorem 2 and 3 to build a consonante credal measure linked to a Gaussian pdf. Unfortunately, it is not always easy to find the analytic expression of $BetP \circ f_{cs}^I$ and of $\lambda \circ f_{cs}^I$. However, we can compute a numerical approximation of $\lambda(f_{cs}(\alpha))$. We will compute the numerical approximation of the credal measure induced by a Gaussian mixture. The results will be compared with the ones obtain in [1].

Example 2 (Application to a Gaussian mixture). In [1], F. Caron *et al.* give an expression of bbd induced by a Gaussian pdf on $\overline{\mathbb{R}}^n$. To build a bbd induced by a Gaussian mixture $f = \sum_i \beta_i f_i$, they decide to create it in a such way that the plausibility verifies the equality $pl = \sum_i \beta_i pl_i$. Hence we obtain a belief function which is isopignistic to f. However, its focal elements are not the α-cuts of f but those of f_i. This method does not build the consonant belief function induced by f. That has an influence on the value taken by pl. We will work on the Gaussian mixture plotted in figure 1. The numerical approximations of $BetP \circ f_{cs}^I$ and $\lambda \circ f_{cs}^I$ are plotted in figure 2. The obtained plausibility of the belief function with theorem 3 is clearly bigger than the one obtained thanks to [1] and its shape is different (*cf.* figure 3). We can deduce that it implies less commitment on singleton and if we want to use it to make classification by using the generalized Bayes theorem, we will obtain a different result.

6 Conclusion

As presented in this paper, we can extend the approach proposed in [10, 12] to describe complex focal elements. With this extended model, it is possible to induce a consonant belief function from a multimodal continuous pdf. In further development, it would be interrresting to prove that these functions are the least committed on singleton for an isopignistic set as it is in [10]. Moreover, the study of the computational cost would be interresting in the perpective of a practical implementation. In a first time, we can imagine to use it in Joint Tracking and Classification problems (*cf.* [1, 11]). We could also use this approach to develop some methods to estimate continuous parameters.

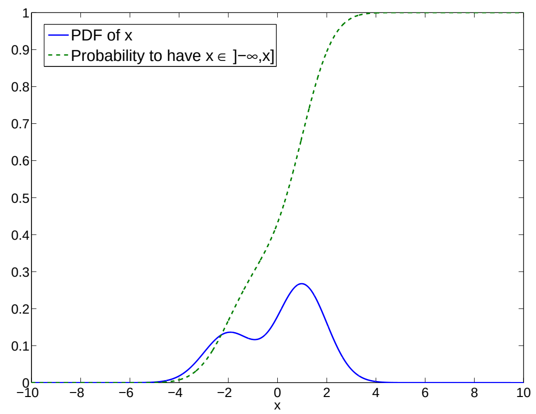

Figure 1: Gaussian mixture

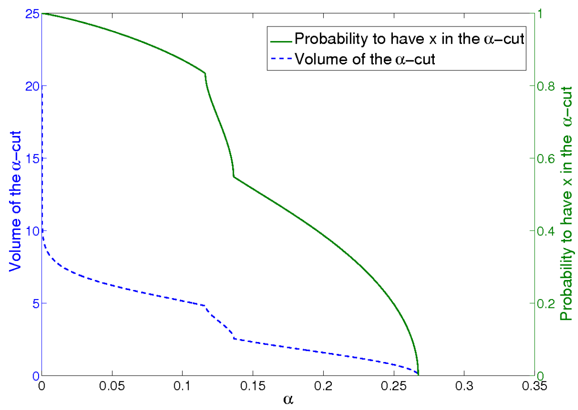

Figure 2: Study of α-cuts

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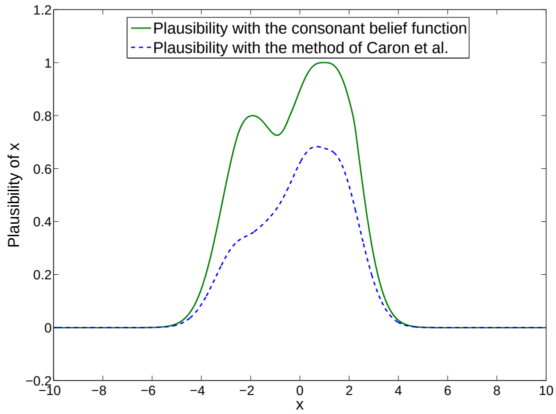

Figure 3: Comparaison of plausibility functions

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