Classical hardness of the Learning with Errors problem

Adeline Langlois

Aric Team, LIP, ENS Lyon

Joint work with
Z. Brakerski, C. Peikert, O. Regev and D. Stehlé

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Our main result

Not quantum

GapSVP in dimension $\sqrt{n}$

A classical reduction from a worst-case lattice problem to the Learning with Errors problem with small modulus.

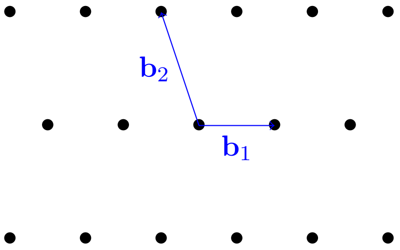
Dimension n

Polynomial in n

Outline

1. Lattices: definitions and problems
2. Lattice-based cryptography:
   LWE and a public-key encryption
3. Our main result:
   classical hardness of LWE for polynomial modulus
4. Other results on LWE.

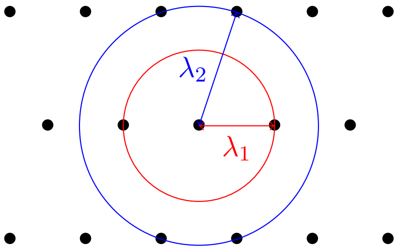
Lattices and problems



Lattice

$\mathcal{L}(\mathbf{B}) = \{\sum_{1=i}^{n} a_i \mathbf{b}_i, a_i \in \mathbb{Z}\}$, where the $(\mathbf{b}_i)_{1 \leq i \leq n}$'s, linearly independent vectors, are a basis of $\mathcal{L}(\mathbf{B})$.

Lattices and problems



Definitions:

- 1st minimum;
- 2nd minimum.

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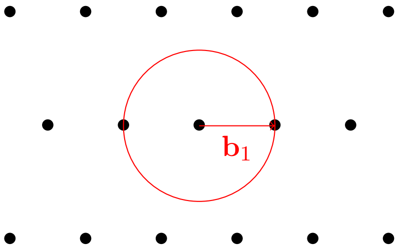

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Problems :

- Shortest Vector Pbm. (computational or decisional version)

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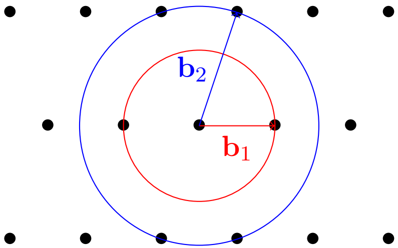

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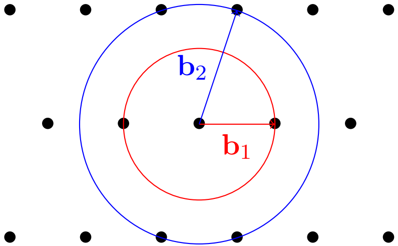
Lattices and problems



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Problems :

- Shortest Vector Pbm. (computational or decisional version)
- Shortest Independent Vectors Pbm.
- Approximation factor: γ.

Conjecture

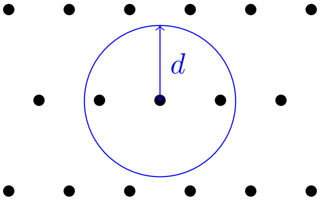
There is no polynomial time algorithm that approximates these lattice problems to within polynomial factors.

GapSVP

Gap Shortest Vector Problem (GapSVP$_\gamma$)

Input : a basis $\mathbf{B}$ of a lattice Λ and a number d,

Output :
- YES: there is $\mathbf{z} \in \Lambda$ non-zero such that $\|\mathbf{z}\| < d$,
- NO: for all non-zero vectors $\mathbf{z} \in \Lambda$: $\|\mathbf{z}\| \geq d$.



Best known algorithm: complexity $2^{\Omega(\frac{n \log \log n}{\log n})}$.

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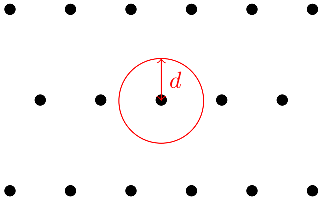

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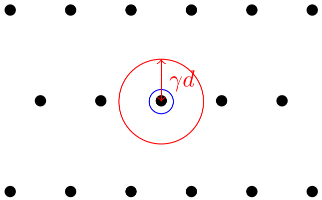
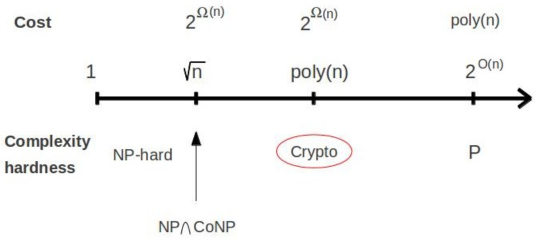


Approximation factor: γ.

Best known algorithm: complexity $2^{\Omega\left(\frac{n \log \log n}{\log n}\right)}$.

Hardness of GapSVP_γ



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LWE-based cryptography

From basic to very advanced primitives

- Public key encryption
  [Regev 2005, ...];
- Identity-based encryption
  [Gentry, Peikert and Vaikuntanathan 2008, ...];
- Fully homomorphic encryption
  [Brakerski and Vaikuntanathan 2011, ...].

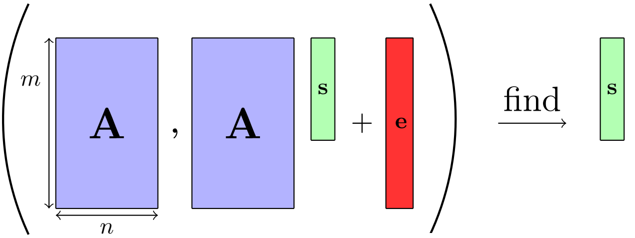
Advantages of LWE-based primitives

- Efficient, especially when the **modulus is polynomial**;
- Security proofs **from the hardness of LWE**;
- Likely to resist attacks from quantum computers.

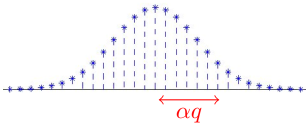
The Learning With Errors problem [Regev05]

LWE_q^n

Given



- $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m \times n})$,
- $\mathbf{s} \leftarrow U(\mathbb{Z}_q^{n})$,
- $\mathrm{e} \sim D_{\mathbb{Z}^m, \alpha q}$ with $\alpha = o(1)$.



Discrete Gaussian error

Decision version: Distinguish from $(\mathbf{A}, \mathbf{b})$ with $\mathbf{b}$ uniform.

An example of Public-Key Encryption[Regev 2005]

- **Parameters**: $n, m, q \in \mathbb{Z}$, $\alpha \in \mathbb{R}$,
- **Keys**: $\text{sk} = \mathbf{s}$ and $\text{pk} = (\mathbf{A}, \mathbf{b})$, with $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$ where $\mathbf{s} \hookleftarrow U(\mathbb{Z}_q^n)$, $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m \times n})$, $\mathbf{e} \hookleftarrow D_{\mathbb{Z}^m, \alpha q}$.
- **Encryption** $(M \in \{0, 1\})$**:** Let $\mathbf{r} \hookleftarrow U(\{0, 1\}^m)$,

$$\mathbf{u}^T = \mathbf{r} \mathbf{A}, \quad v = \mathbf{r} \mathbf{b} + \lfloor q/2 \rceil \cdot M$$

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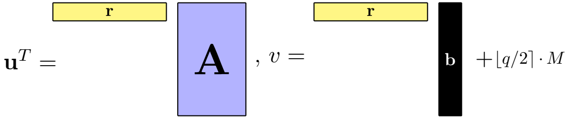
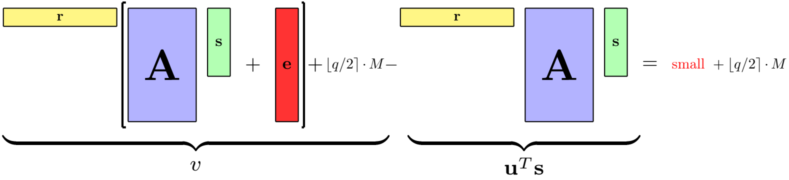


- **Decryption** of $(\mathbf{u}, v)$: compute $v - \mathbf{u}^T\mathbf{s}$,



If close from 0: return 0, if close from $\lfloor q/2 \rceil$: return 1.

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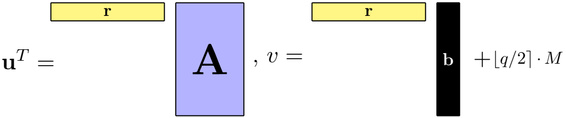

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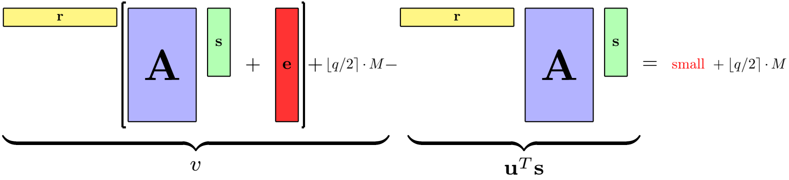

LWE hard $\Rightarrow$ Regev's scheme is "secure".

Reminders

- Hard problem on lattices: GapSVP.
- Lattice-based cryptography:
  Security proof based on reduction from GapSVP to a problem (= a protocol attacker).
- Learning With Errors problem:
  Distinguish between $(\mathbf{A}, \mathbf{b})$ uniform and $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q)$, where $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m \times n})$, $\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$ is secret, and $\mathbf{e}$ Gaussian.
- Public-key encryption: security based on hardness of LWE.

Prior reductions from worst-case lattice problems to LWE

- **[Regev05]**
  - A **quantum** reduction;
  - with q polynomial.

Quantum computer?

- **[Peikert09]**
  - A **classical** reduction;
  - with q exponential,

Inefficient primitives

- **[Peikert09]**
  - A **classical** reduction;
  - based on a non-standard lattice problem;
  - with q polynomial.

Hardness?

Prior reductions from worst-case lattice problems to LWE

- **[Regev05]**
  - A **quantum** reduction;
  - with q polynomial.
- **[Peikert09]**
  - A **classical** reduction;
  - with q exponential,
- **[Peikert09]**
  - A **classical** reduction;
  - based on a non-standard lattice problem;
  - with q polynomial.

Our main result

- A **classical** reduction,
- from a standard worst-case lattice problem,
- with q polynomial.

Main component in the proof: a self reduction

- Recall that [Peikert09] already showed hardness of LWE with q exponential.

How do we obtain a hardness proof for q polynomial?

Main component in the proof: a self reduction

- Recall that [Peikert09] already showed hardness of LWE with q exponential.

How do we obtain a hardness proof for q polynomial?

- All we have to do is show the following reduction:

From LWE	in dimension n with modulus q^k,
to LWE	in dimension nk with modulus q.

Modulus Switching

A reduction from LWE with modulus q to LWE with modulus p.

How to map $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e}) \bmod q$ to $(\mathbf{A}', \mathbf{A}'\mathbf{s} + \mathbf{e}') \bmod p$?

- Transform $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m \times n})$ to $\mathbf{A}' \hookleftarrow U(\mathbb{Z}_p^{m \times n})$;
  First idea: $\mathbf{A}' = \lfloor \frac{p}{q} \mathbf{A} \rceil$?

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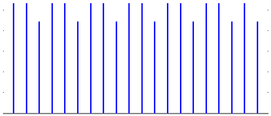
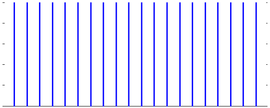
  First idea: $\mathbf{A}' = \lfloor \frac{p}{q} \mathbf{A} \rceil$?
- Two main problems:
  1. The distribution is not uniform:



A naive rounding introduces artefacts.

solution



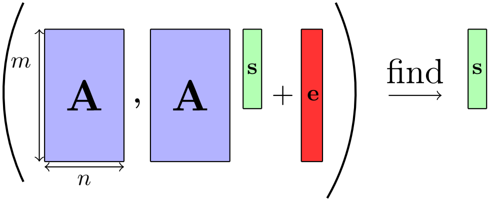
Add a **Gaussian rounding** to smooth the distribution: $\mathbf{A}' = \frac{p}{q}\mathbf{A} + \mathbf{R}$.

  2. In $\mathbf{A}'\mathbf{s} + \mathbf{e}'$, $\mathbf{e}' = \mathbf{R}\mathbf{s} + \frac{p}{q}\mathbf{e}$: the rounding errors gets multiplied by the secret $\mathbf{s}$ (which is uniform is $\mathbb{Z}_q^n$).

From large to small secret

From LWE with arbitrary secret to LWE with binary secret.

- Inspired by ideas from cryptography (prior reduction by **[Goldwasser, Kalai, Peikert and Vaikuntanathan 2010]**) ; but different and stronger techniques.
- Definition of LWE:



- From $\mathbf{s}$ uniform in $\mathbb{Z}_q^n$ to $\mathbf{s}$ uniform in $\{0,1\}^n$.
- **Consequence:** this reduction expands the dimension from n to $n \log q$.

Summary of our new hardness proof of LWE

Our main result

A classical reduction from GapSVP in dimension $\sqrt{n}$ to LWE in dimension n with poly(n) modulus.

Reductions of the proof:

Problem	Dimension	Modulus	Secret	
GapSVP	$\sqrt{n}$			
$\downarrow_0$				**[Peikert09]**
LWE	$\sqrt{n}$	large	$\mathbb{Z}_q^{\sqrt{n}}$	
$\downarrow_1$				**New**
LWE	n	large	small	
$\downarrow_2$				**New**
LWE	n	poly(n)	in $\mathbb{Z}_q^n$	

Other main contributions

Hardness of LWE:

- **Shrinking modulus / Expanding dimension**:
  A reduction from $\mathrm{LWE}_{q^k}^{n}$ to LWE_{q}^{nk}.
- **Expanding modulus / Shrinking dimension**:
  A reduction from LWE_{q}^{n} to $\mathrm{LWE}_{q^k}^{n/k}$.

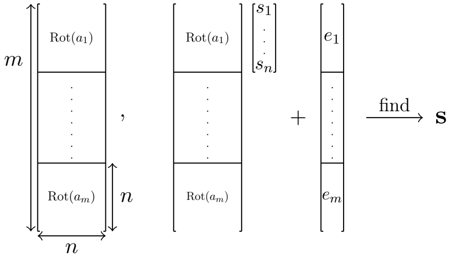
  $\Rightarrow$ The hardness of LWE_q^n is a function of $n \log q$.

Consequences:

- Hardness of $\mathrm{LWE}_{2^n}^{1}$ (Hidden Number Problem).

Ring-LWE

Idea: $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$ (with $n = 2^k$) instead of $\mathbb{Z}^n$.



Why ?

Size of keys and cost of operations.

Luybashevsky, Peikert and Regev 2010

Quantum reduction from Ideal-SIVP to Ring-LWE for q prime such that $q = 1 \bmod 2n$.

Other main contributions

Hardness of LWE:

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  $\Rightarrow$ The hardness of LWE_q^n is a function of $n \log q$.

Consequences:

- Hardness of $\mathrm{LWE}_{2^n}^{1}$ (Hidden Number Problem).
- The Ring-LWE problem in dimension n with exponential modulus is hard under hardness of general lattices problems (not ideal lattices).

Conclusion

Our main result

A classical reduction from GapSVP in dimension $\sqrt{n}$
to LWE in dimension n with poly(n) modulus.

Open problems:

Is there a classical reduction as good as the one in **[Regev05]**?

1. We lose a quadratic term in the dimension;
2. We only get GapSVP and not SIVP.

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1. We lose a quadratic term in the dimension;

   Recall that the **[Peikert09]** reduction is from GapSVP in dimension $\sqrt{n}$ to LWE with dimension $\times$ log(modulus) $= n$.

   Is this reduction sharp?

Conclusion

Our main result

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Open problems:

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1. We lose a quadratic term in the dimension;
2. We only get GapSVP and not SIVP.

In (quantum) **[Regev05]** the worst-case lattice problem is SIVP.

SIVP feels like a harder problem than GapSVP